Sign idempotent sign patterns similar to nonnegative sign patterns

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Abstract

It is shown that not all sign idempotent sign patterns are similar to nonnegative sign patterns. We present two classes of sign idempotent sign patterns that are similar to nonnegative sign patterns. An open problem posed by C. Eschenbach is answered.

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1. Preliminaries, the two theorems, and comments

A matrix whose entries consist of +, − and 0 is called a sign pattern. A matrix (or vector) \( A \) is called constantly signed if it is of the form \( A = \alpha J \) where \( \alpha \in \{+, -, 0\} \) and \( J \) is the all ones matrix (vector). For a sign pattern \( A = (a_{ij}) \), \( A^2 \) is defined as a sign pattern if no two nonzero terms in the sum

\[
\sum_k a_{ik}a_{kj}
\]

are oppositely signed for all \( i \) and \( j \); otherwise \( A^2 \) is not a sign pattern. If \( A = A^2 \), then \( A \) is called sign idempotent. Obviously, the class of sign idempotent sign patterns is closed under signature
similarity, permutation similarity and transposition. So we always assume that a sign idempotent sign pattern $A$ is in the Frobenius normal form, i.e.,
$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{22} & \cdots & A_{2k} \\ \ddots & \ddots & \ddots \\ A_{kk} & & & \end{pmatrix},$$
(1)
where each $A_{ii}$ is square and irreducible, or $A_{ii}$ is the $1 \times 1$ zero matrix, denoted $(0)$.

A positive sign pattern (or matrix) is a sign pattern (or matrix) all of whose entries are positive. A signature matrix $S$ is a diagonal matrix with all diagonal entries belonging to $\{+, -\}$. If $B = S^T A S$, then $A$ is signature similar to $B$. In [1], Eschenbach first characterized irreducible sign idempotent sign patterns, and showed that an irreducible sign pattern $A$ is idempotent if and only if $A$ is signature similar to a positive sign pattern. Thus, we will assume that each nonzero block $A_{ii}$ in the form (1) is positive. The following interesting results from [1] control the blocks $A_{ij}$ above the block diagonal:

**Lemma 1.1.** Suppose $A$ is an $n \times n$ reducible sign pattern in Frobenius normal form (1). If $A_{ii}$ and $A_{jj}$ are positive blocks, then $A$ is sign idempotent only if $A_{ij}$ is constantly signed.

**Lemma 1.2.** Suppose $A$ is an $n \times n$ reducible sign pattern in Frobenius normal form (1). If $A_{ii}$ is positive and $A_{jj} = (0)$, then $A$ is sign idempotent only if $A_{ij}$ is constantly signed.

**Lemma 1.2 (ii).** Suppose $A$ is an $n \times n$ reducible sign pattern in Frobenius normal form (1). If $A_{ii} = (0)$ and $A_{jj}$ is positive, then $A$ is sign idempotent only if $A_{ij}$ is constantly signed.

If $A$ is a sign idempotent sign pattern in Frobenius normal form (1) that has $t$ consecutive $1 \times 1$ zero blocks on the diagonal, then those blocks can be collected into a $t \times t$ zero block. Thus, in view of Lemmas 1.1, 1.2 and 1.2(ii), Eschenbach defined the modified Frobenius normal form of a sign idempotent sign pattern $A$ as follows:
$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{22} & \cdots & A_{2r} \\ \ddots & \ddots & \ddots \\ A_{rr} & & & \end{pmatrix},$$
(2)
where each $A_{ii}$ is $t_i \times t_i$ and either positive or entrywise zero; $A_{ij}$ is constantly signed if $A_{ii}$ and $A_{jj}$ are positive by Lemma 1.1; the columns of $A_{ij}$ are constantly signed if $A_{ii}$ is positive and $A_{jj}$ is a zero block by Lemma 1.2; the rows of $A_{ij}$ are constantly signed if $A_{ii}$ is a zero block and $A_{jj}$ is positive by Lemma 1.2(ii); and $A_{ij}$ is unrestricted if both $A_{ii}$ and $A_{jj}$ are zero blocks (in which case, $i < j - 1$).

