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Asymptotic finite-strain thin-plate theory for elastic solids

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Abstract

We offer some observations on recent efforts to extract models for the stretching and bending of thin plates from three-dimensional finite elasticity. Using an asymptotic argument like that advanced by Ciarlet and his school, we show that recent work purporting to derive a non-standard bending theory generates instead a correction to membrane theory of order thickness squared.

Keywords: Plate theory; Asymptotic analysis

1. Introduction

The problem of deriving two-dimensional models from three-dimensional elasticity to describe the bending and stretching of plates and shells is one of the major open problems of Mechanics, with a history nearly two centuries old. In recent years, advances in the analytical foundations of variational theory, particularly those known collectively as the method of Gamma convergence [1], have been used to shed light on the structure of such models and to furnish a rigorous foundation for those originally proposed on the basis of more formal reasoning. In a representative example of this approach [2], it has been shown that under appropriate constitutive hypotheses the classical Kirchhoff bending theory for plates is recovered in the limit of small thickness. Specifically, it is shown that if $E(h)$ is the total strain energy of a thin plate-like body of thickness h, then there exists a sequence of three-dimensional deformations $\mathbf{x}_h$ such that the limit

$$E_3 = \lim_{h\to 0} h^{-3} E(h) \tag{1}$$

exists, where $E(h)$ is evaluated on members of the sequence and E_3 is the classical bending energy of a thin plate. This is an important result but should not be construed as furnishing a solution to the central problem of estimating $E(h)$ for small h. Indeed, rigorous derivations of

$$E_k = \lim_{h\to 0} h^{-k} E(h), \tag{2}$$

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while of significant interest in their own right, furnish, at best, expansions of the form

$$E(h) \sim hE_1 + h^2E_2 + h^3E_3 + \cdots \tag{3}$$

having unknown convergence properties, and thus leave unresolved the issue of rigorous small-thickness estimates of the energy.

A second main line of development, advanced to its modern standard by Ciarlet and his school [3,4], is based at the outset on the method of asymptotic expansions. In view of the foregoing remarks, it is no less general or rigorous than the approach based on Gamma convergence. The asymptotic approach has proved particularly fruitful and, unlike methods based on Gamma convergence, may be used to extract dynamical models of membranes, plates and shells.

A third principal line of inquiry is based on direct models in which the plate or shell is conceived as a surface endowed a priori with kinematic and constitutive structures, and attendant balance laws, which are deemed to represent the important features of the mechanics of thin bodies [5,6]. This approach is not concerned with the connection between two- and three-dimensional theories and, accordingly, is not discussed further. A fourth idea, intermediate between the derived and direct approaches, is developed in [7]. There, exact necessary conditions for the three-dimensional balance laws are obtained via integration through the thickness of the considered thin body, and constitutive structures pertaining exclusively to the two-dimensional theory are developed.

In the present work we adopt the asymptotic method, which affords a systematic analysis of the questions of concern to us here. In particular we show that recent work [8], based on ideas used in the method of Gamma convergence and purporting to discover a non-standard *bending* energy, in fact furnishes an order h^2 correction to the leading-order *membrane* energy. Further developments concerned with the structure of a genuine bending energy valid for finite elastic strains are discussed in a forthcoming work [9].

2. Preliminary elasticity theory

Our development is based on the standard purely mechanical theory of finite elasticity according to which

$$\mathrm{Div}\, \mathbf{P}(\tilde{\mathbf{F}}) = \mathbf{0} \tag{4}$$

if the body is in equilibrium without body force, where the Piola stress $\mathbf{P}$ is given by

$$\mathbf{P}(\tilde{\mathbf{F}}) = U_{\tilde{\mathbf{F}}}, \tag{5}$$

the gradient with respect to the deformation gradient $\tilde{\mathbf{F}}$ of the strain energy $U(\tilde{\mathbf{F}})$ per unit reference volume. This is assumed for the sake of simplicity to be independent of $\mathbf{X}$, the position in a reference configuration κ_r of a material point of the elastic body, and Div is the divergence with respect to $\mathbf{X}$. The deformation gradient satisfies $\mathrm{d}\mathbf{x} = \tilde{\mathbf{F}}\mathrm{d}\mathbf{X}$, where $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is the position after deformation of the same material point and $\boldsymbol{\chi}$ is the deformation function. The local Eq. (4) is equivalent, under suitable smoothness conditions, to the partwise global equation

$$\int_{\partial P} \mathbf{p}(\mathbf{N})\mathrm{d}A = \mathbf{0}, \tag{6}$$

where P is an arbitrary subvolume of κ_r with boundary ∂P having exterior unit normal $\mathbf{N}$, and $\mathbf{p}(\mathbf{N}) = \mathbf{P}(\tilde{\mathbf{F}})\mathbf{N}$.

