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Uniform PL Approximations of Isotopies and Extending PL Isotopies in Low Dimensions

MARY-ELIZABETH HAMSTROM

Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801

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1. Introduction

In [13] and [14], Hudson and Zeeman proved in the PL category that if M is a manifold properly embedded in a manifold N and $F: M \times I^n \to N$ is a locally unknotted n-isotopy (see definition below), then F extends to an ambient isotopy $H: N \times I^n \to N \times I^n$. If the codimension of M is at least 3, then the local unknottedness condition is automatically satisfied. My purpose here is to prove that this condition is satisfied for codimension 1 if dim $N \leqslant 3$ and to apply this result to obtain uniform versions of some theorems of Bing and Sanderson on approximating homeomorphisms by PL ones.

I assume that the reader has some acquaintance with the two papers of Hudson and Zeeman cited above. The isotopy extension theorems in the topological category were proved in several of my own papers [7, 9, 10]. Many of the ideas involved are discussed in my expository paper [12]. In particular, outlines of the proofs of the approximation theorems appear there.

2. Definitions and Background

I work in the two categories PL and TOP. In the first, all objects are polyhedral manifolds and all morphisms are piecewise linear (PL) maps; in the second, everything is topological. In either category, let M be a manifold properly embedded in a manifold N, i.e., $M \cap \partial N = \partial M$. I suppose once and for all that M has codimension 1, $\dim N \leqslant 3$ and that M and N are compact.

The symbol I denotes the unit interval, I^n the unit n-cube, Δ^n the standard n-simplex, and S^{n-1} the $(n-1)$-sphere bounding either I^n or Δ^n. As usual, ∂ denotes boundary and $\circ$ denotes interior. If $f\colon M \to N$ is an embedding, I write $f\colon M \hookrightarrow N$; if $f\colon N \hookrightarrow N$ is onto, I write $f\colon N \circlearrowleft$. A homeomorphism $f\colon M \times I^n \to N \times I^n$ is level preserving (with respect to I^n) if f commutes with p_2, projection on the second factor. In this case, f determines, for each $t \in I^n$, an embedding $f_t\colon M \hookrightarrow N$ defined by $f_t(x) = p_1(x, t)$. A homeomorphism such as f above is called an *n-isotopy*. An S^n-isotopy is similarly defined by replacing I^n by S^n in the definition of n-isotopy. In all cases, 1_N denotes the identity homeomorphism on N, the subscript being dropped if there is little danger of confusion.

In TOP, the symbol $\mathrm{Top}(N)$ denotes the identity component of the space of homeomorphisms of N onto itself with the sup norm metric and $\mathrm{Top}(N; M)$ denotes the subspace consisting of all those homeomorphisms leaving M pointwise fixed. In PL, the symbol $\mathrm{PL}(N)$ denotes the semisimplicial complex whose k-simplices are level preserving PL homeomorphisms $f\colon N \times \Delta^k \circlearrowleft$, each f_t PL isotopic to 1 and $\mathrm{PL}(N; M)$ denotes that whose k-simplices are level preserving PL homeomorphisms $f\colon N \times \Delta^k \circlearrowleft$ leaving the points of $M \times \Delta^k$ fixed, each f_t PL isotopic to 1 under an isotopy leaving M pointwise fixed. Since $\mathrm{Top}(N)$ is a topological space, homotopy is defined there in the usual way. There is a homotopy theory in PL, the details of which can be found in [16, 18]. It suffices here to note than an element of $\pi_k(\mathrm{PL}(N))$ is represented by a level preserving PL homeomorphism $f\colon N \times \Delta^k \circlearrowleft$ such that $f \mid N \times \partial\Delta^k = 1$ and that f and g represent the same element of π_k if there is a level preserving $F\colon N \times \Delta^k \times I \circlearrowleft$, $F_0 = f$, $F_1 = g$ and $F \mid N \times \partial\Delta^k \times I = 1$. It is also possible to look at homotopy in TOP in this same way.

Also, it is useful to consider $\mathrm{PL}_l(N \times I^n)$. Here, the k-simplices are PL level preserving (with respect to I^n and Δ^k) homeomorphisms $f\colon N \times I^n \times \Delta^k \circlearrowleft$. Similar definitions hold for $\mathrm{Top}_l(N \times I^n)$, $\mathrm{PL}_l(N \times \partial I^n)$ etc.

In either category, *n-connected* means that $\pi_k = 0$ for $k \leqslant n$. The set $\mathrm{PL}(N)$ is LC^n if for each $\epsilon > 0$, there is a $\delta > 0$ such that if $f\colon N \times S^k \circlearrowleft$, $k \leqslant n$, is a level preserving homeomorphism that moves no point as much as δ, then there is a level preserving PL homeomorphism $F: N \times S^k \times I \circlearrowleft$ that moves no point as much as ϵ and is such that $F_0 = f$ and $F_1 = 1$. Similar definitions hold in TOP and for PL$(N; M)$ and Top$(N; M)$.