To characterize reducible sign idempotent sign patterns, Eschenbach presented the following algorithm (algorithm 2.6 in [1]):

**Algorithm 1 (The upper superdiagonal completion process).** Suppose $A = (A_{ij})$ is an $m \times m$ reducible, partially specified block sign pattern in modified Frobenius normal form (2). Let $P = A^2 = (P_{ij})$. Determine each off-diagonal block $A_{ij}$ as follows:
(i) Start with the 1st superdiagonal. Determine each off-diagonal block $A_{i,i+1}$ using Lemma 1.1 if $A_{ii}$ and $A_{i+1,i+1}$ are positive, using Lemma 1.2 if $A_{ii}$ is positive and $A_{i+1,i+1}$ is a zero
block, using Lemma 1.2(ii) if $A_{ii}$ is a zero block and $A_{i+1,i+1}$ is positive. Move up to the next superdiagonal (if there is one).

(ii) Determine each off-diagonal block $A_{i,i+k}$ on the $k$th superdiagonal using step (i) with $A_{i+k,i+k}$ replacing $A_{i+1,i+1}$ if $P_{i,i+k} = A_{ii}A_{i,i+k} + A_{i,i+k}A_{i+k,i+k}$; otherwise let $A_{i,i+k} = A_{i,i+1}A_{i+1,i+k}$. When all blocks are specified on this superdiagonal, move up to the next superdiagonal, if there is one, increase $k$ by 1 for all $k = 2, 3, \ldots, m - 2$, and repeat (ii).

A matrix $A = (a_{ij})$ is said to be transitive if $a_{ik} \neq 0$ and $a_{kj} \neq 0$ imply $a_{ij} \neq 0$; and $A$ is transitively closed if any two of $a_{ik}, a_{kj}$ and $a_{ij}$ are nonzero, then the third is also nonzero. Using Algorithm 1, Eschenbach obtained two main results, i.e., the two following characterizations of reducible sign idempotent sign patterns:

**Theorem 1.3.** A reducible sign pattern $A$, in modified Frobenius normal form (2), each of whose nonzero diagonal blocks is positive, is sign idempotent if and only if each off-diagonal block is obtained using the upper superdiagonal completion process (Algorithm 1).

**Theorem 1.4.** A reducible sign pattern $A$, in modified Frobenius normal form (2), each of whose nonzero diagonal blocks is positive, is sign idempotent if and only if the reduced matrix of $A$ is transitively closed (the definition of the reduced matrix of $A$ can be seen in [1, p. 162]).

Unfortunately, we easily give some counterexamples to show that both Theorem 1.3 and Theorem 1.4 are incorrect. For Theorem 1.3, let $A = (a_{ij})$ be a partially specified sign pattern:

$$
\begin{pmatrix}
+ & ? & ? & ? \\
+ & ? & ? & \\
0 & ? & + \\
\end{pmatrix}.
$$

Next we use Algorithm 1 to complete the partially specified sign pattern $A$. By step (i), $A$ can be completed into the form

$$
\begin{pmatrix}
+ & + & ? & ? \\
+ & 0 & ? & \\
0 & + & + \\
\end{pmatrix}.
$$

By step (ii), we can get

$$
\begin{pmatrix}
+ & + & - & ? \\
+ & 0 & + & \\
0 & + & + \\
\end{pmatrix}.
$$

Thus, by step (ii) again, $a_{14} = a_{12}a_{24} = +$, i.e.,

$$
A = \begin{pmatrix}
+ & + & - & + \\
+ & 0 & + & \\
0 & + & + \\
\end{pmatrix}.
$$
Obviously, the obtained matrix $A$ is not sign idempotent. So Theorem 1.3 does not hold. For Theorem 1.4, let
\[
A = \begin{pmatrix}
+ & + & + \\
+ & 0 & \\
+ & & \\
\end{pmatrix}.
\]
Obviously, $A$ is sign idempotent, but $A$ is not transitively closed. So Theorem 1.4 does not hold.

