Representation of Even Integers as Sums of Squares of Primes and Powers of 2

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As an extension of the Linnik-Gallagher results on the “almost Goldbach” problem, we prove that every large even integer is a sum of four squares of primes and 8330 powers of 2.

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1. INTRODUCTION

In 1951 and 1953, Linnik [11, 12] successfully solved the “almost Goldbach” problem by proving that each large even integer $N$ is a sum of two primes and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \quad (1.1)$$

where (and throughout) $p$ and $\nu$, with or without subscripts, denote a prime number and a positive integer respectively. Linnik’s result was generalized by A. I. Vinogradov [23] in several directions. Later Gallagher [2] established, by a different method, a stronger result from which the theorems in [11, 12, 23] can be deduced.

The results mentioned above have recently been extended in [17] to the representation of $N$ as a sum of four squares of primes and powers of 2, i.e.

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \quad (1.2)$$

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Let
\[ r_k(N) = \sum_{N = p_1^2 + \cdots + p_4^2 + 2^{\alpha} + \cdots + 2^{\alpha}} (\log p_1) \cdots (\log p_4). \]

It has been proved in [17] that for \( N \equiv 4 \pmod{8} \), we have
\[ r_k(N) = \frac{\pi^2}{16} \left( \sum_{n \in \mathcal{S}(N, k)} 1 + O\left( \frac{1}{k} \right) \right). \quad (1.3) \]

Here \( \mathcal{S}(n, k) = \{ n \geq 2 : n = N - 2^{\alpha} - \cdots - 2^{\alpha} \} \), \( \mathcal{S}(n) \) is the singular series related to the representation of \( n \) as four squares of primes (see (6.3) below), and the \( O \)-constant is absolute. Since \( \mathcal{S}(n) \gg 1 \) for \( n \equiv 4 \pmod{24} \) (see Lemma 6.2 below), one deduces from (1.3) that there exists an absolute \( k \) such that every large even integer can be expressed as (1.2).

Noting that for any even integer \( N \) there exist \( +1, +2 \), \( +1 \), or \( +3 \) such that \( N - 2^{\alpha} - 2^{\alpha} \equiv 4 \pmod{8} \), one deduces further that every large even integer \( N \) can be written as four squares of primes and a bounded number of powers of 2.

Our asymptotic formula (1.3) is similar in style to those obtained in [11, 12, and 2], in the sense that the term \( O(1/k) \) arises. The implied \( O \)-constant in [17], as in [11, 12, 2], is not known at all, so it is not clear that how many powers of 2 are needed to ensure \( r_k(N) > 0 \). What one knows about this \( O \)-constant is that it depends on several intricate matters including the distribution of zeros of Dirichlet \( L \)-functions, and therefore it must be large. So one may anticipate that a small \( k \) in (1.3) is not sufficient to give the positiveness of \( r_k(N) \).

In this paper we establish the following result.

**Theorem 1.** For any integer \( k \geq 8330 \) there exists a positive constant \( N_k \) depending on \( k \) only, such that each even integer \( N \geq N_k \) is a sum of four squares of primes and \( k \) powers of 2.

The idea of the proof of Theorem 1 will be given in Section 2. Here we remark that our proof of Theorem 1 actually gives lower bounds for \( r_k(N) \) uniformly for \( k \geq 8330 \), for which the reader is referred to (6.9) and (6.10).

Our Theorem 1 can be compared with those in [14–16] concerning the “almost Goldbach” problem (1.1), which state that under the generalized Riemann hypothesis \( k = 200 \) is acceptable, and unconditionally one can take \( k = 54,000 \). However, it should be pointed out that our approach leading to Theorem 1 is essentially different from those used in [14–16]. We also remark that this paper is not just a quantitative version of [17], because our arguments here depart from [17] in two main aspects. First,
our major arcs \( \mathcal{M} \) in Lemma 2.1 below are enlarged considerably (note that in Lemma 2.1 we have \( P = N^{1/15 - \varepsilon} \) in (2.1) while \( P = N^{1/25} \) in [17]). Second, to get the bound for \( c_1 \) in Theorem 2 below we apply in our Section 4 some recent results on vector-sieve in [1] instead of the four-dimensional sieve used in [17].

Our investigation on (1.2) is not only motivated by the “almost Goldbach” results mentioned above, but also by the Lagrange Theorem of four squares, and the following works in [6, 4, 22, 18, 10, 1, and 13]. In 1938, Hua [6] proved that each large integer congruent to 5 (mod 24) can be written as a sum of five squares of primes. In view of this result and Lagrange’s theorem of four squares, it seems reasonable to conjecture that each large integer \( n \equiv 4 \) (mod 24) is a sum of four squares of primes,

\[
n = p_1^2 + p_2^2 + p_3^2 + p_4^2.
\]

Our Theorem 1 may be viewed as an approximation to this conjecture. There are other approximations, and our result can be compared with them. Greaves [4] gave a lower bound for the number of representations of an integer as a sum of two squares of integers and two squares of primes. Later Shields [21], Plaksin [18], and Kovalchik [10] obtained, among other things, an asymptotic formula in this problem. Brüdern and Fouvry [1] proved that all large \( n \equiv 4 \) (mod 24) is the sum of four squares of integers with each of their prime factors greater than \( n^{1/6.86} \). Very recently the authors [13] proved that, with at most \( O(N^{1/15 + \varepsilon}) \) exceptions, all positive integers \( n \equiv 4 \) (mod 24) not exceeding \( N \) can be written as (1.4). For history and references in this direction, see [13].

Notation. As usual, \( \varphi(n) \) and \( \mu(n) \) stand for the function of Euler and Möbius respectively. \( N \) is a large integer and \( L = \log_2 N \). If there is no ambiguity, we express \( a/b + \theta \) as \( a/b + \theta \) or \( \theta + a/b \). The same convention will be applied for quotients. The letter \( \varepsilon \) denotes a positive constant which is arbitrarily small.