The celebrated Alexander trick proves that PL$(I^n; S^{n-1})$ and Top$(I^n; S^{n-1})$ are k-connected and LC^k for each k ([2; 12, Sec. 2, chap. II). Since PL$([\frac{1}{2}, 1]; \{\frac{1}{2}\})$, PL$([0, \frac{1}{2}]; \{\frac{1}{2}\})$, and PL$(S^1; \{p\})$, $p \in S^1$, are isomorphic to PL$(I; \partial I)$, it is an immediate consequence that PL$(I; \{\frac{1}{2}\})$ and PL$(S^1; \{p\})$ are k-connected and LC^k for each k. It also follows that PL(I^n) and Top(I^n) are homotopically equivalent to PL(S^{n-1}) and Top(S^{n-1}), respectively. For $\dim N \leqslant 3$, PL(N) and Top(N) are LC^k for each k [8, 10, 15]. Also, PL(S^1) and Top(S^1) are homotopically equivalent to S^1; PL$(S^1 \times I; S^1 \times 0)$ and Top$(S^1 \times I; S^1 \times 0)$ are k-connected for each k [1, 8, 20].

Finally, I list some facts that follow readily from results in [7, 10, 11]. First, PL$_l(I^p \times S^n; \partial I^p \times S^n)$ and Top$_l(I^p \times S^n; \partial I^p \times S^n)$ are k-connected and LC^k for each k. Second, since PL(N, M) is LC^k for each k, PL$_l(N \times S^n)$, PL$_l(N \times S^n; M \times S^n)$, PL$_l(N \times I^k; M \times I^k)$, and PL$_l(N \times I^k; (M \times I^k) \cup (N \times \partial I^k))$ are LC^0. In general, if PL(N) is k-connected for each k, PL$_l(N \times S^n)$ is also and in any case, PL$_l(N \times I^n)$ is connected if PL(N) is connected.

I now consider the local unknottedness condition of Hudson and Zeeman (see their papers for more details). This is a PL definition. Everything in this paragraph is PL. A ball pair (B^q, B^m), $q > m$, is a pair of balls with B^m properly embedded in B^q. It is unknotted if it is PL homeomorphic to the standard pair $(\Sigma \Delta^m, \Delta^m)$, where Σ denotes $(q - m)$-fold suspension. If $f: M \hookrightarrow N$ is a proper PL embedding, f is *locally unknotted* if for some triangulations K, L of M, N, for which f is simplicial, the ball pair $(\mathrm{st}(f(v), L), f(\mathrm{st}(v, K))$ is unknotted for each vertex v of K. (Here st denotes the closed star.) This is equivalent to saying that the pair $(lk(f(v), L), f(lk(v, K))$ is PL homeomorphic to $(\Sigma S^{m-1}, S^{m-1})$ if $v \in \mathring{M}$ or $(\Sigma \Delta^{m-1}, \Delta^{m-1})$ if $v \in \partial M$. A proper isotopy $f: M \times I^n \hookrightarrow N \times I^n$ is locally unknotted if for each $t \in I^n$, f_t is locally unknotted and for every simplex σ linearly embedded in I^n, $f \mid M \times \sigma \hookrightarrow N \times \sigma$ is locally unknotted.

3. Lemmas on Fitting Isotopies Together

Lemma 1. *Suppose that $f: M \times S^n \to N \times S^n$ is a locally unknotted PL S^n-isotopy and that PL$(N; M)$ is k-connected for each k. Then, f extends to a PL S^n-isotopy F of N.*

Proof. Consider S^n as the union of two PL balls, D_1 and D_2, with $D_1 \cap D_2 = S^{n-1}$. By the Hudson-Zeeman theorems, $f \mid M \times D_1$ and $f \mid M \times D_2$ extend to a level preserving homeomorphisms F_i: $N \times D_i \circlearrowleft$. The homeomorphisms F_1 and F_2 must be fitted together on $N \times S^{n-1}$. Let $S^{n-1} \times J$, $J = [0, 1]$, denote a standard collar of S^{n-1} in D_2, $x \times 0$ being identified with x for each x in S^{n-1}. It is easy to see that $F_2^{-1}F_1 \mid M \times S^{n-1} = 1$, so that $F_2^{-1}F_1 \in \mathrm{PL}_l(N \times S^{n-1}; M \times S^{n-1})$. Thus, by the remarks in Section 2, there is a level preserving (with respect to S^{n-1} and J) homeomorphism Φ: $N \times S^{n-1} \times J \circlearrowleft$, such that $\Phi \mid M \times S^{n-1} \times J = 1$, $\Phi_0 = F_2^{-1}F_1$, and $\Phi_1 = 1$. Then, F_2': $N \times D_2 \circlearrowleft$, defined by $F_2' \mid N \times (D_2 - (S^{n-1} \times J)) = F_2 \mid N \times (D_2 - (S^{n-1} \times J))$ and $F_2' \mid N \times S^{n-1} \times J = F_2\Phi \mid N \times S^{n-1} \times J$ extends $f \mid M \times D_2$ and equals F_1 on $N \times S^{n-1}$.