We assume here that equilibria satisfy the well-known strong-ellipticity condition

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathcal{M}(\tilde{\mathbf{F}})[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \text{for all } \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}, \tag{7}$$

where

$$\mathcal{M}(\tilde{\mathbf{F}}) = U_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}} \tag{8}$$

is the fourth-order tensor of elastic moduli. It is well known that this is a necessary condition for the stability of a homogeneously deformed equilibrium state against infinitesimal plane harmonic waves. In general the strain-energy function is subject to further restrictions associated with frame invariance and material symmetry but these are not germane to the issues that concern us here.

3. Energy density for plates

A plate is regarded as a body whose reference configuration κ_r is a prismatic region generated by the parallel translation of a simply-connected plane Ω with piecewise smooth boundary curve $\partial\Omega$. The body itself occupies the volume $\bar{\Omega} \times (-h/2, h/2)$ where $\bar{\Omega} = \Omega \cup \partial\Omega$ and h is the (uniform) thickness. Let l be another length scale such as the diameter of a hole in Ω or a typical spanwise dimension. We assume that $\epsilon \doteq h/l \ll 1$ and proceed to generate a formal asymptotic expansion, in powers of ϵ, of the potential energy in the presence of various conservative loads. We regard l as a fixed scale and simplify the notation by setting $l = 1$. Our methods are similar to those pioneered by Goldenveizer [10], Ciarlet [4] and Fox et al. [11], except that we allow the expansions of certain gradients to be non-uniform over $\bar{\Omega}$. Non-uniformity is crucial to the description of edge effects that arise in the solution of typical boundary-value problems [9].

Solutions to three-dimensional boundary-value problems in general depend parametrically on the scale ϵ via the boundary conditions. Thus we assume that the three-dimensional deformation and its gradient admit asymptotic expansions in powers of ϵ. This implies that the surface gradient of the deformation field and its normal derivative with respect to a through-thickness coordinate, the *director* field, are uniformly regular. It is significant that the director field is unrelated to the surface gradient of the deformation. It is this fact which allows the expansion of the surface gradient of the *director* to be decoupled from that of the deformation itself and hence to be non-uniform. This in turn enables boundary and loading data to select various interior scalings, and vice versa, including a balance of the leading-order membrane and bending energies. Various other scalings are explored and interpreted in [9].

Thus we assume, for $\mathbf{X} \in \Omega \times (-\epsilon/2, \epsilon/2)$ and ϵ sufficiently small, that the three-dimensional deformation and deformation gradient admit uniformly valid asymptotic expansions of the form

$$\boldsymbol{\chi}(\mathbf{X}; \epsilon) = \boldsymbol{\chi}^0 + \epsilon\boldsymbol{\chi}^1 + \cdots + \epsilon^n\boldsymbol{\chi}^n + \cdots, \qquad \tilde{\mathbf{F}}(\mathbf{X}; \epsilon) = \tilde{\mathbf{F}}^0 + \epsilon\tilde{\mathbf{F}}^1 + \cdots + \epsilon^n\tilde{\mathbf{F}}^n + \cdots, \tag{9}$$

in which the coefficients of ϵ^k are regarded as functions of position $\mathbf{R}$ in $\bar{\Omega}$ and $\varsigma(\doteq\varsigma/l) \in (-\epsilon/2, \epsilon/2)$ in the normal-coordinate parameterization

$$\mathbf{X}(\mathbf{R}, \varsigma) = \mathbf{R} + \varsigma\mathbf{N} \tag{10}$$

of the reference volume, where $\mathbf{N}$, a fixed unit vector, is the orientation of Ω. This assumption is motivated by the scheme used in [3] to show that the leading-order deformation in the expansion of the virtual-work principle is of order $O(1)$ with respect to ϵ.