2. OUTLINE OF THE METHOD

Our proof of Theorem 1 depends essentially on the following Lemma 2.1 and Theorem 2. Lemma 2.1 was proved by the circle method in [13, Theorem 2], while Theorem 2 will be established in this paper by sieve methods. In order to introduce our Lemma 2.1, we set

\[
P = N^{2/15 - \varepsilon}, \quad Q = N/(PL^{14}).
\]
By Dirichlet’s lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form
\[
\alpha = a/q + \lambda, \quad |\lambda| \leq 1/(qQ) \tag{2.2}
\]
for some integers $a, q$ with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. We denote by $\mathcal{M}(a, q)$ the set of $\alpha$ satisfying (2.2) and define the major arcs $\mathcal{M}$ and the minor arcs $C(\mathcal{M})$ as follows:
\[
\mathcal{M} = \bigcup_{q \leq P} \bigcup_{a = 1}^{q} \mathcal{M}(a, q), \quad C(\mathcal{M}) = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathcal{M}. \tag{2.3}
\]
It follows from $2P \leq Q$ that the major arcs $\mathcal{M}(a, q)$ are mutually disjoint. Let
\[
T(x) = \sum_{p \leq N} (\log p) e(p^2x), \quad G(x) = \sum_{2^v \leq N} e(2^v x) = \sum_{v \leq \log N} e(2^v x). \tag{2.4}
\]
Then $r_k(N)$ can be written as
\[
r_k(N) = \int_{0}^{1} T^4(x) G^k(x) e(-Nx) \, dx = \left( \int_{\mathcal{M}} + \int_{C(\mathcal{M})} \right) T^4(x) G^k(x) e(-Nx) \, dx. \tag{2.5}
\]
One sees from (2.1) that our major arcs are quite large. However, we manage to give an asymptotic formula for the integral on the major arcs, by using the following result of our earlier paper [13, Theorem 2].

**Lemma 2.1.** Let $\mathcal{M}$ be as in (2.3) with $P$ determined by (2.1). Then for $2 \leq n \leq N$, we have
\[
\int_{\mathcal{M}} T^4(x) e(-N\chi) \, dx = \frac{\pi^2}{16} \Xi(n) n + O\left( \frac{N}{\log N} \right). \tag{2.6}
\]
Here $\Xi(n)$ is defined in (6.3) and satisfies $\Xi(n) \gg 1$ for $n \equiv 4 \pmod{24}$.

The lemma only gives an $O$-result if $n$ is much smaller than $N$ but is useful for our purpose even in its weak form.

Thus the main difficulty lies in the minor arcs. Here a crucial step is to get an upper bound for the number of solutions of the equation
\[
n = p_1^3 + p_2^3 - p_3^2 - p_4^2, \quad |n| \leq N, \ p_j^2 \leq N. \tag{2.7}
\]
The following Theorem 2 will serve for this purpose.
THEOREM 2. Let \( n \neq 0 \) be an integer with \( n \equiv 0 \) (mod 24), and \( r_-(n) \) the number of representations of \( n \) in the form (2.7). Then we have

\[
    r_-(n) \leq c_1 \tilde{\mathcal{Z}}_-(n) \frac{\pi^2 N}{16 \log^4 N}
\]

with \( c_1 \leq (1 + \varepsilon)^6 11^4 n^{24}/2^{24} \) and

\[
    \tilde{\mathcal{Z}}_-(n) = \left( 2 \frac{1}{2^{\beta_0}} - \frac{1}{2^6} \right) \prod_{p \equiv 3 \pmod{\beta_0}} \left( 1 + \frac{1}{p^2 - 1} - \frac{1}{p^{p-2}} \right)
\]

where \( \beta_0 \) satisfies \( 2^{\beta_0} \mid n \).

For the definition of \( \tilde{\mathcal{Z}}_-(n) \) via infinite series, see (3.4), (3.5), and (3.7). Here we remark that \( \tilde{\mathcal{Z}}_-(n) \) is the singular series associated with the representation \( n = m_1^2 + m_2^2 - m_3^2 - m_4^2 \) where all the \( m_i \) are positive integers. Hence \( \tilde{\mathcal{Z}}_-(n) \) is different from the \( \mathcal{Z}(n) \) in Lemma 2.1.

Theorem 2 will be proved in Sections 3 and 4 by the “vector sieve” of [1].

The four-dimensional upper bound sieve as used in [17] also manages to establish Theorem 2, but with a larger \( c_1 \). It turns out that the bound for \( k \) in Theorem 1 depends mainly on the size of \( P \) in Lemma 2.1 and the \( c_1 \) in Theorem 2. A larger \( P \) will give a better \( k \), and a smaller \( c_1 \) will also give a better \( k \).

3. PRELIMINARIES FOR THE VECTOR SIEVE

This and the following section are devoted to the proof of Theorem 2. In order to sieve the set

\[
    \mathcal{A} = \{ (x_1, x_2, x_3, x_4) \in \mathbb{N}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = n, 1 \leq |n| \leq N, 1 \leq x_i^2 \leq N \},
\]

we require information concerning the distribution of the sequence \( \mathcal{A} \) in arithmetic progressions.

In what follows, boldface symbols denote four-dimensional vectors, \( \mathbf{d} = (d_1, d_2, d_3, d_4) \), for example. The letter \( e \) is reserved for \((1, 1, 1, 1)\). Also, we define \( |\mathbf{d}| = \max|d_i| \) and \( \overline{\mathbf{d}} = d_1 d_2 d_3 d_4 \). For a vector \( \mathbf{d} \), we write \( \mu(\mathbf{d}) \) for \( \mu(d_1) \mu(d_2) \mu(d_3) \mu(d_4) \).
Let \( x \equiv 0 \pmod{d} \) denote the simultaneous condition \( x_j \equiv 0 \pmod{d_j} \) for \( j = 1, \ldots, 4 \). We need an asymptotic formula for the cardinality of
\[
\mathcal{A}_d = \{ x \in \mathcal{A} : x \equiv 0 \pmod{d} \},
\]
i.e. the number of solutions of the equation
\[
d_1^2 x_1^2 + d_2^2 x_2^2 - d_3^2 x_3^2 - d_4^2 x_4^2 = n, \quad 1 \leq |n| \leq N, 1 \leq d_j^2 x_j^2 \leq N. \tag{3.2}
\]
Now one recalls Kloosterman’s refinement of the Hardy-Littlewood method (see for example [1, Section II]), which is capable of treating the equation
\[
a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 = n
\]
with coefficients \( a_1, a_2, a_3, a_4 \). This refinement suggests that the main term of \(|\mathcal{A}_d|\) should be
\[
\frac{1}{d} \pi \mathfrak{Z}(n, d) \mathfrak{F} \left( \frac{n}{N} \right) N, \tag{3.3}
\]
where
\[
\mathfrak{Z}(n, d) = \sum_{q=1}^{\infty} q^{-4} \sum_{a \equiv 1 \pmod{(a, q)}} e \left( \frac{am}{q} \right) S(q, ad_1^2) S(q, ad_2^2) S(q, -ad_3^2) S(q, -ad_4^2) \tag{3.4}
\]
with
\[
S(q, a) = \sum_{m=1}^{q} e \left( \frac{am^2}{q} \right), \tag{3.5}
\]
and where
\[
\mathfrak{F}(x) = 2 \left[ \min(1, 1-x) \right] v^{-1/2} (1 - v)^{1/2} \text{div}. \tag{3.6}
\]
Clearly, for \( 1 \leq |n| \leq N \) we have \( 0 \leq \mathfrak{Z}(n/N) \leq \pi \). By an argument similar to that leading to \([1, (2.44) \text{ and } (2.45)]\), one sees that \( \mathfrak{Z}(n, d) \) is absolutely convergent. The singular series in Theorem 2 is defined by
\[
\mathfrak{Z}(n) = \mathfrak{Z}(n, e). \tag{3.7}
\]
Also, we define
\[ \omega(d) = \frac{\pi}{16} \Xi(n, d) \Xi(n), \]
so that (3.3) becomes
\[ \frac{\omega(d)}{d} = \frac{\pi}{16} \Xi(n, d) \Xi \left( \frac{n}{N} \right) N. \]

The difference between \( |a_d| \) and its main term expected above can be estimated on average, by using Kloosterman's refinement.