The notation in the proof of Lemma 1 appears below.

LEMMA 2. *Suppose that f_1 and f_2 are* PL *level preserving homeomorphisms of $M \times D_1$ and $M \times D_2$ into $N \times D_1$ and $N \times D_2$ such that $f_1(M \times D_1) \cap f_2(M \times D_2) = f_1(M \times S^{n-1})$. If* PL($M$) *is k-connected for each k, then there is a* PL *homeomorphism f of $M \times D_2$ onto $f_2(M \times D_2)$ such that $f \mid M \times S^{n-1} = f_1 \mid M \times S^{n-1}$.*

Proof. The proof is essentially the same as that for Lemma 1.

LEMMA 3. *For each $\epsilon > 0$, there is a $\delta > 0$ such that if F_1 and F_2 in the proof of Lemma* 1 *move no point as much as δ, then F can be chosen so as to move no point as much as ϵ.*

Proof. Since $\mathrm{PL}_l(N \times S^{n-1}; M \times S^{n-1})$ is LC^0 (see Section 2), there is a δ, $0 < \delta < \epsilon/2$, such that any element of $\mathrm{PL}_l(N \times S^{n-1}; M \times S^{n-1})$ within 2δ of the identity is isotopic to the identity in an $\epsilon/2$-neighborhood of the identity. Thus, if F_1 and F_2 move no point as much as δ, then $F_2^{-1}F_1 \mid N \times S^{n-1}$ moves no point as much as 2δ, so that Φ can be chosen so as to move no point as much as $\epsilon/2$. Thus, $F_2\Phi$ and, therefore, F move no point as much as ϵ.

There are several useful variations of these lemmas whose proofs are virtually identical to the proofs above. These will be mentioned as needed.

4. THE LOCAL UNKNOTTEDNESS OF ISOTOPIES IN LOW DIMENSIONS

In this section, everything is PL.

PROPOSITION 1. *If P is a polyhedron properly embedded in $I \times Q$, where $Q = I^n$ or S^n and P meets each $I \times \{x\}$ in a point of $\mathring{I} \times \{x\}$, then P is locally unknotted.*

Proof. Suppose P has a triangulation T. For each vertex v of T, $\mathrm{st}(v, T)$ is a ball. The union of the $I \times \{x\}$ that meet $\mathrm{st}(v, T)$ is clearly a ball B and $(B, \mathrm{st}(v, T))$ is an unknotted ball pair.

COROLLARY. *If $f: \{\frac{1}{2}\} \times Q \hookrightarrow I \times Q$ is proper, then f extends to an ambient isotopy $F: I \times Q \circlearrowleft$.*

Proof. Since (see Section 2), $\mathrm{PL}(I; \{\frac{1}{2}\})$ is LC^k and k-connected for each k, this follows from Proposition 1 and Lemma 1 of Section 3.

PROPOSITION 2. *Proposition 1 is true with I replaced by S^1.*

Proof. Let v be a vertex of T and x its Q coordinate. There is a 1-cell U in S^1 that is a union of two 1-simplices such that $U \times \{x\}$ is a neighborhood of v in $S^1 \times \{x\}$. There is a ball neighborhood B of x in Q such that $(\partial U \times B) \cap P = \varnothing$. If T' is a refinement of T so fine that $\mathrm{st}(v, T') \subset U \times B$, then, as in the proof of Proposition 1, the ball B' that is the union of the segments $U \times \{y\}$ that meet $\mathrm{st}(v, T')$ is such that $(B', \mathrm{st}(v, T'))$ is an unknotted ball pair.

COROLLARY. *If $f: \{p\} \times Q \hookrightarrow S^1 \times Q$ is a proper isotopy, then f extends to an ambient isotopy $F: S^1 \times Q \circlearrowleft$.*

Proof. Since $\mathrm{PL}(S^1; p)$ is k-connected for each k (see Section 2), the proof is like that of the corollary to Proposition 1.

THEOREM 1. *Let P be a polyhedron properly embedded in $I^2 \times I^n$ such that for each $q \in I^n$, $P \cap (I^2 \times \{q\})$ is an arc. Then,* (1) *P is locally unknotted and* (2) *there is a level preserving* PL *homeomorphism of $I \times I^n$ onto P.*

Proof. This is done by induction on n and is evident if $n = 0$. Suppose the theorem is true for $n \leqslant k$ and that we have $n = k + 1$. Let v be a vertex of P with second coordinate q. The worst possible situation is when $q \in \partial I^{k+1}$ and $v \in \partial I^2 \times \{q\}$, so I consider this case and leave the other cases to the reader. Throughout, I use the linear structure of $I^2 \times I^{k+1}$.