The deformation and its gradient are related exactly by

$$\mathrm{d}\boldsymbol{\chi} = \tilde{\mathbf{F}}\mathrm{d}\mathbf{X} = \tilde{\mathbf{F}}(\mathrm{d}\mathbf{R} + \mathbf{N}\mathrm{d}\varsigma). \tag{11}$$

Therefore,

$$\mathrm{d}\boldsymbol{\chi} = \mathbf{F}\mathrm{d}\mathbf{R} + \mathbf{d}\mathrm{d}\varsigma, \tag{12}$$

where

$$\mathbf{F} = \tilde{\mathbf{F}}\mathbf{1} \quad \text{and} \quad \mathbf{d} = \tilde{\mathbf{F}}\mathbf{N}, \tag{13}$$

are the surface gradient of the deformation and the director field, respectively, and $\mathbf{1}$ is the (two-dimensional) identity on the translation space of Ω. Both depend on $\mathbf{R}$ and ς via (10) and we note that $\mathbf{F}$ maps the translation space of Ω to that of three-dimensional space.

The representation

$$\mathbf{I} = \mathbf{1} + \mathbf{N} \otimes \mathbf{N} \tag{14}$$

of the three-dimensional identity may be used with $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{I}$ to derive

$$\tilde{\mathbf{F}} = \mathbf{F} + \mathbf{d} \otimes \mathbf{N}. \tag{15}$$

It follows immediately from $(9)_2$ and (13) that $\mathbf{F}$ and $\mathbf{d}$ admit the expansions

$$\mathbf{F} = \mathbf{F}^0 + \epsilon\mathbf{F}^1 + \cdots + \epsilon^n\mathbf{F}^n + \cdots, \qquad \mathbf{d} = \mathbf{d}^0 + \epsilon\mathbf{d}^1 + \cdots + \epsilon^n\mathbf{d}^n + \cdots, \tag{16}$$

in which the coefficients depend on $\mathbf{R}$ and ς, and from (15) that

$$\tilde{\mathbf{F}}^k = \mathbf{F}^k + \mathbf{d}^k \otimes \mathbf{N}. \tag{17}$$

The foregoing expansion scheme generates the associated energy expansion

$$U(\tilde{\mathbf{F}}(\epsilon)) = U(\tilde{\mathbf{F}}(0)) + \epsilon U_\epsilon + \frac{1}{2}\epsilon^2 U_{\epsilon\epsilon} + o(\epsilon^2) \tag{18}$$

in which $\tilde{\mathbf{F}}(\epsilon)$ is given by (9)$_2$ with the dependence on $\mathbf{X}$ suppressed for notational simplicity and $U_\epsilon, U_{\epsilon\epsilon}$ are the derivatives of $U(\tilde{\mathbf{F}}(\epsilon))$ with respect to ϵ, evaluated at $\epsilon = 0$. Thus,

$$U_\epsilon = \mathbf{P}(\tilde{\mathbf{F}}^0)\cdot\tilde{\mathbf{F}}^1 \quad \text{and} \quad U_{\epsilon\epsilon} = \mathcal{M}(\tilde{\mathbf{F}}^0)[\tilde{\mathbf{F}}^1]\cdot\tilde{\mathbf{F}}^1 + 2\mathbf{P}(\tilde{\mathbf{F}}^0)\cdot\tilde{\mathbf{F}}^2. \tag{19}$$

The strain energy per unit area of Ω is then given by

$$\begin{aligned} W &= \int_{-\epsilon/2}^{\epsilon/2} U(\tilde{\mathbf{F}})\mathrm{d}\varsigma = \int_{-\epsilon/2}^{\epsilon/2} U(\tilde{\mathbf{F}}^0)\mathrm{d}\varsigma + \epsilon\int_{-\epsilon/2}^{\epsilon/2}\mathbf{P}(\tilde{\mathbf{F}}^0)\cdot\tilde{\mathbf{F}}^1\mathrm{d}\varsigma \\ &\quad + \epsilon^2\int_{-\epsilon/2}^{\epsilon/2}\mathbf{P}(\tilde{\mathbf{F}}^0)\cdot\tilde{\mathbf{F}}^2\mathrm{d}\varsigma + \frac{1}{2}\epsilon^2\int_{-\epsilon/2}^{\epsilon/2}\mathcal{M}(\tilde{\mathbf{F}}^0)[\tilde{\mathbf{F}}^1]\cdot\tilde{\mathbf{F}}^1\mathrm{d}\varsigma + o(\epsilon^3). \end{aligned} \tag{20}$$