The following result is [17, Lemma 9.1], which is a minor modification of [1, Theorem 3].

**Lemma 3.1.** Let \( \omega(d) \) be as in (3.8) and \( D = N^{1/22 + 2\varepsilon} \). Define \( R(n, N, d) \) by
\[ |a_d| = \frac{\omega(d)}{d} \Xi(n, d) \Xi \left( \frac{n}{N} \right) N + R(n, N, d), \]
where \( \Xi(n, d) \) and \( \Xi(n/N) \) are as in (3.7) and (3.6) respectively. Then for arbitrary \( A > 0 \), we have
\[ \sum_{|d| \leq D} \mu(d) |R(n, N, d)| \ll NL^{-A}. \]

The behavior of the function \( \omega(d) \) is crucial for the sieve method. It turns out that, although \( \omega(d) \) is multiplicative for each variable \( d_j \), in general,
\[ \omega(d) \neq \omega(d_1) \omega(d_2) \omega(d_3) \omega(d_4) \]
where \( d_1 = (d_1, 1, 1, 1) \), \( d_2 = (1, d_2, 1, 1) \), etc. This problem can be solved by the method of [1]. Following [1, Section II.4], we investigate \( \omega(d) \) carefully in the following four lemmas.

We suppose throughout that \( \mu^2(d) = 1 \). For \( u, v = 0, 1, 2 \), let \( e_u, v(p) \) denote the following four-dimensional vector
\[ (p_{u}, \ldots, p_{u}, 1, \ldots, 1, p_{v}, \ldots, p_{v}, 1, \ldots, 1). \]

Clearly, \( e_{0,0}(p) = e \). It has been proved in [17, (4.7)] that \( \omega(d) \) has the decomposition
\[ \omega(d) = \prod_{p^m \nmid d} \omega_{u, v}(p). \]
Trivially, we have \( \omega_{0,0}(p) = 1 \). For \( 1 \leq u + v \leq 4 \), the values of \( \omega_{u,v}(p) \) are given in [17, Lemmas 8.1 and 8.2].

**Lemma 3.2.** With \( \omega_{u,v}(p) \) defined as in (3.11), we put

\[
\Omega(p) = 2\omega_{1,0}(p) + 2\omega_{0,1}(p) - \frac{\omega_{2,0}(p)}{p} - \frac{\omega_{0,2}(p)}{p} - \frac{4\omega_{1,1}(p)}{p^2} + \frac{2\omega_{1,2}(p)}{p^2} + \frac{2\omega_{1,2}(p)}{p^2},
\]

and

\[
W(z) = \prod_{p \leq z} \left( 1 - \frac{\Omega(p)}{p} \right).
\]

Then we have

\[
\log^{-4}\! z \ll W(z) \leq c_4 e^{-4\gamma} \log^{-4}\! z
\]

with \( c_4 \leq (1 + e) \pi^{24}/2^{32} \), where \( \gamma \) denotes the Euler constant.

*Proof.* The second inequality in (3.14) has been established in the proof of [17, Proposition 2.2, between (9.7) and (9.8)] except for the upper bound for \( c_4 \). We postpone the evaluation of \( c_4 \) and the proof for the first inequality in (3.14) until Section 7.

The following lemma can be proved in the same way as that of [1, Lemma 12].

**Lemma 3.3.** For square-free \( d \), let

\[
\omega'(d) = \prod_{p \mid d} \omega_{1,0}(p), \quad \omega''(d) = \prod_{p \mid d} \omega_{0,1}(p).
\]

For \( d = (d_1, d_2, d_3, d_4) \) with \( \mu(d) = 1 \), we put \( d_{i,j} = (d_i, d_j), \ 1 \leq i < j \leq 4 \).

Then

(i) there exists a function \( g \) of the six variables \( d_{i,j} \), such that for any \( d \) we have

\[
\omega(d) = \omega'(d_1) \omega'(d_2) \omega''(d_3) \omega''(d_4) g((d_{i,j}));
\]

(ii) there exists an absolute constant \( c \), independent of \( n \), such that for any \( d \) we have

\[
g((d_{i,j})) \leq (\max d_{i,j})^c;
\]
for any \( d \) we have the inequality
\[
| \hat{o}(d) | \leq | \hat{o}(d_1) \hat{o}(d_2) \hat{o}(d_3) \hat{o}(d_4) |
\]
where \( \hat{o} \) is the multiplicative function defined on square-free \( m \) by
\[
\hat{o} = \begin{cases} 
2, & \text{if } p \nmid m, \\
|p^{1/2}|, & \text{if } p \mid m.
\end{cases}
\]
The following is essentially [1, Section III.2, Proposition], which serves as a fundamental lemma for the vector sieve.

**Lemma 3.4.** Let \( z_0 \geq 2 \). Let \( A \) satisfy \( \mu(A) = 1 \) and \( p \mid A \Rightarrow p \geq z_0 \) or \( p = 2 \). Let
\[
S(A, z_0) = \left| \{ x \in A : p \mid x \Rightarrow p \geq z_0 \} \right|.
\]
Let \( W(z_0) \) be defined by (3.13). Then for \( D_0 \geq z_0, A \geq 1 \), one has
\[
S(A, z_0) = \left| \left\{ x \in A : p \mid x \Rightarrow p \geq z_0 \right\} \right| + O\left( H^5(n) A^{-1/2} \log^{13} D_0 + A^{e^{-s}} \right)
\]
\[
\times \frac{\omega(A)}{I} \frac{\pi}{16} \int \zeta(n) \frac{N}{N} N
\]
\[
+ O\left( \sum_{p \mid d : 2 < p < z_0} \mu^2(d) |R(n, N, dl)| \right).
\]
The implied constants are absolute, \( dl = (d_1, \ldots, d_4) \), and we have written
\[
s_0 = \frac{\log D_0}{\log z_0}, \quad H(n) = \prod_{p \mid n} \left( 1 + p^{-1/2} \right).
\]

4. APPLICATION OF THE VECTOR SIEVE: PROOF OF THEOREM 2

We fix a positive number \( D \). Let \{ \hat{x}_y^+ \} be one of the two sequences of Rosser’s weights related to \( D \); for details see Iwaniec [8, 9; or 1, p. 84]. Let \( \mathcal{P} \) be a set of primes and put
\[
P(z) = \prod_{p \in \mathcal{P}, p < z} p.
\]
For two arithmetical functions $g$ and $h$, we denote their Dirichlet convolution by $g \ast h$, i.e.,

$$(g \ast h)(n) = \sum_{d|n} g(d) h(n/d).$$

The arithmetical function which is identically 1 is denoted by $1$. The next lemma is [9, Lemma 3]. See also [1, Lemma 10].