The source of this difficulty is Lemma 3.1 in [1, p. 160] which is needed to prove Theorem 1.3 and Theorem 1.4. This lemma is:

**Lemma 1.5.** Suppose $A$ is a reducible sign pattern in modified Frobenius normal form (2) containing $m$ diagonal blocks. If each nonzero diagonal block is positive, and if each off-diagonal block is determined using Algorithm 1, then $A$ is signature similar to an $m \times m$ upper block triangular matrix, each of whose blocks is positive or entrywise zero.

We provide a counterexample to show that Lemma 1.5 is incorrect. Let
\[
A = \begin{pmatrix}
+ & 0 & + & - \\
+ & + & + & \\
+ & 0 & & \\
\end{pmatrix}.
\]
It is easy to check that the sign idempotent sign pattern $A$ is obtained using Algorithm 1, and $A$ is not signature similar to a nonnegative sign pattern. Next we identify the errors in the proof of Lemma 1.5. In the proof [1, p. 161], there is a statement that “Consequently
\[
(S_{i+1}PS_{i+1})_{hk} = Q_{nh}P_{hk}Q_{nk} = P_{hk}
\]
is positively signed or a 0-block”. We observe that the equation (3) does not always hold. By the definition of $S_{i+1}$, if $P_{i+1,k}$ is negatively signed, then (3) should be
\[
(S_{i+1}PS_{i+1})_{hk} = Q_{nh}P_{hk}(-Q_{nk}) = -P_{hk},
\]
which means that the key statement that “Thus the first $i$ rows of $S_{i+1}PS_{i+1}$ consist of positively signed matrices or 0-blocks” is not true. So Lemma 1.5 does not hold. This also means that not all sign idempotent sign patterns are similar to nonnegative sign patterns.

2. Sign idempotent sign patterns similar to nonnegative sign patterns

In this section, we present two classes of sign idempotent sign patterns that are similar to nonnegative sign patterns. Let $I_n$ denote the identity sign pattern. A generalized permutation pattern $P$ is either a permutation sign pattern or a sign pattern obtained by replacing some or all of the + entries in a permutation sign pattern with −’s. Obviously, $P^{-1} = P^T$. Denote the $k$th column of $A$ as $A^{(k)}$. Let $\mathbb{R}^n$ be the set of all $n \times n$ real matrices. For a sign pattern $A$, define
\[
Q(A) = \{B \in \mathbb{R}^n | \text{sign}(B) = A\}.
\]
We easily get the following equivalent relation on sign idempotent sign patterns.

**Theorem 2.1.** Let $A$ be a sign pattern. Then $B^2 \in Q(A)$ for all matrices $B \in Q(A)$ if and only if $A^2 = A$. 
Here are two further results from [1]. We give a simple proof of the second because it provides a tool we will use later.

**Lemma 2.2** 1. If $A$ is an irreducible sign idempotent sign pattern, then $A$ is entrywise nonzero.

**Theorem 2.3** 1. If $A$ is an $n \times n$ irreducible sign idempotent sign pattern, then there exists a signature matrix $S$ such that $S^T AS$ is positive.

**Proof.** Let $A$ be an irreducible sign idempotent sign pattern. By Lemma 2.2, $A$ is entrywise nonzero. Denote the first row of $A$ as $\alpha = (\alpha_1, \ldots, \alpha_n)$. Since $A^2 = A$, it is easy to verify that $A = \alpha^T \alpha$ where $\alpha_1 = +$. Set $S = \text{diag}\{\alpha_1, \ldots, \alpha_n\}$. Then

$$S^T AS = S^T \alpha^T \alpha S = \begin{pmatrix} + & \cdots & + & \cdots & \cdots & + \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ + & \cdots & + & \cdots & \cdots & + \\ \end{pmatrix} = \alpha_1 J. \quad \Box$$

**Lemma 2.4.** Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a sign idempotent sign pattern, where each $A_{ii}$ is $t_i \times t_i$ and either irreducible or entrywise zero. Then there exists a signature matrix $S$ such that $S^T AS$ is nonnegative.