Our assumptions imply that every integral in this expansion is of the form

$$G(\epsilon) = \int_{-\epsilon/2}^{\epsilon/2} g(\varsigma)\mathrm{d}\varsigma \tag{21}$$

in which $g(\varsigma)$ has continuous derivatives. Thus, by Leibniz' rule,

$$G(\epsilon) = \epsilon g_0 + \frac{1}{24}\epsilon^3 g_0'' + o(\epsilon^3), \tag{22}$$

where the primes refer to ς-derivatives and the subscript $_0$ to evaluation at $\varsigma = 0$. For example, if the material is uniform in the through-thickness direction then the first term on the right-hand side of (20) admits the expansion

$$\begin{aligned} \int_{-\epsilon/2}^{\epsilon/2} U(\tilde{\mathbf{F}}^0)\mathrm{d}\varsigma &= \epsilon U(\mathbf{F}_0 + \mathbf{d}_0\otimes\mathbf{N}) + \frac{1}{24}\epsilon^3\{\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0\otimes\mathbf{N})\cdot(\mathbf{G} + \mathbf{h}\otimes\mathbf{N}) \\ &\quad + \mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0\otimes\mathbf{N})[\mathbf{D} + \mathbf{g}\otimes\mathbf{N}]\cdot(\mathbf{D} + \mathbf{g}\otimes\mathbf{N})\} + o(\epsilon^3), \end{aligned} \tag{23}$$

where

$$\mathbf{F}_0 + \mathbf{d}_0\otimes\mathbf{N} = (\tilde{\mathbf{F}}^0)_0, \qquad \mathbf{D} + \mathbf{g}\otimes\mathbf{N} = (\tilde{\mathbf{F}}^0)_0' \quad \text{and} \quad \mathbf{G} + \mathbf{h}\otimes\mathbf{N} = (\tilde{\mathbf{F}}^0)_0''. \tag{24}$$

Comparison with (16) and (17) furnishes

$$\mathbf{g} = (\mathbf{d}^0)_0' = (\boldsymbol{\chi}^0)_0'', \qquad \mathbf{h} = (\mathbf{d}^0)_0'' = (\boldsymbol{\chi}^0)_0''' \tag{25}$$

and

$$\mathbf{D} = \mathbf{F}_0' = \nabla\mathbf{d}_0, \qquad \mathbf{G} = \mathbf{F}_0'' = \nabla\mathbf{g}, \tag{26}$$

where ∇ is the (two-dimensional) surface gradient with respect to position $\mathbf{R}$ on Ω. Further,

$$\mathbf{F}_0 = \nabla\mathbf{r}_0, \tag{27}$$

where

$$\mathbf{r}_k = (\boldsymbol{\chi}^k)_0 \tag{28}$$

is the restriction to Ω of the kth term in the asymptotic expansion of $\boldsymbol{\chi}$.

Using (22), we find that the second and third terms in (20) are approximated by

$$\epsilon \int_{-\epsilon/2}^{\epsilon/2} \mathbf{P}(\tilde{\mathbf{F}}^0) \cdot \tilde{\mathbf{F}}^1 \mathrm{d}\varsigma = \epsilon^2 \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}) + O(\epsilon^4), \tag{29}$$

and

$$\epsilon^2 \int_{-\epsilon/2}^{\epsilon/2} \mathbf{P}(\tilde{\mathbf{F}}^0) \cdot \tilde{\mathbf{F}}^2 \mathrm{d}\varsigma = \epsilon^3 \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_2 + \mathbf{d}_2 \otimes \mathbf{N}) + o(\epsilon^3), \tag{30}$$

respectively, where

$$\mathbf{d}_k = (\mathbf{d}^k)_0 \quad \text{and} \quad \mathbf{F}_k = (\mathbf{F}^k)_0 = \nabla \mathbf{r}_k. \tag{31}$$

The final term in (20) is approximated by

$$\frac{1}{2}\epsilon^2 \int_{-\epsilon/2}^{\epsilon/2} \mathcal{M}(\tilde{\mathbf{F}}^0)[\tilde{\mathbf{F}}^1] \cdot \tilde{\mathbf{F}}^1 \mathrm{d}\varsigma = \frac{1}{2}\epsilon^3 \mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}] \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}) + o(\epsilon^3). \tag{32}$$

Altogether, the strain energy density on Ω is formally approximated by

$$\begin{aligned} W = {} & \epsilon U(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) + \epsilon^2 \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}) \\ & + \epsilon^3 \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_2 + \mathbf{d}_2 \otimes \mathbf{N}) + \frac{1}{2}\epsilon^3 \mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}] \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}) \\ & + \frac{1}{24}\epsilon^3 \{\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{G} + \mathbf{h} \otimes \mathbf{N}) + \mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})[\mathbf{D} + \mathbf{g} \otimes \mathbf{N}] \cdot (\mathbf{D} + \mathbf{g} \otimes \mathbf{N})\} + o(\epsilon^3). \end{aligned} \tag{33}$$

The strain energy stored in any region $\Pi \subset \bar{\Omega}$ is

$$S_\Pi = \int_\Pi W \mathrm{d}A, \tag{34}$$

in which W is regarded as depending implicitly on $\mathbf{R}$ through the terms indicated in (33) and also explicitly if the material is not homogeneous.