**Lemma 4.1.** For any set of primes $\mathcal{P}$, any $n \geq 1$ and any $z \geq 2$, we have

$$(\mu \ast 1)((n, P(z))) \leq (\lambda^+ \ast 1)((n, P(z))). \quad (4.1)$$

Let $F(s)$ be one of the classical functions of the linear sieve (so $F(s) = 2e^s/s$ for $1 \leq s \leq 3$). Let $\omega$ be any multiplicative function satisfying

$$0 < \omega(p) < p \text{ if } p \in \mathcal{P}, \quad \omega(p) = 0 \text{ if } p \notin \mathcal{P},$$

and

$$\prod_{w_1 < p < w_2} \left(1 - \frac{\omega(p)}{p}\right) \leq \left(\frac{\log w_2}{\log w_1}\right) \left(1 + \frac{K}{\log w_1}\right)$$

for all $2 \leq w_1 \leq w_2$. Then we have, uniformly for $\omega$,

$$\sum_{d \in \mathcal{H}(z)} \frac{\omega(d)}{d} \leq \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right) \left\{ F(s) + O(e^{\sqrt{\log z}} \log^{-1/3} D) \right\}$$

whenever $z \leq D$. Here we have written $s = (\log D)/(\log z)$.

Now we can give

**Proof of Theorem 2.** The proof follows the same line as that of [1, Theorem 1] (see [1, Section III.3]), so we may be brief. Let

$$P(z_0, z) = \prod_{z_0 \leq p < z} p$$

and define $T(\mathcal{A}, z_0)$ to be the set of all $x \in \mathcal{A}$ such that $p | x \Rightarrow p = 2$ or $p > z_0$. Then we have

$$S(\mathcal{A}, z) = \sum_{x \in T(\mathcal{A}, z_0)} \prod_{j=1}^{4} (\mu \ast 1)((x_j, P(z_0, z))). \quad (4.2)$$
Now we introduce a parameter $D$ and the sequences \( \{ \lambda_d^+ \} \) related to $D$. So $\lambda^+ = 0$ if $d \geq D$, and $| \lambda^+ | \leq 1$. Applying (4.1) to (4.2), we have the bound

\[
S(\mathcal{A}_z, z) \leq \sum_{l_1 \mid P(z_0, z)} \sum_{l_2 \mid P(z_0, z)} \sum_{l_3 \mid P(z_0, z)} \sum_{l_4 \mid P(z_0, z)} \lambda_{l_1}^+ \lambda_{l_2}^+ \lambda_{l_3}^+ \lambda_{l_4}^+ S(\mathcal{A}_z, z_0).
\]

The above $S(\mathcal{A}_z, z_0)$ can be estimated by Lemma 3.4, which gives

\[
S(\mathcal{A}_z, z_0) \leq W(z_0)(\Sigma(D, z_0, z) + E) + O\left( \frac{\pi}{16} \mathcal{E}_- (n) \mathcal{E}(n/N) N \right)
\]

\[
+ O\left\{ \sum_{\|d\| \leq D_0} \sum_{|l| \leq D} \mu^2(d) \mu^2(l) |R(n, N, d)| \right\},
\]

where

\[
\Sigma(D, z_0, z) = \sum_{l_1 \mid P(z_0, z)} \sum_{l_2 \mid P(z_0, z)} \sum_{l_3 \mid P(z_0, z)} \sum_{l_4 \mid P(z_0, z)} \lambda_{l_1}^+ \lambda_{l_2}^+ \lambda_{l_3}^+ \lambda_{l_4}^+ \frac{\omega(l)}{l},
\]

and where $E$ corresponds to terms arising from the $O$-term in Lemma 3.4.

Clearly,

\[
E \ll \{ H^5(n) D^{-1/2} \log^{17} D_0 + D^4 e^{-\eta \log z_0} \} \sum_{l_1 \mid P(z_0, z)} \sum_{l_2 \mid P(z_0, z)} \sum_{l_3 \mid P(z_0, z)} \sum_{l_4 \mid P(z_0, z)} \frac{\omega(l)}{l},
\]

where the first inequality in (3.14) has been used. The above three formulae correspond to [1, (3.28) (cf. also (3.12)), (3.29), and (3.30)] respectively.

In the following we suppose $\log^3 N \leq z_0 \leq \log^{30} N$. By the argument leading to [1, (3.36) and (3.37)], we can derive from the above two formulae respectively that

\[
\Sigma(D, z_0, z) = \left\{ \sum_{l \mid P(z_0, z)} \lambda_l^+ \frac{\omega(l)}{l} \right\}^2 \left\{ \sum_{l \mid P(z_0, z)} \lambda_l^+ \frac{\omega(l)}{l} \right\}^2 + O(\beta_0^{-1/2} \log^8 z),
\]

and

\[
E \ll \{ H^6(n) D^{-1/2} \log^{17} D_0 + D^4 e^{-\eta \log z_0} \} \log^8 z.
\]

To estimate the $O$-term in (4.3), we note that every integer $m$ with the property that $p \mid m \Rightarrow 2 < p < z$ can be decomposed uniquely as $m = m_1 m_2$.
where \( p | m_1 \Rightarrow 2 < p < z_0 \) and \( p | m_2 \Rightarrow z_0 < p < z \). Thus the \( O \)-term in (4.3) is bounded by

\[
O \left( \sum_{d \leq D_0} \mu^2(d) \left| R(n, N, d) \right| \right). \tag{4.6}
\]

Now we fix \( D = N^{1/12 - 2\varepsilon} \) and \( D_0 = N^\varepsilon \) where \( \varepsilon > 0 \) is very small. By Lemma 3.1, the \( O \)-term in (4.6) is \( O(N \log^{-10} N) \) which is certainly acceptable. Choosing \( \Delta = H^{10}(n) \log^{200} N \) and \( z_0 = \log^{20} N \), and noting that \( H(n) \ll \log^{1/2} N \), we deduce from (4.5) that

\[
E \ll \log^{-10} N. \tag{4.7}
\]

Applying [17, Lemmas 8.1 and 8.2], one sees that all conditions in Lemma 4.1 are satisfied, and that the \( \omega'(p) \) and \( \omega''(p) \) defined as in Lemma 3.3 are \( p(p+1) \). Thus (4.4) and Lemma 4.1 give

\[
\Sigma(D, z_0, z) \leq \left( F(2) + O(\log^{-1/3} D) \right)^4 \times \prod_{\substack{z_0 < p < z}} \left( 1 - \frac{\omega'(p)}{p} \right)^2 \left( 1 - \frac{\omega''(p)}{p} \right)^2 + O(z_0^{-1/2} \log^8 z).
\]