Let L be a triangulation of $I^2 \times I^{k+1}$ and K a triangulation of P such that v is a vertex of K, K is a subcomplex of L, K restricted to $P \cap (I^2 \times \{q\})$ is a subcomplex K_0, and L restricted to $I^2 \times \{q\}$ is a subcomplex L_0. Consider st(v, K_0) and st(v, L_0). The first consists of a single 1-simplex, σ; the second has two edges, σ_1 and σ_2, on $\partial I^2 \times \{q\}$ and is the cone of a polyhedral arc that meets $\partial I^2 \times \{q\}$ in two points. There is a convex polyhedral disc D in I^2 with $D \times \{q\}$ a neighborhood of v in int st(v, L_0) such that $D \times \{q\}$ has on its boundary three edges τ, τ_1, τ_2, perpendicular to $\sigma, \sigma_1, \sigma_2$, respectively, τ crossing σ. If ρ is a $(k + 2)$-simplex of K having σ as an edge and ρ meets some $I^2 \times \{x\}$ in a 1-simplex, then that 1-simplex is parallel to σ. Thus, if $\underset{\sim}{Q}$ is a sufficiently small convex polyhedral neighborhood of q in I^{k+1}, $D \times \underset{\sim}{Q} \subset$ int st(v, L) and for each x in $\underset{\sim}{Q}$, $\partial D \times \{x\}$ consists of an arc $D_1 \times \{x\}$ in $\partial I^2 \times \{x\}$ and an arc $D_2 \times \{x\}$ in $I^2 \times \{x\}$ that meets P in just one point and $\partial I^2 \times \{x\}$ in two points. Note that if $X =$ cl$[\partial(D \times \underset{\sim}{Q}) \cap \text{int}(I^2 \times I^{k+1})]$ and $Y =$ cl$(X \cap P)$, then (X, Y) is a $((k + 2), (k + 1))$ ball pair, $D \times \underset{\sim}{Q} = v_* X$, and $(D \times \underset{\sim}{Q}) \cap P = v_* Y$. If (X, Y) is an unknotted ball pair, then, since $D \times \underset{\sim}{Q}$ is convex and lies in st(v, L), $(lk(v, L), lk(v, K))$ is an unknotted ball pair.

Note that $\partial \underset{\sim}{Q}$ is the union of nonoverlapping balls, $\underset{\sim}{Q}_1 \subset \partial I^{k+1}$ and $\underset{\sim}{Q}_2$ lying except for $\partial \underset{\sim}{Q}_2$ in $\mathring{I}^{k+1}$. Then, $\partial(D \times \underset{\sim}{Q}) = (\partial D \times \underset{\sim}{Q}) \cup (D \times \partial \underset{\sim}{Q}) = ((D_1 \times \underset{\sim}{Q}) \cup (D \times \underset{\sim}{Q}_1)) \cup ((D_2 \times \underset{\sim}{Q}) \cup (D \times \underset{\sim}{Q}_2))$. The set X is $(D_2 \times \underset{\sim}{Q}) \cup (D \times \underset{\sim}{Q}_2)$ and Y meets each $D_2 \times \{x\}$ in an interior point of $D_2 \times \{x\}$ and each $D \times \{x\}$ in an arc with one endpoint in $\mathring{D}_2 \times \{x\}$, the other in $\mathring{D}_1 \times \{x\}$. By the induction hypothesis, $(D \times \underset{\sim}{Q}_2, Y \cap (D \times \underset{\sim}{Q}_2))$ is an unknotted ball pair and by the corollary to Proposition 1, $(D_2 \times \underset{\sim}{Q}, Y \cap (D_2 \times \underset{\sim}{Q}))$ is as well. These fit together properly so that (X, Y) is an unknotted ball pair. The other cases are handled similarly. Thus, (1) is proved.