**Proof.** If $A_{11}$ and $A_{22}$ are zero blocks, obviously $A = 0$ since $A$ is sign idempotent.

Case (i) Both $A_{11}$ and $A_{22}$ are nonzero. Without loss of generality, we may assume that both $A_{11}$ and $A_{22}$ are positive by Theorem 2.3. If $A_{12}$ is a zero block, obviously the result holds. If $A_{12}$ is nonzero, by Lemma 1.1, $A_{12} = \alpha J$ where $\alpha \in \{+, -\}$. Set $S_1 = \alpha I_{t_2}$. Then

$$\begin{pmatrix} I_{t_1} & S_1^T \\ \end{pmatrix} A \begin{pmatrix} I_{t_1} \\ S_1 \end{pmatrix}$$

is nonnegative.

Case (ii) $A_{22}$ is a zero block and $A_{11}$ is nonzero. Without loss of generality, we may assume that $A_{11}$ is positive by Theorem 2.3. By Lemma 1.2, the $i$th column of $A_{12}$ is constantly signed for all $1 \leq i \leq t_2$, i.e., $A_{12}^{(i)} = \alpha_i J^{(i)}$ where $\alpha_i \in \{+, -, 0\}$. Set $S_1 = \text{diag}\{\alpha'_1, \ldots, \alpha'_n\}$, where $\alpha'_i = \alpha_i$ if $\alpha_i \neq 0$ and $\alpha'_i = +$ if $\alpha_i = 0$. Then

$$\begin{pmatrix} I_{t_1} & S_1^T \\ \end{pmatrix} A \begin{pmatrix} I_{t_1} \\ S_1 \end{pmatrix}$$

is nonnegative.

Case (iii) When $A_{11}$ is a zero block but $A_{22}$ is not, the argument is analogous to case (ii). \quad \Box

**Theorem 2.5.** Let $A$ be a sign idempotent sign pattern with no zero diagonal entries. If $A$ is transitively closed, then there exists a generalized permutation pattern $P$ such that $P^T AP$ in the form (2) is nonnegative, and further, every block in $P^T AP$ above the block diagonal is positive or entrywise zero.

**Proof.** Note that the transitive closure property of $A$ still holds after permutation similarity. Since $A$ is a sign idempotent sign pattern with no zero diagonal entries, there exists a generalized
permutation pattern $P_1$ such that $A' = P_1^T A P_1$ is in the form (2) with each diagonal block $A_{ii}$ being positive.

To get the result, we use induction on $r$, the number of diagonal blocks in $A'$. The case $r = 1$ is trivial. The case $r = 2$ is true by case (i) of Lemma 2.4. Now we assume that the assertion holds for $r - 1$ and prove the result for $r$. By our induction assumption, there exists a generalized permutation pattern $S_1$ such that

$$A_1 = \left( S_1^T I_{t_r} \right) A' \left( S_1 I_{t_r} \right) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} \\ A_{22} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ A_{r-1,r-1} & & & A_{r-1,r} & A_{rr} \end{pmatrix}$$

(4)

where each $A_{ii}$ is $t_i \times t_i$ and positive, and $A_{ij}$ is positive or entrywise zero for $1 \leq i < j \leq r - 1$. If $A_{1r}, \ldots, A_{r-1,r}$ are all zero blocks, then the result holds, so we assume that at least one of these blocks is not zero. That is, there is an $i$ with $1 \leq i < r$ such that $A_{ir} \neq 0$. There are two cases that we need consider.

Case (1) Suppose $A_{1r} \neq 0$. If $A_{ir} = 0$ for all $1 < i < r - 1$, since $A_{1r} \neq 0$ is constantly signed, by the transitive closure property, we get that $A_{1i} = 0$ for all $1 < i < r - 1$. Thus, by (4), $A_1$ is permutation similar to the following form:

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{rr} & 0 & \cdots & 0 \\ A_{22} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots \\ A_{r-1,r-1} & & & A_{rr} \end{pmatrix}$$

where each $A_{ii}$ is positive, and $A_{ij}$ is positive or entrywise zero for all $2 \leq i < j \leq r - 1$. By Lemma 2.4, the result holds.