Finally we choose \( z = D^{1/2} = N^{1/44 - \varepsilon} \), so that \( s = (\log D)/(\log z) = 2 \) and \( F(2) = e' \). By this and the estimate \( \prod_{p < z} (1 - 1/p) \sim e^{-7} \log z \), the above formula becomes

\[
\Sigma(D, z_0, z) \leq \left( e' + \epsilon \right)^4 \left( \frac{\log^4 z_0}{\log^4 z} \right) \left( \frac{44^4 e^4 (1 + \epsilon)}{\log^4 N} \right) \leq 44^4 e^4 (1 + \epsilon) \frac{\log^4 z_0}{\log^4 N}. \tag{4.8}
\]

Inserting (4.6), (4.7), and (4.8) into (4.3), and then using Lemma 3.2 to estimate \( W(z_0) \), we conclude that

\[
S(A, z) \leq W(z_0) 44^4 e^4 (1 + \epsilon) \frac{\log^4 z_0}{\log^4 N} \frac{\pi}{16} N \frac{\log^4 N}{N} \]

\[
\times \Xi(n) \Xi \left( \frac{n}{N} \right) \left( \frac{N}{\log^4 N} \right) + O \left( \frac{N}{\log^4 N} \right)
\]

\[
\leq \frac{11^3 e^{24}}{2^{24}} (1 + \epsilon) ^5 \cdot \Xi(n) \frac{\pi^2}{16} \frac{N}{\log^4 N},
\]

where in the last inequality we have used \( \Xi(n/N) \leq \pi \).
The number of solutions of (2.7) with every \( p_j > z \) is clearly \( \leq S(A, z) \), while other solutions counted by \( r_-(n) \) are clearly \( \ll N^{1/2} N^{1/44} N^e \ll N^{2/3} \).

Hence,

\[
r_-(n) \leq S(A, z) + N^{2/3} \leq \frac{11^4 \pi^{24}}{2^{24}} (1 + \varepsilon)^6 \cdot \mathbb{Z}_-(n) \frac{\pi^2 N}{16 \log^4 N},
\]

which is the desired upper bound for \( r_-(n) \). This proves Theorem 2.

5. ESTIMATION OF AN INTEGRAL

In this section we prove the following result.

**Lemma 5.1.** Let \( T(x) \) and \( G(x) \) be as in (2.4). Then

\[
\int_0^1 |T(x) G(x)|^4 \, dx \leq c_5 \frac{\pi^2 N}{16} L^4,
\]

where

\[
c_5 \leq \left( \frac{11^4 \cdot 43 \cdot \pi^{24}}{2^{24} \cdot 5^2} + \frac{3}{\pi^2 \log^2 2} \right) (1 + \varepsilon)^9.
\]

To this end, we need

**Lemma 5.2.** For odd \( q \), let \( \varphi(q) \) be the smallest positive integer \( \varphi \) such that \( 2^\varphi \equiv 1 \pmod{q} \). Then the series \( \sum_{q=1}^{\infty} p^2 q/\varphi(q) \) is convergent, and its value \( c_6 < 43/25 \).

**Proof.** The convergence of the series was established by Romanoff, and a shorter proof was later given by Erdős; for these see [19, Section V.8]. The bound \( c_6 < 1.7196 < 43/25 \) is due to [14, p. 396].

**Proof of Lemma 5.1.** One easily sees that

\[
\int_0^1 |T(x) G(x)|^4 \, dx \leq (\log \sqrt{N})^4 Z(N),
\]

where \( Z(N) \) denotes the number of solutions of the equation

\[
p_1^2 + p_2^2 - p_3^2 - p_4^2 = 2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}
\]
with
\[ p_j \ll N^{1/2}, \quad m_j \ll L. \quad (5.3) \]

Now we distinguish between two cases.

**Case 1.** In this case we treat the contribution from those \((m_1, m_2, m_3, m_4)\) such that
\[ 2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4} \neq 0. \quad (5.4) \]

Let \((m_1, m_2, m_3, m_4)\) be fixed and satisfy, in addition to (5.4), \(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4} \equiv 0 \pmod{24}\). For these \((m_1, m_2, m_3, m_4)\), one trivially has \(|2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}| \leq 2N\), so one deduces from Theorem 2 that
\[ \left| \left( p_1^2 + p_2^2 + p_3^2 + p_4^2 : p_j \text{satisfies (5.2) and (5.3)} \right) \right| \leq c_1 \frac{\pi^2}{16} \frac{N}{\log^4 N} \sum_{1 \leq m_1, \ldots, m_4 \leq L} g(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}). \]

where \(g(h) = \prod_{p | h, p \geq 3} (1 + \frac{1}{p})\). On the other hand, for \((m_1, m_2, m_3, m_4)\) satisfying (5.4) and \(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4} \not\equiv 0 \pmod{24}\), we have
\[ \left| \left( p_1^2 + p_2^2 + p_3^2 + p_4^2 : p_j \text{satisfies (5.2) and (5.3)} \right) \right| \ll N^{1/2 + \varepsilon}. \]

Therefore \(Z_1(N)\), the number of solutions of (5.2) with \(p_j, m_j\) satisfying both (5.3) and (5.4), can be estimated as
\[ Z_1(N) \ll 4c_1 (1 + \varepsilon) \frac{\pi^2}{16} \frac{N}{\log^4 N} \sum_{1 \leq m_1, \ldots, m_4 \leq L} g(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}). \]

Denote by \(\Sigma\) the sum above. Noting that \(g(h) = g(-h)\) for \(h \not= 0\) and that
\[ \sum_{1 \leq m_1, \ldots, m_4 \leq L} 1 \leq 4(1 + \varepsilon) \sum_{1 \leq m_1, m_2, m_3 \leq L} 1, \]
we get
\[ \Sigma \ll 4(1 + \varepsilon) \sum_{\Delta} g(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}). \]
Here the condition $3m_4 < m_1 + m_2 + m_3$ in the above sum guarantees (5.4). For a fixed integral vector $(h_1, h_2, h_3)$ with $1 \leq h_j \leq L$, we have

$$|\{(m_1, m_2, m_3, m_4) : 1 \leq m_j \leq L, m_1 - m_4 = h_1, m_2 - m_4 = h_2, m_3 - m_4 = h_3\}| \leq L.$$

Thus,

$$\Sigma \leq 4(1 + \varepsilon) \sum_{1 \leq m_4 < m_1 + m_2 + m_3 < L} g(2m_1) g(2m_2 - m_4 - 2m_3 - m_4 - 1) \leq 4(1 + \varepsilon) L \sum_{h_j + h_2 > 0} g(2^{h_1} + 2^{h_2} - 2^{h_3} - 1).$$