4. Uniform regular expansion and the membrane problem

It is of interest to consider the implications of the assumption that the coefficients of ϵ^k in (33) are independent of ϵ, at all points $\mathbf{R} \in \bar{\Omega}$. This means that all functions appearing therein are of order $O(1)$, together with their gradients, uniformly over the plate. If no loads are applied, then the potential energy is the strain energy defined by (34), with Π replaced by Ω. We assume that $\boldsymbol{\chi}(\mathbf{X})$ is assigned, and is thus independent of ϵ, on a part $\partial\Omega_x \times (-\epsilon/2, \epsilon/2)$ of the cylindrical generating surface of the three-dimensional body. We also assume provisionally that the traction vanishes on the surface $\partial\Omega_p \times (-\epsilon/2, \epsilon/2)$, where $\partial\Omega_x$ and $\partial\Omega_p$ are complementary parts of $\partial\Omega$.

The deformation is equilibrated if and only if $\dot{E} = 0$, where $E = S_\Omega$ and the superposed dot denotes the Gateaux derivative, evaluated at an equilibrium state, with respect to the real parameter in a kinematically possible one-parameter family of deformations. From (33) it follows that

$$E/\epsilon = E_0 + \epsilon E_1 + \epsilon^2 E_2 + o(\epsilon^2), \tag{35}$$

and thus that

$$\dot{E}_0 + \epsilon \dot{E}_1 + \epsilon^2 \dot{E}_2 + o(\epsilon^2) = 0, \tag{36}$$

where

$$E_0 = \int_\Omega U(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \mathrm{d}A, \qquad E_1 = \int_\Omega \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}) \mathrm{d}A, \tag{37}$$

and

$$E_2 = B + \int_\Omega \{\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{F}_2 + \mathbf{d}_2 \otimes \mathbf{N}) + \frac{1}{2}\mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}] \cdot (\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N})\} \mathrm{d}A \tag{38}$$

with

$$B = \frac{1}{24}\int_\Omega \{\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\mathbf{G} + \mathbf{h} \otimes \mathbf{N}) + \mathcal{M}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})[\mathbf{D} + \mathbf{g} \otimes \mathbf{N}] \cdot (\mathbf{D} + \mathbf{g} \otimes \mathbf{N})\} \mathrm{d}A. \tag{39}$$

Treating (36) as an asymptotic expansion of the virtual-work principle, we have [4]

$$\dot{E}_k = 0; \quad k = 0, 1, 2, \ldots. \tag{40}$$

For the $O(1)$ problem ($k = 0$) we define $\hat{U}(\mathbf{F}_0, \mathbf{d}_0) = U(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})$ and find that

$$\dot{E}_0 = \int_{\partial\Omega} \dot{\mathbf{r}}_0 \cdot \hat{U}_{\mathbf{F}_0} \boldsymbol{\nu} \mathrm{d}S + \int_\Omega (\dot{\mathbf{d}}_0 \cdot \hat{U}_{\mathbf{d}_0} - \dot{\mathbf{r}}_0 \cdot \operatorname{div} \hat{U}_{\mathbf{F}_0}) \mathrm{d}A, \tag{41}$$

wherein div is the divergence with respect to $\mathbf{R}$, $\boldsymbol{\nu}$ is the rightward unit normal to $\partial\Omega$ when traversed in the sense of increasing S, and the *variations* $\dot{\mathbf{r}}_0$ and $\dot{\mathbf{d}}_0$ are independent. Further,

$$\hat{U}_{\mathbf{F}_0} \cdot \dot{\mathbf{F}}_0 + \hat{U}_{\mathbf{d}_0} \cdot \dot{\mathbf{d}}_0 = \dot{U} = \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) \cdot (\dot{\mathbf{F}}_0 + \dot{\mathbf{d}}_0 \otimes \mathbf{N}), \tag{42}$$

yielding

$$\hat{U}_{\mathbf{F}_0} = \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\mathbf{1} \quad \text{and} \quad \hat{U}_{\mathbf{d}_0} = \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\mathbf{N}. \tag{43}$$