If $A_{ir} \neq 0$ for some $1 < i \leq r - 1$, since $A_{1r} \neq 0$, by the transitive closure property, we get $A_{1i} \neq 0$. By Lemma 1, $A_{1r}, A_{ir}$ and $A_{1i}$ are each constantly signed. By the induction assumption, $A_{1i}$ is positive. By sign idempotence, $A_{1r} = A_{1i} A_{ir}$, so $A_{1r} = \alpha J_{Ir}$ and $A_{ir} = \alpha J_{Ii}$ for some $\alpha \in \{+, -\}$. Let $S = \alpha I_{t_r}$. Then $A_{1r} S$ and $A_{ir} S$ are positive, and $S^T A_{rr} S = A_{rr}$. If $A_{jr} \neq 0$ for some $j$ with $j \neq i$ and $1 < j < r$, then the same argument shows that $A_{jr} S$ is positive. Thus

$$\begin{pmatrix} I_{n-t_r} \\ S^T \end{pmatrix} A_1 \begin{pmatrix} I_{n-t_r} \\ S \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} S \\ A_{22} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ A_{r-1,r-1} & & & A_{r-1,r} & A_{rr} \end{pmatrix}$$

which means that the result holds.

Case (2) Suppose $A_{1r} = 0, \ldots, A_{sr} = 0$ and $A_{s+1,r} \neq 0$ for some $s$ with $s < r - 1$. Then $A_{s+1,r}$ is positive or entrywise negative by Lemma 1.1. Thus, by the transitive closure property, $A_1$ is in the form
where each $A_{ii}$ is positive, and $A_{ij}$ is positive or entrywise zero for $1 \leq i < j \leq r - 1$. Now we only need consider the submatrix of $A$ as follows:

$$
\begin{pmatrix}
A_{s+1,s+1} & \cdots & A_{s+1,r} \\
\vdots & \ddots & \vdots \\
A_{rr} & & A_{rr}
\end{pmatrix}.
$$

According to the proof of the previous case (1), we conclude that the result holds. This completes the proof. □

**Theorem 2.6.** Let $A$ be a sign idempotent sign pattern in modified Frobenius normal form (2) with no block above the block diagonal containing zero entries. Then there exists a signature matrix $S$ such that $S^TAS$ is nonnegative, and further, every block in $S^TAS$ above the block diagonal is positive.

**Proof.** To get the result, we use induction on $r$, the number of diagonal blocks in $A$. The case $r = 1$ is trivial. The case $r = 2$ is true by Lemma 2.4. Now we assume that the assertion holds for $r - 1$ and prove the result for $r$. Note that no block above the block diagonal of $A$ contains zero entries. By our induction assumption, there exists a signature matrix $S_1$ such that

$$
A_1 = \left( S_1^T \right) A \left( S_1 \right) = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} \\
A_{21} & A_{22} & \cdots & A_{2,r-1} & A_{2r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{r-1,1} & A_{r-1,2} & \cdots & A_{r-1,r-1} & A_{r-1,r} \\
A_{rr} & & & \cdots & A_{rr}
\end{pmatrix},
$$

where each $A_{ii}$ is $t_i \times t_i$ and either positive or entrywise zero, and $A_{ij}$ is positive for all $1 \leq i < j \leq r - 1$.

Since $A_1$ is sign idempotent,

$$
A_{1r} = A_{11}A_{1r} + \cdots + A_{1r}A_{rr}.
$$

Note that $A_{1i}$ is positive for $i = 2, \ldots, r - 1$. Since $A_{1r}$ is nonzero entrywise for $i = 2, \ldots, r - 1$, we get $A_{1r} = A_{i1}A_{ir}$, which means that the $k$th columns of $A_{1r}$ and $A_{1r}$ are constantly signed for all $1 \leq k \leq t_r$, i.e.,

$$
A_{1r}^{(k)} = \alpha_k J_{1r}^{(k)}, \quad A_{ir}^{(k)} = \alpha_k J_{ir}^{(k)},
$$

where $\alpha_k \in \{+, -\}$. Set $S_2 = \text{diag}[\alpha_1, \ldots, \alpha_r]$. Then $A_{ir}S_2$ is positive for $i = 1, \ldots, r - 1$. Next we need consider two cases:
Case (i) If $A_{rr} = 0$, then

\[
\left( I_{n-t_r} \right) A_1 \left( I_{n-t_r} \right)^T S_2 = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} S_2 \\ A_{22} & A_{2,1} & \cdots & A_{2,r-1} & A_{2r} S_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{r-1,r-1} & A_{r-1,1} & \cdots & A_{r-1,r-2} & A_{r-1,r} S_2 \\ A_{rr} & 0 & \cdots & \cdots & 0 \end{pmatrix},
\]

which means that the result holds.