(5.9)

Here the condition $h_1 + h_2 + h_3 > 0$ indicates that $h_1, h_2, h_3$ cannot vanish at the same time. Obviously, there are at most $O(L^2)$ terms in the last sum such that one or two of $h_1, h_2, h_3$ vanishes, and the total contribution of these terms to $\Sigma$ is $\ll L^3 \log \log N \ll L^3 \log L$, on using the elementary bound $g(d) \ll \log \log d$. Hence (5.5) becomes

$$\Sigma \leq 4(1 + \varepsilon) L \sum_{1 \leq h_j, h_2, h_3 < L} g(2^{h_1} + 2^{h_2} - 2^{h_3} - 1) + O(L^3 \log L).$$

Since for any fixed odd integer $t$, there is at most one solution of the equation $2^{h_1} - 2^{h_3} - 1 = -t$, one deduces further that

$$\Sigma \leq 4(1 + \varepsilon) L^3 \max_{|t| \leq N} \sum_{1 \leq h \leq L} g(2^h - t) + O(L^3 \log L).$$

(5.6)

The sum on the right hand side of (5.6) can be estimated as

$$\sum_{1 \leq h \leq L} g(2^h - t) = \sum_{1 \leq h \leq L} \sum_{d | 2^h - t} \frac{\mu^2(d)}{d} \sum_{2^h \equiv t \pmod{d}} \frac{\mu^2(d)}{d} \sum_{1 \leq k \leq L} 1.$$

It follows from $d | 2^h - t$ that $t \equiv 2^h \pmod{d}$. Let $h_0$ be the least positive integer such that $t \equiv 2^h \pmod{d}$. Then we have $2^h \equiv 2^{h_0} \pmod{d}$, or $2^{h_1 - h_0} \equiv 1 \pmod{d}$, and consequently $g(d) | h - h_0$. Hence, by Lemma 5.2,

$$\sum_{1 \leq h \leq L} g(2^h - t) = \sum_{d \leq 2N} \frac{\mu^2(d)}{d} \sum_{1 \leq h \leq L} \sum_{1 \leq h \leq L} \frac{\mu^2(d)}{d} \frac{1}{g(d)(h - h_0)} \leq c_6 L,$$

$$\sum_{d \leq 2N} \frac{\mu^2(d)}{d} \sum_{1 \leq h \leq L} \sum_{1 \leq h \leq L} \frac{\mu^2(d)}{d} \frac{1}{g(d)(h - h_0)} \leq c_6 L.$$
uniformly for all possible $t$. Inserting this into (5.6), we get $\Sigma \leq 4c_6(1+\varepsilon)^3 L^4$, and consequently,

$$Z_1(N) \leq c_1c_6(1+\varepsilon)^3 \pi^2 NL^4/\log^4 N. \quad (5.7)$$

**Case 2.** It remains to estimate $Z_2(N)$, the number of solutions of (5.2) with $p_j, m_j$ satisfying (5.3) but not (5.4). Clearly, $Z_2(N)$ is the number of solutions of

$$p_1^2 + p_2^2 = p_3^2 + p_4^2 \quad (5.8)$$
times that of

$$2^m_1 + 2^m_2 = 2^m_3 + 2^m_4, \quad (5.9)$$
where $p_j, m_j$ are as in (5.3). By [20, Satz 3], the number of solutions of (5.8) with $p_1 p_2 \neq p_3 p_4$ is $O(N \log^{-3} N)$. Also by the prime number theorem, (5.8) has approximately $2(\sqrt{N}/\log \sqrt{N})^2 = 8N \log^{-2} N$ trivial solutions, namely those satisfying $p_1 p_2 = p_3 p_4$. Therefore, the total number of solutions of (5.8) is $\leq 8(1+\varepsilon) \log^{-2} N$. To investigate (5.9), one fixes $m_1, m_3$ arbitrarily, then one finds that there is at most one choice for $m_2, m_4$. It follows that (5.9) has at most $L^2$ solutions, and consequently

$$Z_2(N) \leq 8(1+\varepsilon) NL^2/\log^2 N. \quad (5.10)$$

This finishes the discussion of Case 2.

We can now conclude from (5.7) and (5.10) that

$$Z(N) = Z_1(N) + Z_2(N) \leq \left( c_1c_6 + \frac{8}{\pi^2} \log^2 2 \right) (1+\varepsilon)^3 \frac{\pi^2 NL^4}{\log^4 N},$$
which in combination with (5.1) gives Lemma 5.1.

### 6. PROOF OF THEOREM 1

We need the following three lemmas.

**Lemma 6.1.** Let $\mathcal{Z}(N,k)$ be as in (1.3) with $k \geq 2$. Then for $N \equiv 4 \pmod{8}$, $n \geq 1$, $1/4 \geq 1 - \varepsilon$. Then for $N \equiv 4 \pmod{24}$,

$$\sum_{n \equiv 2(N,k)} \frac{1}{4} (1-\varepsilon) NL^k.$$
Proof. We have

\[ \sum_{n \equiv \pm N(k) \text{ (mod 24)}} n = \sum_{(v)} (N - 2^n - \cdots - 2^n), \]

where the conditions \((v)\) in the above \(\sum_{(v)}\) are

\[ 1 \leq v_1, \ldots, v_k \leq L, \quad 2^{v_1} + \cdots + 2^{v_k} \leq N - 2 \]

and

\[ 2^{v_1} + \cdots + 2^{v_k} \equiv N - 4 \text{ (mod 24)}. \]

Note that \(N \equiv 4 \text{ (mod 8)}\) and \(2^n \equiv 0 \text{ (mod 8)}\) for each \(v \geq 3\). So the above condition \((v)\) will be satisfied if \(v_1, \ldots, v_k\) satisfies the stronger conditions \((v')\):

\[ 3 \leq v_1, \ldots, v_k \leq \log_2(N/(kL)) \quad \text{and} \quad 2^{v_1} + \cdots + 2^{v_k} \equiv N - 4 \text{ (mod 3)}. \]

Therefore,

\[ \sum_{n \equiv \pm N(k) \text{ (mod 24)}} n \geq \sum_{(v')} (N - 2^{v_1} - \cdots - 2^{v_k}) \geq \left( N - \frac{N}{L} \right) \sum_{(v')} 1. \tag{6.1} \]

Now note that \(2^r \equiv 1 \text{ or } -1 \text{ (mod 3)}\) according to whether \(r\) is even or odd. Thus, if we fix arbitrarily the \(v_1, \ldots, v_k\) in the last sum of (6.1), then there are at least

\[ \left( \frac{\log_2(N/(kL))}{2} - 2 \right)^2 \geq \left( \frac{1}{4} - O\left( \frac{\log_2(kL)}{L} \right) \right) L^2 \]

choices for \(v_1, v_2\). Consequently the last sum in (6.1) is \(\geq \{1/4 - o(1)\} L^k\).

