An effective and sharp lower bound on Seshadri constants on surfaces with Picard number 1

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Abstract

On an algebraic surface with Picard number 1 we compute in terms of the generator of the ample ray a lower bound for Seshadri constant valid at every point of the surface. We show that this bound cannot be improved in general.

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1. Introduction

Seshadri constants were introduced by Demailly [4]. They measure the local positivity of an ample line bundle at a point. Though they are defined locally, they depend on the global geometry of the underlying variety and vice versa.

Definition 1. Let X be a smooth projective variety and L an ample line bundle on X. Then

$$\varepsilon(L, x) := \inf_{x \in C} \frac{L.C}{\text{mult}_x C}$$

where the infimum is taken over all curves $C \subset X$ passing through x is the Seshadri constant at the point $x \in X$ (it is enough to consider irreducible curves).

By the ampleness criterion of Seshadri $\varepsilon(L, x)$ is a positive real number. If L is very ample, then it is easy to see that $\varepsilon(L, x) \geqslant 1$ for all points $x \in X$.

Shortly after Seshadri constants became an object of an independent study, Ein and Lazarsfeld [5] proved a remarkably theorem that the above bound holds for a general point and a merely ample line bundle on an algebraic surface. In fact it is conjectured that the same is true in any dimension.

Theorem 2 *(Ein–Lazarsfeld). Let S be a smooth projective surface and L an ample line bundle on S. Then*

$$\varepsilon(L, x) \geqslant 1$$

for all points $x \in S$ away of at most countably many.

On the other hand, Miranda [7] provided examples showing that for any $\varepsilon > 0$ there exists a surface S, a point $x_0 \in S$ and an ample line bundle L on S such that $\varepsilon(L, x_0) < \varepsilon$. In these examples the surfaces change as ε gets smaller and smaller. Moreover their Picard numbers grow reciprocally to ε.

2. The problem and the result

Ein and Lazarsfeld raised a natural question if there exists a single surface S and sequences L_n of ample line bundles and x_n of points on S such that

$$\varepsilon(L_n, x_n) \to 0.$$

This is not known up to now and it is conjectured that this is not possible, i.e. that on a given surface there should be a universal lower bound on Seshadri constants of all ample line bundles. For surfaces with Picard number 1 this follows easily form the following improvement of Theorem 2 obtained by Oguiso in [9].

Theorem 3 *(Oguiso). Let S be a smooth projective surface and let L be an ample line bundle on S. Then for an arbitrary $\delta > 0$ the set of points x such that*

$$\varepsilon(L, x) \leqslant 1 - \delta$$

is finite.

If the Picard number of S is 1, then there is only the ample generator L one has to take care of. In particular it follows from the above corollary that there exists a lower bound for $\varepsilon(L, x)$, namely the minimum over all points. However Oguiso theorem is noneffective.

Corollary 4. *Let S be a surface with Picard number 1 with an ample generator L. Then there exists a number ε_0 such that*

$$\varepsilon(L, x) \geqslant \varepsilon_0$$

for all points $x \in S$.

In order to make an effective statement one could revoke instead the big theorem of Matsusaka whose effective version on surfaces was proved by Fernandez del Busto [6].

Theorem 5 *(Fernandez del Busto). Let L be an ample line bundle on a smooth projective surface S with $a = L^2$ and $b = (K_S + L)L$. Then the line bundle mL is globally generated (in particular $\varepsilon(mL, x) \geqslant 1$) provided*

$$m > \frac{(b+1)^2}{2a} - 1.$$

Applying this result on a surface with Picard number 1 yields the following effective statement.

Corollary 6. *Let S be a smooth projective surface with Picard number* 1 *with an ample generator L and let r be an integer such that $K_S = rL$. Then*

$$\varepsilon(L, x) \geqslant \frac{2L^2}{1 + (r+4)^2(L^2)^2 + 2(r+3)L^2}$$

for every point $x \in S$.

While effective, this bound is not sharp. The following theorem tackles this problem. This is the main result of this note.

Theorem 7. *Let S be a smooth projective surface with $\rho(S) = 1$ and let L be an ample line bundle on S. Then for any point $x \in S$*

(S) $\varepsilon(L, x) \geqslant 1$ *if S is not of general type, and*
(G) $\varepsilon(L, x) \geqslant \dfrac{1}{1 + \sqrt[4]{K_S^2}}$ *if S is of general type.*

Moreover both bounds are sharp.

Remark 8. It seems worth to note that for surfaces with Picard number 1 the result of Theorem 2 was improved to

$$\varepsilon(L, x) \geqslant \left\lfloor \sqrt{L^2} \right\rfloor$$

by Steffens [13].

Proof of Theorem 7, case (S). For the proof we go first through the Enriques–Kodaira classification of surfaces. Taking into account the assumption $\rho(S) = 1$, there are only few cases.

If $\kappa(S) = -\infty$, then $S = \mathbb{P}^2$ and it is well known that $\varepsilon(\mathcal{O}(1), x) = 1$ for any point $x \in \mathbb{P}^2$. This verifies in particular that the bound stated in this part of the theorem is sharp.