Case (ii) If $A_{rr} \neq 0$, considering the fact that $A_{1r} = A_{1r} A_{rr}$ by (5), then the rows of $A_{1r}$ are constantly signed. Thus, by (6), each $A_{ir}$ is constantly signed, i.e., $\alpha_1 = \cdots = \alpha_t = \alpha$ where $\alpha \in \{+, -\}$. Hence, $S_2^T A_{rr} S_2 = A_{rr}$. Thus

\[
\left( I_{n-t_r} \right) A_1 \left( I_{n-t_r} \right)^T S_2 = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,r-1} & A_{1r} S_2 \\ A_{22} & A_{2,1} & \cdots & A_{2,r-1} & A_{2r} S_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{r-1,r-1} & A_{r-1,1} & \cdots & A_{r-1,r-2} & A_{r-1,r} S_2 \\ A_{rr} & 0 & \cdots & \cdots & 0 \end{pmatrix},
\]

which means that the result holds. This completes the proof. \[\square\]

**Example 1.** Let a sign idempotent sign pattern

\[
A = \begin{pmatrix} + & + & - & + & - & - \\ + & + & - & + & - & - \\ 0 & 0 & + & + \\ 0 & 0 & - & - \\ + & + \\ + & + \end{pmatrix}.
\]

Then $S^T A S$ is nonnegative, where $S = \text{diag}[+, +, -, +, -, -]$.

### 3. Sign idempotent sign patterns that allow idempotence

A sign pattern $A$ is said to allow idempotence if there exists an idempotent matrix $B \in Q(A)$. In [1, p. 164], it is shown that not all sign idempotent sign patterns allow idempotence. For example, $A_1 = \begin{pmatrix} + & - \\ 0 & + \end{pmatrix}$ does not allow idempotence. Thus, identifying sign idempotent sign patterns that allow idempotence is an open problem posed by C. Eschenbach in [1]. We define the minimum rank of $A$ as

\[\text{mr}(A) = \min\{\text{rank}(B) \mid B \in Q(A)\} \]

**Theorem 3.1.** Let $A$ be an irreducible sign idempotent sign pattern. Then $A$ allows idempotence. Moreover, $\text{mr}(A) = 1$. 
Proof. By Theorem 2.3, there exists a signature matrix $S$ such that $S^TAS$ is positive. Let $D \in Q(S)$ with all nonzero entries belonging to $\{1, -1\}$. Set

$$B_1 = \alpha \beta^T,$$

where $\alpha$ and $\beta$ are positive column vectors and $\beta^T\alpha = 1$. Then $B = D^{-1}B_1D \in Q(A)$ and $B^2 = B$. Hence $A$ allows idempotence. Obviously rank$(B) = 1$, so mr$(A) = 1$ for $A$ to be entrywise nonzero. □

Theorem 3.2. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ be a sign idempotent sign pattern, where each $A_{ii}$ is $t_i \times t_i$ and either irreducible or entrywise zero. Then $A$ allows idempotence if and only if at least one of $A_{11}$, $A_{12}$ and $A_{22}$ is a zero block.

Proof. Assume that $A$ allows idempotence, which is realized by an idempotent matrix $B \in Q(A)$. Suppose that $A_{11}$, $A_{12}$ and $A_{22}$ are nonzero. By Lemma 2.4, there exists a diagonal matrix $D$ with all diagonal entries belonging to $\{1, -1\}$ such that

$$B_1 = D^{-1}BD = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

where $B_{11}$, $B_{12}$ and $B_{22}$ are positive. Since $B_1$ is idempotent,

$$B_{11}B_{12} + B_{12}B_{22} = B_{12}. $$

Since $B_{11}^2 = B_{11}$, multiplying by $B_{11}$ yields

$$B_{11}B_{12} + B_{11}B_{12}B_{22} = B_{11}B_{12}. $$

So $B_{11}B_{12}B_{22} = 0$, which means that $B_{12} = 0$. We get a contraction. So at least one of $A_{11}$, $A_{12}$ and $A_{22}$ is a zero block. Conversely, we consider the following cases:

Case (i) If both $A_{11}$ and $A_{22}$ are zero blocks, then $A = 0$ for $A$ to be sign idempotent.