To see that (2) is true, note that the induction hypothesis and Proposition 1 tell us more. There is a PL level preserving (with respect to $\underset{\sim}{Q}_2$ and $\underset{\sim}{Q}$) homeomorphism h on $(I \times \underset{\sim}{Q}_2) \cup (\{0\} \times \underset{\sim}{Q})$ taking $I \times \underset{\sim}{Q}_2$ onto $Y \cap (D \times \underset{\sim}{Q}_2)$ and $\{0\} \times \underset{\sim}{Q}$ onto $Y \cap (D_2 \times \underset{\sim}{Q})$. Get a triangulation of $I \times \underset{\sim}{Q}$ isomorphic to that induced on $v_* Y$ by st(v, K) and use this to star h and thus, extend F to a level preserving homeomorphism of $I \times \underset{\sim}{Q}$ onto $P \cap (D \times \underset{\sim}{Q})$. This construction can be made at any point of P. Therefore, for each $q \in I^{k+1}$, there are a sequence $\underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_p$ of PL $(k + 1)$-ball neighborhoods of q in I^{k+1} and a sequence $D_1, \ldots, D_p$ of discs in I^2 such that the balls $B_i = D_i \times \underset{\sim}{Q}_i$ are nonoverlapping, $\bigcup B_i$ is a ball neighborhood of $P \cap (I^2 \times \{q\})$, and for each i, $B_i \cap B_{i+1}$ is a ball, $B_i \cap B_j = \emptyset$ unless $|i - j| \leqslant 1$, and $P \cap (D_i \times \underset{\sim}{Q}_i)$ is level preserving homeomorphic to $I \times Q_i$. By fitting the B_i together, we get a PL ball neighborhood Q of q such that $P \cap (I^2 \times Q)$ is level preserving homeomorphic to $I \times Q$. Now, I^{k+1} is a union of balls $Q_1, Q_2, \ldots, Q_r$, such that for each i, $Q_{i+1} \cap \bigcup_{j=1}^{i} Q_j$ is a ball. Since the Q_i can be chosen, as above, so that $P \cap (I^2 \times Q_i)$ is level preserving homeomorphic to $I \times Q_i$ and $\mathrm{Pl}_l(I \times I^{k+1})$ is connected (see Section 2), a suitable variation of Lemma 2 can be applied to fit the $P \cap (I^2 \times Q_i)$ together to obtain the required homeomorphism of P onto $I \times I^{k+1}$.

COROLLARY. *There is a* PL *level preserving homeomorphism of $I^2 \times I^n$ onto itself taking $\{\frac{1}{2}\} \times I \times I^n$ onto P.*

THEOREM 2. *If P is a polyhedron properly embedded in $N \times I^n$, where N is a compact 2-manifold, such that each $P \cap (N \times \{q\})$ is an arc, or each $P \cap (N \times \{q\})$ is a simple closed curve, then P is locally unknotted.*

Proof. The proof of Theorem 1, although it uses the linear structure of $I^2 \times I^n$, is essentially local. Let Δ be a ball neighborhood of v in N. There is a PL homeomorphism of Δ onto I^2. We use this homeomorphism and apply the proof of Theorem 1 locally in $\Delta \times I^n$.

COROLLARY. *If M is a 1-manifold properly embedded in the 2-manifold N and $f\colon M \times I^n \hookrightarrow N \times I^n$ is a proper n-isotopy, then f extends to an n-isotopy $F\colon N \times I^n \circlearrowleft$.*

THEOREM 3. *If Q is S^n or I^n and P is a polyhedron in $I^2 \times Q$ such that $P \cap (I^2 \times \{x\})$ is a simple closed curve for each x in Q, then there is a* PL *level preserving homeomorphism of $S^1 \times Q$ onto P.*

Proof. The proof of Theorem 1 shows that P is locally unknotted and that for each $x \in Q$, there is a PL ball neighborhood B of x such that $P \cap (I^2 \times B)$ is level preserving homeomorphic to $S^1 \times B$. Since I^n is the union of balls $B_1, \ldots, B_k$, such that for each i, $B_{i+1} \cap \bigcup_{j=1}^{i} B_j$ is a ball and $\mathrm{PL}_l(S^1 \times I^n)$ is connected, a suitable variation of Lemma 2 can be applied to prove Theorem 3 in the case $Q = I^n$.

If $Q = S^n$, then Q is the union of two balls D_1 and D_2 such that $D_1 \cap D_2 = S^{n-1}$. The Hudson–Zeeman theorems and the proof above imply for S^1 in $\mathring{I}^2$, that there are PL level preserving homeomorphisms F_i, $i = 1, 2$, of $I^2 \times D_i$ onto $I^2 \times D_i$ taking $S^1 \times D_i$ onto $P \cap (I^2 \times D_i)$ and leaving each point of $\partial I^2 \times D_i$ fixed. These may not agree on $I^2 \times S^{n-1}$. However, if A is the annulus in I^2 bounded by S^1 and ∂I^2,

$\mathrm{PL}(A; \partial I^2)$ is k-connected for each k (see Section 2) so that $F_1 \mid A \times D_1$ and $F_2 \mid A \times D_2$ may be fitted together to obtain a level preserving homeomorphism F of $A \times S^n$ onto itself taking $S^1 \times S^n$ onto P. The Hudson–Zeeman theorems and Lemma 1 imply that F extends to all of $I^2 \times S^n$.