These results combine with (5) to yield a decomposition of the stress which is analogous to (15):

$$\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N}) = \hat{U}_{\mathbf{F}_0} + \hat{U}_{\mathbf{d}_0} \otimes \mathbf{N}. \tag{44}$$

The Euler equations for the $O(1)$ problem may be read off from (41). Thus,

$$\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\mathbf{N} = 0, \qquad \operatorname{div}[\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\mathbf{1}] = \mathbf{0} \quad \text{in } \Omega, \tag{45}$$

and the boundary conditions are

$$\mathbf{r}_0 = \mathbf{x}_0 \quad \text{on } \partial\Omega_x, \qquad \mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \partial\Omega_p, \tag{46}$$

where $\mathbf{x}_0$ is the restriction of the assigned position field to $\partial\Omega_x$. The second of (46) is of course the $O(1)$ contribution to the zero-traction condition. Taken together, these equations constitute a determinate differential-algebraic system for the functions $\mathbf{r}_0(\mathbf{R})$ and $\mathbf{d}_0(\mathbf{R})$. The latter is eliminated by using $(45)_1$ to obtain $\mathbf{d}_0 = \bar{\mathbf{d}}_0(\mathbf{F}_0)$, where $\bar{\mathbf{d}}_0$ is uniquely determined in the presence of strong ellipticity [9,12].

These results simplify the $O(\epsilon)$ term ($k = 1$) dramatically. This is due to the fact that $\mathbf{r}_0$ and $\mathbf{d}_0$ satisfy the $O(1)$ problem and thus are not subject to further variation. The associated Euler equations are given by (40), yielding

$$\dot{E}_1 = \int_{\partial\Omega_x} \dot{\mathbf{r}}_1 \cdot \mathbf{P}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})\boldsymbol{\nu} \mathrm{d}S. \tag{47}$$

In view of our remarks about the assigned position field, the $O(\epsilon^k)$ corrections to its values on $\partial\Omega_x$ vanish in all configurations of the body, implying that $\dot{\mathbf{r}}_k = \mathbf{0}$ on $\partial\Omega_x$ for $k > 0$. Thus $\dot{E}_1 \equiv 0$ and we conclude that E_1 is degenerate. In the same way it follows that the first term in the integral of (38) is also degenerate.

Using the major symmetry of $\mathcal{M}$, the $O(\epsilon^2)$ problem ($k = 2$) reduces to

$$\dot{E}_2 = \dot{B} + \int_\Omega (\dot{\mathbf{F}}_1 \cdot \mathbf{Q}_1 + \dot{\mathbf{d}}_1 \cdot \mathbf{q}_1) \mathrm{d}A, \tag{48}$$

where

$$\mathbf{Q}_1 = \mathcal{M}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}]\mathbf{1} \quad \text{and} \quad \mathbf{q}_1 = \mathcal{M}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \mathbf{d}_1 \otimes \mathbf{N}]\mathbf{N}. \tag{49}$$

From $(31)_2$ we immediately obtain

$$\int_{\Omega} \dot{\mathbf{F}}_1 \cdot \mathbf{Q}_1 \mathrm{d}A = \int_{\partial\Omega_p} \dot{\mathbf{r}}_1 \cdot \mathbf{Q}_1 \boldsymbol{\nu} \mathrm{d}S - \int_{\Omega} \dot{\mathbf{r}}_1 \cdot \operatorname{div} \mathbf{Q}_1 \mathrm{d}A. \tag{50}$$

Further, using (39) we proceed in similar fashion to derive

$$\begin{aligned} 24\dot{B} = & \int_{\Omega} \{\dot{\mathbf{g}} \cdot (R_{\mathbf{g}} - \operatorname{div}[\mathbf{P}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})\mathbf{1}]) + \dot{\mathbf{h}} \cdot \mathbf{P}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})\mathbf{N}\} \mathrm{d}A \\ & + \int_{\partial\Omega_x} \dot{\mathbf{g}} \cdot \mathbf{P}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})\boldsymbol{\nu} \mathrm{d}S, \end{aligned} \tag{51}$$

where

$$R = \mathcal{M}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})[\bar{\mathbf{D}} + \mathbf{g} \otimes \mathbf{N}] \cdot (\bar{\mathbf{D}} + \mathbf{g} \otimes \mathbf{N}). \tag{52}$$