Case (ii) If both $A_{11}$ and $A_{12}$ are nonzero, and $A_{22}$ is a zero block, then the columns of $A_{12}$ are constantly signed by Lemma 1.2. By lemma 2.4, we can assume, without loss of generality, that $A_{11}$ is positive and $A_{12}$ is nonnegative. Set

$$B_{11} = \begin{pmatrix} 1/t_1 & \cdots & 1/t_1 \\ \vdots & \ddots & \vdots \\ 1/t_2 & \cdots & 1/t_2 \end{pmatrix} \in t_1 \times t_2,$$

and $B_{12} \in Q(A_{12})$ is a $t_1 \times t_2$ nonnegative matrix with all nonzero entries being 1. Then $B \in Q(A)$ is idempotent. Hence $A$ allows idempotence. Similarly, $A$ allows idempotence if both $A_{22}$ and $A_{12}$ are nonzero, and $A_{11}$ is a zero block.

Case (iii) If both $A_{11}$ and $A_{22}$ are nonzero, and $A_{12}$ is a zero block, by lemma 2.4, then we can assume, without loss of generality, $A_{11}$ and $A_{22}$ are positive. Set

$$B_{ii} = \alpha_i \beta_i^T, \quad i = 1, 2,$$

where each $\alpha_i$ and $\beta_i$ are positive column vectors and $\beta_i^T\alpha_i = 1$. Then $B \in Q(A)$ is idempotent. Hence $A$ allows idempotence. Similarly, $A$ allows idempotence if both $A_{22}$ and $A_{12}$ are zero blocks, or both $A_{11}$ and $A_{12}$ are zero blocks. This completes the proof. □
Theorem 3.3. Let $A$ be a sign idempotent sign pattern with no zero diagonal entries. Then $A$ allows idempotence if and only if there exists a generalized permutation pattern $P$ such that

$$P^TAP = \begin{pmatrix} A_{11} & \cdots & 0 & A_{1r} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{22} & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{r-1,r-1} \\ 0 & \cdots & 0 & A_{r-1,r} \\ A_{rr} & \end{pmatrix},$$

where each $A_{ii}$ is square and positive. Moreover, $mr(A) = r$.

Proof. Since $A$ is a sign idempotent sign pattern with no zero diagonal entries, then there exists a generalized permutation pattern $P$ such that $A' = P^TAP$ is in the form (2) with all diagonal blocks being positive. Suppose $A$ allows idempotence, which is realized by an idempotent matrix $B \in Q(A)$. We use induction on $r$, the number of diagonal blocks in $A'$.

The cases that $r = 1, 2$ are true by Theorem 3.1 and Theorem 3.2. Now we assume that the assertion holds for $r - 1$ and prove the result for $r$. By our induction assumption, $A'$ has the following form

$$A' = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & A_{1r} \\ A_{22} & \cdots & 0 & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r-1,r-1} & 0 & \cdots & A_{r-1,r} \\ A_{rr} & \end{pmatrix},$$

where each $A_{ii}$ is square and positive. Since $A$ is sign idempotent, by Lemma 1.1, $A_{ir}$ is constantly signed for all $1 \leq i \leq r - 1$, i.e., $A_{ir} = \alpha_i I_r$ where $\alpha_i \in \{+, -, 0\}$. Set $S = \text{diag} \{\alpha'_1 I_1, \ldots, \alpha'_{r-1} I_{r-1}, I_r\}$, where $\alpha'_i = \alpha_i$ if $\alpha_i \neq 0$ and $\alpha'_i = +$ if $\alpha_i = 0$. Then $A'_{ir} = \alpha'_i A_{ir}$ is nonnegative, and

$$A_1 = S^T A' S = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & A_{1r} \\ A_{22} & \cdots & 0 & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r-1,r-1} & 0 & \cdots & A_{r-1,r} \\ A_{rr} & \end{pmatrix},$$