COROLLARY. *If $S^1 \subset \mathring{I}^2$ and $f: S^1 \times S^n \to \mathring{I}^2 \times S^n$ is a* PL *S^n-isotopy, then f extends to a* PL *isotopy $F: I^2 \times S^n \circlearrowleft$.*

THEOREM 4. *Suppose that M is the* disc$\{\frac{1}{2}\} \times I^2$ *in I^3 and that $f: M \times I^n \hookrightarrow I^3 \times I^n$ is a proper level preserving embedding. Then, f is locally unknotted.*

Proof. The proof is by induction on n and is evident if $n = 0$. Suppose the theorem is true for $n \leqslant k$ and we have $n = k + 1$. Let Δ be a simplex linearly embedded in I^{k+1}. By the induction hypothesis, $f \mid M \times \Delta$ is locally unknotted if $\dim \Delta \leqslant k$, so suppose Δ has dimension $k + 1$. Let K and L be triangulations of $M \times \Delta$ and $N \times \Delta$ with respect to which $f \mid M \times \Delta$ is simplicial and let $v = (p, q)$ be a vertex of K. Again, the worst situation is when $q \in \partial\Delta$ and $p \in \partial M$. However, the proof is much like that of Theorem 1 in spirit, so, for variation, I consider the case $v \in \mathring{M} \times \mathring{I}^{k+1}$ and use, as before, the linear structure of $M \times I^{k+1}$ and $I^3 \times I^{k+1}$. Suppose further, that $L \mid I^3 \times \{q\}$ is a subcomplex L_0 of L, $K \mid M \times \{q\}$ is a subcomplex K_0 of K and that for each $x \in M$, $f(x, q) = (x, q)$. This last is assumed for notational convenience and is made possible by composing f with $f_q^{-1} \times 1$.

Consider $f(\mathrm{st}(v, K_0))$ and $\mathrm{st}(f(v), L_0)$. The first is the cone of a polyhedral simple closed curve; the second is the cone of a 2-sphere and $(\mathrm{st}(f(v), L_0), f(\mathrm{st}(v, K_0))$ is an unknotted ball pair. There is a convex (triangulated) polyhedral ball D in I^3 such that (1) $D \times \{q\} \subset \mathrm{int}(\mathrm{st}(f(v), L_0))$, (2) neither L_0 nor D contains a vertex of the other, (3) no edge of D meets an edge of L and (4) any face of $\partial D \times \{q\}$ that meets $f(M \times \{q\})$ is perpendicular to $f(M \times \{q\})$. It follows, then, that $D \times \{q\}$ meets $f(\mathrm{st}(v, K_0))$ in a disc $E \times \{q\}$ and that $\partial E \times \{q\}$ meets each interior 1-simplex of $f(\mathrm{st}(v, K_0))$ in exactly one point, that point interior to the edge of $\partial E \times \{q\}$ containing it. Suppose that σ is a $(k + 3)$-simplex of K having a 1- or 2-simplex τ of $f(\mathrm{st}(v, K_0))$ as a face. If τ is a 2-simplex and x is near q, then $f(\sigma)$ meets $I^3 \times \{x\}$ in a 2-simplex; if τ is a 1-simplex, then $f(\sigma)$ meets $I^3 \times \{x\}$ in a 2-cell.

Thus, $f(M \times \{x\})$ has an obvious cellular decomposition. If Q is a sufficiently small convex polyhedral neighborhood of q in Δ, $D \times Q$ lies in $\mathrm{st}(f(v), L)$ and for each x in Q, $\partial D \times \{x\}$ meets $f(M \times \{x\})$ in a simple closed curve. To see that this intersection is a simple closed curve, note that (1) if σ is a simplex of ∂D, $\sigma \times \{x\}$ meets $f(M \times \{x\})$ if and only if $\sigma \times \{q\}$ meets $f(M \times \{q\})$, (2) neither $\partial D \times \{x\}$ nor $f(M \times \{x\})$ contains a vertex of the other and (3) no 1-simplex of $\partial D \times \{x\}$ meets a 1-simplex of $f(M \times \{x\})$. Thus, if a 2-simplex of $\partial D \times \{x\}$ meets a 2-cell in $f(M \times \{x\})$, the intersection is a straight line interval. It readily follows that $(\partial D \times \{x\}) \cap f(M \times \{x\})$ is a simple closed curve. By Theorem 3, there is a PL level preserving homeomorphism $h\colon \partial E \times Q \hookrightarrow f(M \times Q)$ taking $\partial E \times \{x\}$ onto $(\partial D \times \{x\}) \cap f(M \times \{x\})$. By Theorems 1 and 2 and the corollaries, h extends to a PL level preserving

$$h_1\colon (E \cup \partial D) \times Q \hookrightarrow (\partial D \times Q) \cup ((D \times Q) \cap f(M \times Q)).$$

By the induction hypothesis, Hudson–Zeeman theorems, and Lemma 1, $h_1 \mid (E \cup \partial D) \times \partial Q$ extends to $H_1\colon D \times \partial Q \circlearrowleft$. Now,

$$\partial(D \times Q) = (\partial D \times Q) \cup (D \times \partial Q) \cong S^{k+3},$$
$$f(M) \cap \partial(D \times Q) = h(\partial E \times Q) \cup h_1(E \times \partial Q) \cong S^{k+2},$$

and $(\partial(D \times Q), f(M) \cap \partial(D \times Q))$ is an unknotted sphere pair. As in the proof of Theorem 1, $(lk(f(v), L), f(lk(v, K))$ is an unknotted sphere pair. Thus, the isotopy f is unknotted.