Inserting this into (6.1), one gets Lemma 6.1.

Lemma 6.2. Let

\[ C(q, a) = \sum_{m = 1 \atop (m, q) = 1}^{q} e \left( \frac{am^2}{q} \right), \quad B(n, q) = \sum_{a = 1 \atop (a, q) = 1}^{q} C^4(q, a) e \left( -\frac{am}{q} \right). \tag{6.2} \]
and

\[ A(n, q) = \frac{B(n, q)}{\varphi^4(q)} \quad \Xi(n) = \sum_{q=1}^{\infty} A(n, q). \] (6.3)

Then for \( n \equiv 4 \pmod{24} \), one has

\[ c_7 < \Xi(n) \ll (\log \log n)^{11} \]

with \( c_7 = 4/5 \); while for \( n \not\equiv 4 \pmod{24} \), one has \( \Xi(n) = 0 \).

**Proof.** This is [17, Proposition 4.3] except for the value of \( c_7 \); we postpone the evaluation of \( c_7 \) until the next section.

The following estimate for \( G(x) \) is quoted from [14, Lemma 3].

**Lemma 6.3.** Let \( \eta < 1/(7\varepsilon) \). Then the set \( \mathcal{E} \) of \( x \in (0,1] \) for which \( |G(x)| \geq (1-\eta) L \) has measure \( \ll L^{5/2} N^{\eta - 1} \), where

\[
\Theta = \Theta(\eta) = \frac{1}{\log 2} \eta \csc^2(\pi/8) \log \frac{1}{\eta \csc^2(\pi/8)} 
+ \frac{1}{\log 2} (1-\eta \csc^2(\pi/8)) \log \frac{1}{1-\eta \csc^2(\pi/8)}. \]

Now we prove the main result of this paper.

**Proof of Theorem 1.** We distinguish between two cases according to whether \( N \equiv 4 \pmod{8} \) or not.

**Case 1.** Suppose \( N \equiv 4 \pmod{8} \). Let \( \mathcal{E} \) be as in Lemma 6.3 and \( \mathcal{H} \) as in (2.3) with \( P, Q \) determined by (2.1). Then (2.5) becomes

\[
r_k(N) = \int_0^1 T^4(x) G^k(x) e(-Nx) \, dx = \int_{\mathcal{E}} + \int_{\mathcal{E} \cap \mathcal{H} \cap \mathcal{C}(\mathcal{E})} + \int_{\mathcal{H} \cap \mathcal{C}(\mathcal{E})}. \] (6.4)

Introducing the notation \( \Xi(N, k) \) and then applying Theorem 2, we see that the first integral on the right-hand side of (6.4) is

\[
\geq c_2 \frac{\pi^2}{16} \left\{ \sum_{n \in \Xi(N, k)} n \right\} + O(NL^{k-1}) \geq \frac{1}{5} (1-\varepsilon) \frac{\pi^2}{16} NL^k, \] (6.5)
where in the last two inequalities we have used Lemmas 6.2 and 6.1 respectively.

To estimate the second integral, one notes that each $C(M)$ can be written as (2.2) for some $P < q < Q$ and $1 < a < q$ with $(q, a) = 1$. We now apply Ghosh [3, Theorem 2], which states that, for $x \in C(M)$,

$$T(x) \ll N^{1/2+\epsilon}(P^{-1} + N^{-1/4} + QN^{-1})^{1/4} \ll N^{1/2 - 1/30 + 2\epsilon}. $$

Now we take $\eta = 1/368$ so that the definition of $\Theta$ in Lemma 6.3 gives $\Theta < 0.1333 < 2/15$. Thus the second integral in (6.4) satisfies

$$\int_{C(M) \cap \mathcal{E}} \ll N^{\Theta} - 1 N^{2 - 2/15 + 8\epsilon} L^{k/5/2} \ll NL^{k-1}. \quad (6.6)$$

On using Lemmas 6.3 and 5.1, the last integral in (6.4) can be estimated as

$$\int_{C(M) \cap \mathcal{E}} \leq \left\{ (1 - \eta) L \right\}^{k-4} \int_0^1 |T(x) G(x)|^4 dx \leq c_5 (1 - \eta)^{k-4} \frac{\pi^2}{16} NL^{k}. \quad (6.7)$$

Inserting (6.5), (6.6), and (6.7) into (6.4), we get

$$r_k(N) \geq \frac{\pi^2}{16} NL^{k} \left\{ \frac{1}{5} - c_5 (1 - \eta)^{k-4} \right\} (1 - \varepsilon)^2, \quad (6.8)$$

when $k \geq 4$ and $N \geq N_k$. Also when $k \geq 8328$ and $\varepsilon = 10^{-8}$, one has $\{1/5 - c_5 (1 - \eta)^{k-4}\}(1 - \varepsilon)^2 > 1/90$. Consequently if $k \geq 8328$ and $N \geq N_k$, then (6.8) becomes

$$r_k(N) \geq NL^{k-2}/200. \quad (6.9)$$

It therefore follows from (6.9) that for any $k \geq 8328$, every large even integer $N \geq N_k$ with $N \equiv 4 \pmod{8}$ can be expressed in the form of (1.2).

**Case 2.** Now suppose $N$ is even but $N \not \equiv 4 \pmod{8}$. Since for any even integer $N$ there exist $\mu_1, \mu_2 = 1, 2$ or 3 such that $N - 2^{\mu_1} - 2^{\mu_2} \equiv 4 \pmod{8}$, we deduce from Case 1 that if $k \geq 8330$ then every even integer $N \geq N_k + 16$ can be written in the form of (1.2), and

$$r_k(N) \geq NL^{k-2}/200. \quad (6.10)$$

This completes the proof of Theorem 1.
7. THE EVALUATION OF $c_4$ AND $c_7$

Now it only remains to prove the evaluations for $c_4$ and $c_7$ given in Lemmas 3.2 and 6.2 respectively.

Proof of (3.14). We first evaluate $c_4$. To this end, we should estimate \( 1 - \Omega(p)/p \) from above for all \( p \geq 3 \). We distinguish between two cases according to whether \( p \mid n \) or not. For convenience we write \( x = 1/p \).