Hence, there exists a generalized permutation matrix $D \in Q(PS)$ with all nonzero entries belong to $\{1, -1\}$ such that

$$B_1 = D^{-1} BD = \begin{pmatrix} B_{11} & 0 & \cdots & 0 & B_{1r} \\ B_{22} & \cdots & 0 & B_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r-1,r-1} & 0 & \cdots & B_{r-1,r} \\ B_{rr} & \end{pmatrix},$$

where each $B_{ii}$ is square and positive, and $B_{ir}$ is nonnegative for $i = 1, \ldots, r - 1$. Since $B_1$ is idempotent,

$$B_{ii} B_{ir} + B_{ir} B_{rr} = B_{ir}.$$
Note that $B_{ii}^2 = B_{ii}$. Then

$$B_{ii}B_{ir} + B_{ii}B_{ir}B_{rr} = B_{ii}B_{ir},$$

which implies that $B_{ii}B_{ir}B_{rr} = 0$. Since $B_{ii}$ and $B_{rr}$ are positive, we get $B_{ir} = 0$ for $1 \leq i \leq r - 1$. Hence $A_{ir}' = 0$ for $1 \leq i \leq r - 1$. The result holds.

Conversely, Let $B_{ii} = \alpha_i \beta_i^T$, where $\alpha_i$ and $\beta_i$ are positive column vectors and $\beta_i^T \alpha_i = 1$. Set $B' = \text{diag}\{B_{11}, \ldots, B_{rr}\}$. Then $B' \in Q(P^TAP)$ is idempotent and $\text{rank}(B') = r$. So $A$ allows idempotence, and $\text{mr}(A) = r$.

**Theorem 3.4.** Let the sign idempotent sign pattern $A$ be an $r \times r$ block matrix in modified Frobenius normal form (2) with $r \geq 4$. If no block above the block diagonal contains zero entries, then $A$ does not allow idempotence.

**Proof.** Suppose $A$ allows idempotence, which is realized by an idempotent matrix $B \in Q(A)$. By Theorem 2.6, $B$ is similar to the following form:

$$
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1r} \\
B_{21} & B_{22} & \cdots & B_{2r} \\
\vdots & \ddots & \ddots & \vdots \\
B_{r1} & \cdots & B_{rr}
\end{pmatrix},
$$

where each $B_{ii}$ is square and either positive or entrywise zero, and each $B_{ij}$ is positive for $i < j$. Since $r \geq 4$, at least two blocks $B_{ii}$ are positive. Assume that $B_{kk}$ and $B_{ss}$ are positive for $k < s$. Since $B^2 = B$,

$$B_{ks} = B_{kk}B_{ks} + \cdots + B_{ks}B_{ss}.$$

Thus

$$B_{kk}B_{ks} = B_{kk}B_{ks} + \cdots + B_{kk}B_{ks}B_{ss}.$$

Since each $B_{ij}$ is positive for $i < j$, we have

$$B_{kk}B_{ks}B_{ss} = 0,$

from which we get $B_{ks} = 0$. This is a contraction. So $A$ does not allow idempotence.

**Remark.** If $r = 3$, Theorem 3.4 is not true. Let

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{22} & B_{23} & \vdots \\ B_{33} \end{pmatrix} = \begin{pmatrix} 0 & J_{12} & t_2J_{13} \\ \frac{1}{t_2}J_{22} & J_{23} & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

where each $B_{ii}$ is $t_i \times t_i$. It is easily checked that the sign idempotent sign pattern $A = \text{sign}(B)$ allows idempotence.

**Example 2.** Let a sign idempotent sign pattern

$$A = \begin{pmatrix} + & 0 & 0 \\ 0 & + & - \\ 0 & - & + \end{pmatrix}.$$
Then $A$ is realized by an idempotent matrix $B \in Q(A)$ as follows:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{9}{2} \\ 0 & -\frac{1}{18} & \frac{1}{2} \end{pmatrix}.$$

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**Reference**