THEOREM 5. *If M is a 2-manifold properly embedded in the 3-manifold N, then every proper n-isotopy $f\colon M \times I^n \hookrightarrow N \times I^n$ extends to an ambient n-isotopy $F\colon N \times I^n \circlearrowleft$.*

Proof. Again, the proof of Theorem 4 is essentially local.

5. Approximations of Topological Isotopies by PL Ones

In this section, N is a compact 3-manifold, M is a compact 2-manifold properly embedded in N, and N has a fixed PL structure in which M is a polyhedron. The proofs have been outlined in [12].

In [4] and [17], Moise and Bing prove that if $f \in \mathrm{Top}(N)$ and $\epsilon > 0$, there is an $f^* \in \mathrm{PL}(N)$ such that $d(f, f^*) < \epsilon$. Sanderson proved in

[19], that if $F \in \mathrm{Top}(N)$, $f = F \mid M$, and $f^*\colon M \hookrightarrow N$ is a proper PL embedding sufficiently close to f, then f^* extends to $F^* \in \mathrm{PL}(N)$, a close approximation to F. In [3], Bing proved that every proper embedding f of M in N can be approximated by a proper PL embedding. In this section, I give some uniform versions of these theorems in the form of approximations of isotopies. In [5, 6], Craggs also proves some approximation theorems. In particular, he approximates 1-isotopies.

Remark. All the theorems below are true for $\dim M = 1$ and $\dim N = 2$.

Notation. Suppose we have $N \times I$. If $a, b \in I$, let $\pi_{a,b}\colon N \times \{a\} \to N \times \{b\}$ be the homeomorphism taking (x, a) onto (x, b). Note that $\pi_{a,b}\pi_{b,a} = 1$.

THEOREM 6. *If $f\colon N \times I^n \circlearrowleft$ is a topological n-isotopy, then for each $\epsilon > 0$, there exists a PL n-isotopy $f^*\colon N \times I^n \circlearrowleft$ such that $d(f, f^*) < \epsilon$. If $f \mid N \times \partial I^n = 1$, i.e., f represents an element of $\pi_n(\mathrm{Top}(N))$, then f^* may be chosen to represent an element of $\pi_n(\mathrm{PL}(N))$.*

Proof. The theorem, which is obvious if $n = 0$, is proved by induction on n. Suppose the theorem is true for $n \leqslant k$ and that here, $n = k + 1$, $f \mid N \times \partial I^{k+1} = 1$, and $\epsilon > 0$. For the rest of the proof, all homeomorphisms are 1 on $N \times \partial I^{k+1}$ and explicit mention of this fact will not be made. Consider I^{k+1} as $I^k \times I$. From a slight variation of the proof of Lemma 3, it follows that there is a positive number δ such that if $0 \leqslant a < b < c \leqslant 1$ and F_1 and F_2 are level preserving PL δ-approximations to $f \mid N \times I^k \times [a, b]$ and $f \mid N \times I^k \times [b, c]$, then there is a level preserving ϵ-approximation F to $f \mid N \times I^k \times [a, c]$, such that

$$F \mid N \times I^k \times [a, b] = F_1 \qquad \text{and} \qquad F \mid N \times I^k \times \{c\} = F_2 \mid N \times I^k \times \{c\}.$$

Now, let $0 = t_0 < t_1 < \cdots < t_n = 1$, be such that for each i and each $s, t \in [t_{i-1}, t_i]$, $d(f(x, y, s), f(x, y, t)) < \delta/4$. Let $f_0\colon N \times I^k \times \{0\} \circlearrowleft$ be 1, $f_{n-1}\colon N \times I^k \times \{t_{n-1}\} \circlearrowleft$ be 1, and for $1 \leqslant i \leqslant n - 2$, let f_i be a PL $\delta/4$-approximation to $f \mid N \times I^k \times \{t_i\}$. Let $F_i \mid N \times I^k \times [t_{i-1}, t_i]$ be defined by $F_i(x, y, t) = \pi_{t_{i-1},t} f_{i-1} \pi_{t,t_{i-1}}(x, y, t)$, i.e., $F_i(x, y, t)$ has the same first coordinate as $f_{i-1}(x, y, t_{i-1})$. Then, F_i is a δ-approximation to $f \mid N \times I^k \times [t_{i-1}, t_i]$. Thus, as indicated above, F_2 can be fitted to F_1 to get f_2^*, F_3 can be fitted to f_2^* to get f_3^* and so on until the required f^* is obtained.

Remark. As observed in [12, Sec. 3.1], this argument proves that for compact 3-manifolds N, Top(N) and PL(N) are homotopically equivalent.