If $\kappa(S) = 0$, then S is either abelian or K3. In the first case S is a homogeneous variety, so $\varepsilon(L, x)$ does not depend on x and by Ein–Lazarsfeld theorem we have $\varepsilon(L, x) \geqslant 1$. Actually, Nakamaye [8] showed that if $\varepsilon(L, x) = 1$ on an abelian variety, then the variety is a product of

an elliptic curve and a lower dimensional abelian variety. Note also that for abelian surfaces with Picard number 1 the exact values of Seshadri constants are known [1].

If S is a K3 surface without (-2)-curves and L is an ample line bundle on S, then L is globally generated [11]. This means that the morphism defined by the linear system $|L|$ is finite, hence $\varepsilon(L, x) \geqslant 1$ for all points $x \in S$. □

3. Seshadri constants of the canonical bundle

What remains are surfaces of general type. To complete the proof of Theorem 7 we need some preparations.

First of all if the degree of the canonical divisor is not too small, then Reider's theorem [10] applies. More exactly we have the following lemma.

Lemma 9. *Let S be a surface of general type with $\rho(S) = 1$ (i.e. K_S is ample) and $K_S^2 \geqslant 5$. Then the bicanonical system $|2K_S|$ is base point free.*

Proof. This is just Reider's theorem for K_S. Note, that all exceptional cases in the theorem are immediately excluded under our assumptions. □

As a corollary we get that

$$\varepsilon(K_S, x) \geqslant \frac{1}{2} \tag{1}$$

in this situation.

Now we turn to the case $K_S^2 \leqslant 4$. Then either K_S is a primitive generator of the ample half-line or there exists an ample line bundle L on S with $K_S = 2L$. In the latter situation it must be $L^2 = 1$ and consequently $K_S^2 = 4$. However such numerical invariants contradict the Riemann–Roch theorem for L. So we can assume that K_S is a primitive line bundle. We obtain the following classification, which seems to be of independent interest.

Lemma 10. *Let S be a surface of general type with $\rho(S) = 1$ and such that K_S is primitive. Suppose that there exists a point $x \in S$ such that*

$$\varepsilon(K_S, x) < 1,$$

then $K_S^2 = 1$, $q(S) = 0$ and $p_g(S) \leqslant 2$ or $K_S^2 = 2$ and $\varepsilon(K_S, x) = \frac{2}{3}$.

Proof. By assumption we have that K_S is ample. The algebraicity result of Campana and Peternell [3] implies that there exists an irreducible curve $C \subset S$ computing the constant at x, i.e.

$$\varepsilon(K_S, x) = \frac{K_S.C}{m} < 1$$

with $m = \text{mult}_x C$. There exists a positive integer p such that $C \in |pK_S|$. Thus the above inequality yields $pK_S^2 < m$ which is equivalent to $pK_S^2 + 1 \leqslant m$.

On the other hand, by the genus formula we have

$$p_a(C) = 1 + \frac{p(p+1)}{2} K_S^2.$$

A point of multiplicity m causes the geometric genus of a curve to drop by at least $\binom{m}{2}$, so that

$$1 + \frac{p(p+1)}{2} K_S^2 - \binom{m}{2} \geqslant 0.$$

This gives $p^2(K_S^2)^2 \leqslant 2 + p^2 K_S^2$, which is possible only if $K_S^2 = 1$ or $K_S^2 = 2$ and $p = 1$.

Bombieri [2, Theorem 11] showed that $K_S^2 = 1$ implies $q(S) = 0$. The restriction $p_g(S) \leqslant 2$ follows from the Noether inequality.

If $K_S^2 = 2$, then C is a canonical curve and in the inequalities above we have equality, so that in particular $m = 3$. Then the Seshadri quotient is

$$\frac{K_S.C}{m} = \frac{2}{3} > \frac{1}{2}. \qquad \square$$

Finally we take a closer look to surfaces of general type with $K_S^2 = 1$. They split again in two classes.

3.1. Surfaces with $K_S^2 = 1$ and $p_q \leqslant 1$

If $p_g = 0$ or 1, then by the Riemann–Roch we have at least a pencil of bicanonical divisors. It is easy to check that the base locus of $|2K_S|$ in both cases consists only of points. Moreover for any point $x \in S$ there is an irreducible curve $D_x \in |2K_S|$ passing through x. Let C be any other irreducible curve on S passing through x. Then we have

$$2K_S.C = D_x.C \geqslant \mathrm{mult}_x C.$$

This shows that the Seshadri quotient of C satisfies

$$\frac{K_S.C}{\mathrm{mult}_x C} \geqslant \frac{1}{2}.$$

The curve D itself has arithmetic genus 4, so it can have at most a triple point at x. Hence

$$\frac{K_S.D_x}{\mathrm{mult}_x D_x} \geqslant \frac{2}{3} > \frac{1}{2}$$

and we are done in these cases.

Note that our argument is rather rough, in particular we did not care if surfaces with given invariants and Picard number 1 exist. We will address this question later showing the optimality of the bound stated in the case (G) of Theorem 7.

3.2. Surfaces with $K_S^2 = 1$ and $p_g = 2$

The last case is that of a smooth surface S of general type with $K_S^2 = 1$, $p_g(S) = 2$ and $\rho(S) = 1$.