The Euler equations and boundary conditions are obtained by invoking (40), $(45)_2$ and (48). Thus,

$$\operatorname{div} \mathbf{Q}_1 = \mathbf{0}, \qquad \mathbf{q}_1 = \mathbf{0} \quad \text{in } \Omega \tag{53}$$

and

$$R_{\mathbf{g}} = \mathbf{0} \quad \text{in } \Omega, \tag{54}$$

where $\bar{\mathbf{D}} = \nabla \bar{\mathbf{d}}_0$, together with

$$\mathbf{g} = \mathbf{y}_0(S) \quad \text{on } \partial\Omega_x \quad \text{and} \quad \mathbf{Q}_1 \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega_p. \tag{55}$$

The restriction of $\mathbf{g}$ to $\partial\Omega_x$, denoted by $\mathbf{y}_0$, is equal to the restriction to $\partial\Omega_x$ of the second derivative $\boldsymbol{\chi}''$ of the assigned deformation with respect to ς. This furnishes information about the curvature of the deformed generators of the cylindrical part $\partial\Omega \times (-\epsilon/2, \epsilon/2)$ of the reference surface of the body.

It is easy to show that the equation which holds on $\partial\Omega_p$ represents the $O(\epsilon)$ contribution to the zero-traction condition. Further, in the presence of strong ellipticity, (54) possesses a unique solution of the form [9]

$$\mathbf{g} = \mathbf{K}(\mathbf{F}_0)\bar{\mathbf{D}}, \tag{56}$$

which furnishes $\mathbf{g}$ in terms of $\mathbf{F}_0(\mathbf{R})$ and its spatial derivatives. The continuous extension of this solution to $\partial\Omega_x$ is generally incompatible with $(55)_1$. A similar inconsistency arises in alternative asymptotic treatments [4,11]. To circumvent it, we assume that boundary data are specified for $\boldsymbol{\chi}(\mathbf{X})$ only at $\zeta = 0$, so that $\boldsymbol{\chi}''$ is unrestricted on the boundary. Eqs. $(55)_1$ and (56) may then be used to determine $\mathbf{y}_0$ in terms of the known $O(1)$ solution a posteriori. This is consistent with conventional ideas in membrane theory and requires no adjustment to the foregoing development.

Evidently the sole effect of the contribution B to the order $O(\epsilon^2)$ problem is to fix $\mathbf{g}$ in terms of the $O(1)$ solution. The latter plays no role in the determination of $\{\mathbf{r}_1, \mathbf{d}_1\}$, which is described by a *linear* boundary-value problem. To elaborate, the result $(53)_2$ may be used in $(49)_2$ to eliminate $\mathbf{d}_1$ in favor of $\mathbf{F}_1$. Thus, $\mathbf{d}_1 = \bar{\mathbf{d}}_1(\mathbf{F}_1)$, where

$$\bar{\mathbf{d}}_1(\mathbf{F}_1) = -\mathbf{A}_{N,0}^{-1}(\mathcal{M}_0[\mathbf{F}_1])\mathbf{N} \tag{57}$$

in which $\mathcal{M}_0 = \mathcal{M}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})$ and $\mathbf{A}_{N,0} = \mathbf{A}_N(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})$ is the $O(1)$ acoustic tensor based on $\mathbf{N}$. This yields $\mathbf{Q}_1 = \bar{\mathbf{Q}}_1(\mathbf{F}_1)$, the linear function defined by

$$\bar{\mathbf{Q}}_1(\mathbf{F}_1) = \{\mathcal{M}(\mathbf{F}_0 + \bar{\mathbf{d}}_0 \otimes \mathbf{N})[\mathbf{F}_1 + \bar{\mathbf{d}}_1(\mathbf{F}_1) \otimes \mathbf{N}]\}\mathbf{1}. \tag{58}$$

Eqs. $(53)_1$ and (55) then constitute a homogeneous linear boundary-value problem for $\mathbf{r}_1$. Although it is clear that one solution is $\mathbf{r}_1 \equiv \mathbf{0}$ in $\bar{\Omega}$, the general question of the existence and uniqueness of nontrivial solutions to this problem, and to the nonlinear $O(1)$ problem, is not studied here.

Thus the problems which emerge through $O(\epsilon^2)$ from the uniform scaling assumption furnish the restrictions to $\bar{\Omega}$ of the functions $\boldsymbol{\chi}$ and $\mathbf{d}$ through order $O(\epsilon)$.