Suppose first that \( p^\beta \parallel n \) with \( p \geq 3 \) and \( \beta \geq 1 \). Then by (3.12) and [17, Lemma 8.2],

\[
\frac{\Omega(p)}{p} \geq \frac{4x - 11x^2 + 9x^3 - x^{\beta+1} - x^{\beta+2}}{1 + x - x^{\beta+1} - x^{\beta+2}},
\]

and consequently,

\[
1 - \frac{\Omega(p)}{p} \leq \frac{1 - 3x + 11x^2 - 9x^3}{1 + x - x^{\beta+1} - x^{\beta+2}} \leq (1 - x)^4 \frac{1 - 3x + 11x^2 - 9x^3}{(1 - x^2)^2 (1 - x)^4}.
\]

It is easily seen that for \( x \leq 1/3 \),

\[
1 - 3x + 11x^2 - 9x^3 \leq (1 - x)^3 (1 - x^2)^{-10}.
\] (7.3)

Thus, when \( x \leq 1/3 \),

\[
1 - \Omega(p)/p \leq (1 - x)^4 (1 - x^2)^{-12}.
\] (7.4)

Now suppose \( p \geq 3 \) and \( p \nmid n \). Then by (3.12) and [17, Lemma 8.1], we have

\[
\frac{\Omega(p)}{p} \geq \frac{4}{p+1} - \frac{2}{p(p-1)} - \frac{4}{p(p+1)} = \frac{4p^2 - 10p + 2}{p(p^2 - 1)},
\]

and consequently

\[
1 - \frac{\Omega(p)}{p} \leq \frac{1 - 4x + 9x^2 - 2x^3}{1 - x^2}.
\]

It is easily seen that for \( x \leq 1/3 \), the bound (7.1) still holds in this case.
From this and (7.1) we conclude that
\[
\prod_{3 \leq p \leq z} \left(1 - \frac{O(p)}{p}\right) \leq \prod_{3 \leq p \leq z} \left(1 - \frac{1}{p}\right)^4 \prod_{3 \leq p \leq z} \left(1 - \frac{1}{p^2}\right) - 12
\]
\[
\leq (1 + \varepsilon) \frac{2^4 e^{-4\varepsilon}}{\log^2 z} \cdot \frac{3^{12} z^{12}}{4^{12}}
\]
\[
= \frac{\pi^{24}}{2^5} (1 + \varepsilon) \frac{e^{-4\varepsilon}}{\log^2 z}.
\]
This gives the upper bound for \( c_4 \).

Now we prove the first inequality in (3.14). We should estimate \( 1 - \Omega(p)/p \) from below for all \( p \geq 3 \). Here we still need to distinguish between two cases according to whether \( p \mid n \) or not. Arguing similarly one gets that, for \( p \nmid n \) with \( p \geq 3 \) and \( \beta \geq 1 \),
\[
\frac{\Omega(p)}{p} \leq \frac{4x + 8x^2 + 4x^3}{1 + x - x^\beta + 1 - x^{\beta+2}},
\]
and for \( p \mid n \),
\[
\frac{\Omega(p)}{p} \leq \frac{4x + 4x^2 + 8x^3}{1 - x^2}.
\]
Consequently,
\[
W(z) = \prod_{p \leq z} \left(1 - \frac{\Omega(p)}{p}\right) \gg \prod_{p \leq z} \left(1 - \frac{4}{p}\right) \gg \frac{1}{\log^4 z}.
\]
This completes the proof of (3.14).

The Value of \( c_4 \). It has been given in the proof of [17, Proposition 4.3] that
\[
\Xi(n) = \{1 + A(n, 2) + A(n, 2^2) + A(n, 2^3)\} \prod_{p \geq 3} \{1 + A(n, p)\},
\]
where \( A(n, q) \) is defined as in (6.3). It has also been proved that in [17, Lemma 4.2] that when \( n \equiv 4 \pmod{24} \),
\[
1 + A(n, 2) + A(n, 2^2) + A(n, 2^3) = 8, \quad 1 + A(n, 3) = 3.
\]
Therefore to estimate \( \Xi(n) \) it remains to compute \( 1 + A(n, p) \) for \( p \geq 5 \).
We will use the notation
\[
G(\chi, n) = \sum_{m=1}^{q} \chi(m) e\left(\frac{nm}{q}\right), \quad c_q(n) = \sum_{m=1 \atop (m, q) = 1}^{q} e\left(\frac{nm}{q}\right),
\]
where \(c_q(n)\) is the Ramanujan sum. We will also use the notation \(S(q, a)\) introduced in (3.5). By [7, Theorem 7.5.4], we have for \(p \geq 5\),
\[
C(p, a) = S(p, a) - 1 = \chi(a) S(1, 1) - 1,
\]
where \(\chi\) is the Legendre symbol \((a/p)\). Inserting this into (6.2), one sees that
\[
B(n, p) = S^4(p, 1) c_p(-n) - 4 S^3(p, 1) G(\chi, -n) + 6 S^2(p, 1) c_p(-n) - 4 S(p, 1) G(\chi, -n) + c_p(-n)
\]
Using the well-known formulae (see [7, Theorems 7.5.5 and 7.5.8])
\[
S(p, 1) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod{4}, \\ i \sqrt{p}, & \text{if } p \equiv 3 \pmod{4}, \end{cases}
\]
and
\[
|G(\chi, n)| = \begin{cases} \sqrt{p}, & \text{if } p \nmid n, \\ 0, & \text{if } p \mid n, \end{cases}
\]
\[
c_p(n) = \begin{cases} -1, & \text{if } p \nmid n, \\ 0, & \text{if } p \mid n, \end{cases}
\]
one obtains
\[
B(n, p) \equiv \begin{cases} -5p^2 - 10p - 1, & \text{if } p \nmid n, \\ (p-1)(p^2 - 6p + 1), & \text{if } p \mid n. \end{cases}
\]
Hence by (6.3), we have
\[
\prod_{p \geq 5} \{1 + A(n, p)\} \geq \prod_{p \geq 5 \atop p \nmid n} \left(1 - \frac{5p^2 + 10p + 1}{(p-1)^4}\right) \cdot \prod_{p \geq 5 \atop p \mid n} \left(1 + \frac{p^2 - 6p + 1}{(p-1)^3}\right) > \prod_{p \geq 5} \left(1 - \frac{5p^2 + 10p + 1}{(p-1)^4}\right). \tag{7.4}
\]
To estimate the last product, one notes that for \( p \geq 23 \),
\[
1 - \frac{5p^2 + 10p + 1}{(p - 1)^6} \geq \left(1 - \frac{1}{(p - 1)^2}\right)^6.
\]
Thus, the last product in (7.4) is
\[
\prod_{5 \leq p \leq 23} \left(1 - \frac{5p^2 + 10p + 1}{(p - 1)^6}\right) \cdot \prod_{3 \leq p \leq 23} \left(1 - \frac{1}{(p - 1)^2}\right)^{-6} \cdot \prod_{p \geq 3} \left(1 - \frac{1}{(p - 1)^2}\right)^6
\geq 0.4029 \times (0.6601)^6 > 1/30,
\]
where we have used \( \prod_{p \geq 3} (1 - (p - 1)^{-2}) = 0.6601 \ldots \) (see [5, p. 128]). This in combination with (7.4), (7.3), and (7.2) ensures that one can take \( c_7 = 4/5 \) in Lemma 6.2. The proof is complete.

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