This time we can argue basically as in the preceding case but with the canonical pencil this time. This pencil consists of irreducible and reduced curves of genus 2 all of which pass through a single base point x_0 and meet there transversally. Let $x \in S$ be fixed and let D_x be a curve in the pencil through x. If C is an irreducible curve not in the pencil passing through x, then we have

$$K_S.C = D_x.C \geqslant \mathrm{mult}_x C,$$

so that in this case the Seshadri quotient is actually at least 1. Now, it is not possible that all curves in the pencil are smooth. This can be seen either computing the topological Euler characteristic of the surface or with the argument that with all fibers smooth, the pencil would be an isotrivial family contradicting the assumption that S is of general type. On the other hand, since the members of $|K_S|$ are curves of genus 2 they can carry singularities with multiplicity at most 2. We see that there must exist a canonical curve $D \in |K_S|$ and a point $x \in D$ with $\mathrm{mult}_x D = 2$. Then

$$\varepsilon(K_S, x) = \frac{K_S.D}{\mathrm{mult}_x D} = \frac{1}{2}.$$

Summing up (1), Lemma 10 and the above discussion we have the following

Upshot. *If S is a surface of general type with $\rho(S) = 1$, then*

$$\varepsilon(K_S, x) \geqslant \frac{1}{2}$$

for arbitrary point $x \in S$.

At the end of the proof of case (G) of Theorem 7 we give an example showing that the bound in the Upshot is sharp.

4. Primitive line bundles on surfaces of general type

In order to conclude the proof of Theorem 7 we have to study now the situation of S being a surface of general type with Picard number 1, L an ample generator and r a positive integer such that $K_S = rL$.

From the Upshot stated above we get immediately a naive bound

$$\varepsilon(L, x) = \frac{1}{r}\varepsilon(K_S, x) \geqslant \frac{1}{2r} \geqslant \frac{1}{2\sqrt{K_S^2}}$$

but in fact we can do slightly better.

Proof of Theorem 7, case (G). Assume that $x \in S$ is a point with $\varepsilon(L, x) < 1$ (otherwise there is nothing to prove) and let $C \in |pL|$ be a curve with multiplicity m at x computing this Seshadri constant

$$\varepsilon(L, x) = \frac{L.C}{m} = \frac{pL^2}{m}.$$

We have $p_a(C) = 1 + \frac{1}{2}p(p + r)L^2$ and this gives an upper bound on the multiplicity m:

$$m(m - 1) \leqslant 2 + p(p + r)L^2$$

which is equivalent to

$$m \leqslant \frac{1 + \sqrt{9 + 4p(p + r)L^2}}{2}.$$

Thus we have the following bound

$$\varepsilon(L, x) \geqslant \frac{2pL^2}{1 + \sqrt{9 + 4p(p + r)L^2}}.$$

The function on the right is growing for admissible values of p and L^2. Setting $L^2 = 1$ and $p = 1$ we obtain

$$\varepsilon(L, x) \geqslant \frac{2}{1 + \sqrt{13 + \sqrt{K_S^2}}}. \tag{2}$$

Since the case of $K_S^2 \leqslant 4$ was already discussed in Lemma 10, which in particular implies the bound stated in part (G) of the theorem, we can assume that $K_S^2 \geqslant 5$. But then it is easy to check that the number on the right in (2) is greater or equal to our bound $\frac{1}{1 + \sqrt[4]{K_S^2}}$ and this ends the proof of the inequality.

   Now we show that the bound is sharp. To this end let S be a general surface of degree 10 in the weighted projective space $\mathbb{P}(1, 1, 2, 5)$. By adjunction we have that $K_S^2 = 1$. Moreover, sections of K_S correspond to polynomials of degree 1 in the weighted polynomial ring on 4 variables. Thus $p_g(S) = 2$ (see also [12]).

   Steenbrink [12] checked that a general surface S of degree $d \geqslant 2 + a + b$ in $\mathbb{P}(1, 1, a, b)$ with a and b coprime has Picard number 1. His result applies in our case. The existence of a point x with $\varepsilon(K_S, x) = \frac{1}{2}$ follows now from the discussion in Section 3.2.

   This example also shows that the bound given in the Upshot is in fact optimal. $\quad\square$

5. Final remarks and a challenge

   Looking back at the examples of Miranda we observe that in their case the lower bound of $\frac{1}{1 + \sqrt[4]{|K_S^2|}}$ holds. This somehow gives a concrete effective number which could serve as a lower

bound on arbitrary surface verifying in effect the conjecture stated in Section 2. It could be too much to state it as a conjecture but we dare a little challenge.

Question. *Does there exist a* (*minimal*) *polarized surface* (S, L) *and a point* $x \in S$ *such that*

$$\varepsilon(L, x) < \frac{1}{2 + \sqrt[4]{|K_S^2|}}?$$

The appearance of 2 in the above formulation accounts for the existence of Enriques surfaces which carry an ample line bundle L with $\varepsilon(L, x) = \frac{1}{2}$, see [14].

Acknowledgments

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