In the work of Bhattacharya and James [8] an energy functional of the form

$$E/\epsilon = E_0 + \epsilon^2 E_2^* \tag{59}$$

is derived by an alternative formal procedure, where, apart from a three-dimensional strain-gradient term which is incompatible with our assumptions, E_2^* corresponds to our E_2 but with B and the (degenerate) first integrand in (38) omitted (see [8], Eq. (8.19)). Apart from this difference the problem is treated in the same manner, namely by considering the minimization problems for E_0 and E_2^* separately, with the solution to the first entering the second parametrically. In that work, E_2^* is interpreted as the leading-order *bending* energy. Our results indicate that this term is in fact the lowest-order correction to the $O(1)$ *membrane* energy. Bending effects are accounted for by the functional B through the leading-order director gradient $\bar{\mathbf{D}}$ appearing in the function R. However, this bending energy plays a passive role to the order considered due to the assumption that $\mathbf{D}$ and $\mathbf{d}_0$ are uniformly of the same order with respect to ϵ. To render this functional active it is necessary to admit *non-uniform* coordinate rescalings leading to different orders for $\mathbf{D}$ and $\mathbf{d}_0$, such as occur in localized edge effects that are well known in applications of shell theory [7]. Such rescalings lead to useful algorithms for implementing the method of matched asymptotic expansions arising in singular-perturbation problems. Conventionally, this is associated with localized boundary layers, but recently internal rescalings of this kind have also been used to resolve widespread wavy wrinkle patterns observed in tensioned sheets [13]. The misinterpretation of E_2^* as a bending energy in [8] seems to originate in the fact that this term appears at $O(\epsilon^2)$, which is classically identified with bending in works that are not based on asymptotic methods. Indeed, all works to date based on global asymptotic expansions of the energy or on the method of Gamma convergence suffer from the same inability to distinguish precisely those non-uniform scalings that are of primary interest in applications. These are studied extensively in [9].

It is straightforward to amend the foregoing results to take edge loads into account. These are regarded as dead tractions for illustrative purposes. Thus we assume that the (three-dimensional) traction $\mathbf{p}(\mathbf{X})$ is assigned, and is hence independent of ϵ, on $\partial\Omega_p \times (-\epsilon/2, \epsilon/2)$. This is less general than alternative asymptotic treatments based on the virtual-work principle [4] in which attention is not confined to dead-load traction conditions, and thus $\mathbf{p}$ is permitted to depend on ϵ, but the present restriction nevertheless suffices for purposes of illustration. Assuming for simplicity that $\mathbf{p}$ is independent of ζ, the associated load potential is

$$L = \int_{\partial\Omega_p} \mathbf{p} \cdot \left(\int_{-\epsilon/2}^{\epsilon/2} \chi \,\mathrm{d}\zeta \right) \mathrm{d}S \tag{60}$$

and it admits the expansion

$$L/\epsilon = L_0 + \epsilon L_1 + \epsilon^2 L_2 + o(\epsilon^2), \tag{61}$$

where

$$L_0 = \int_{\partial\Omega_p} \mathbf{p} \cdot \mathbf{r}_0 \mathrm{d}S, \qquad L_1 = \int_{\partial\Omega_p} \mathbf{p} \cdot \mathbf{r}_1 \mathrm{d}S, \qquad L_2 = \int_{\Omega_p} \mathbf{p} \cdot \mathbf{r}_2 \mathrm{d}S + \frac{1}{24} \int_{\partial\Omega_p} \mathbf{p} \cdot \mathbf{g} \mathrm{d}S. \tag{62}$$

The potential energy is given by the difference between the total strain energy and the load potential. With this adjustment the stationary-energy conditions are again given by $\dot{E}_k = 0$, with L_k incorporated. The residual $O(1)$ problem is found to be

$$\int_{\partial\Omega_p} \dot{\mathbf{r}}_0 \cdot (\hat{U}_{\mathbf{F}_0} \boldsymbol{\nu} - \mathbf{p}) \mathrm{d}S, \tag{63}$$

yielding

$$\mathbf{P}(\mathbf{F}_0 + \mathbf{d}_0 \otimes \mathbf{N})\boldsymbol{\nu} = \mathbf{p} \quad \text{on } \partial\Omega_p \tag{64}$$

in place of $(45)_1$. With this satisfied, we find that E_1 remains degenerate and that the $O(\epsilon^2)$ problem is unchanged. In particular, $(55)_2$ remains valid as expected because $\mathbf{p}$ is independent of ϵ.

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