THEOREM 7. *Let $f: M \times I^n \hookrightarrow N \times I^n$ be a topological isotopy that extends to an ambient isotopy $F: N \times I^n \circlearrowleft$ and let ϵ be a positive number. Then, there is a positive number δ such that if $h: M \times I^n \hookrightarrow N \times I^n$ is a proper* PL *isotopy and $d(f, h) < \delta$, then h extends to a* PL *ambient isotopy H such that $d(F, H) < \epsilon$.*

Proof. By the theorem of Sanderson mentioned above, the theorem is evident if $n = 0$. Suppose the theorem is true for $n \leqslant k$ and that here $n = k + 1$ and $\epsilon > 0$. Consider I^{k+1} as $I^k \times I$. Since PL(N; M) is LC^k for each k, $\mathrm{PL}_l(N \times I^k; M \times I^k)$ is LC^0 (see Section 2). It follows from a suitable variation of the proof of Lemma 3, that there is a positive number η, such that if $0 \leqslant a < b < c \leqslant 1$ and F_1 and F_2 are PL level preserving η-approximations to $F \mid N \times I^k \times [a, b]$ and $F \mid N \times I^k \times [b, c]$ that agree on $M \times I^k \times \{b\}$, then there is a PL level preserving ϵ-approximation H to $F \mid N \times I^k \times [a, c]$, such that $H \mid N \times I^k \times [a, b] = F_1$, $H \mid M \times I^k \times [b, c] = F_2 \mid M \times I^k \times [b, c]$ and $H \mid N \times I^k \times \{c\} = F_2 \mid N \times I^k \times \{c\}$. It follows from the induction hypothesis, that for each $t \in I$, there is a $\delta_t < \eta/2$ such that if h_t is a PL δ_t-approximation to $f \mid M \times I^k \times \{t\}$, then h_t extends to a PL $\eta/8$-approximation to $F \mid N \times I^k \times \{t\}$. There is a $\delta_t/4$-neighborhood U of t such that if $s \in U$, $d(F(x, y, s), F(x, y, t)) < \delta_t/4$. Thus, if h_s is a PL $\delta_t/4$-approximation to $f \mid M \times I^k \times \{s\}$, $\pi_{s,t}h_s$ is a PL δ_t-approximation to $f \mid M \times I^k \times \{t\}$ and thus, extends to a PL $\eta/8$-approximation H_t to $F \mid N \times I^k \times \{t\}$. Then, $\pi_{t,s}H_t$ is a PL $\eta/2$-approximation to $F \mid N \times I^k \times \{s\}$. Thus, a compactness argument shows that there is a $\delta > 0$ such that for $t \in I$, if h_t is a PL δ-approximation to $f \mid M \times I^k \times \{t\}$, then h_t extends to a PL $\eta/2$-approximation H_t to $F \mid N \times I^k \times \{t\}$.

Let h be a PL δ-approximation to f. Then, for each t, $h \mid M \times I^k \times \{t\}$ extends to a PL $\eta/2$-approximation H_t to $F \mid N \times I^k \times \{t\}$. For any $s \in I$, Theorem 4 and the Hudson–Zeeman theorems yield a PL level preserving homeomorphism $H_{ts}^*: N \times I^k \times [t, s] \circlearrowleft$, extending H_t and $h \mid M \times I_k \times [t, s]$. If s is close enough to t, H_{ts}^* is an η-approximation to $F \mid N \times I^k \times [t, s]$. Thus, there are numbers $0 = t_0 < t_1 < \cdots <$

$t_n = 1$ and PL η-approximations H_i to $F \mid N \times I^k \times [t_{i-1}, t_i]$ that extend $h \mid M \times I^k \times [t_{i-1}, t_i]$. By the remarks above (see proof of Theorem 3), these fit together to yield the required H.

Remark. Since $\mathrm{PL}(N; M)$ is LC^k for each k, it follows that $\mathrm{PL}_l(N \times I^k; (M \times I^k) \cup (N \times \partial I^k))$ is LC^0 (see Section 2), so that a variation of Lemma 3 can be applied to prove that if $F \mid N \times \partial I^n = 1$ and $h \mid M \times \partial I^n = 1$, then H can be constructed so that $H \mid N \times \partial I^n = 1$.

THEOREM 8. *If $f: M \times I^n \hookrightarrow N \times I^n$ is a topological n-isotopy $f \mid M \times \partial I^n = 1$, then f may be approximated arbitrarily closely by a* PL *n-isotopy h such that $h \mid M \times \partial I^n = 1$.*

Proof. In what follows, all homeomorphisms are 1 on $N \times \partial I^n$. It is convenient to consider I^n as the union of two copies of I^n, $I_1^n \cup I_2^n$, where $I_1^n \cap I_2^n = I^{n-1}$. By the topological version of the Hudson–Zeeman theorems [10, 12, chap. 1], $f \mid M \times I_i^n$ extends to $F_i: N \times I_i^n \circlearrowleft$. Let ϵ be a positive number. Since $\mathrm{PL}(N; M)$ is LC^k for each k, $\mathrm{PL}_l(N \times I^{n-1}; (M \times I^{n-1}) \cup (N \times \partial I^{n-1}))$ is LC^0, as noted in Section 2. Thus, there is a positive number $\delta < \epsilon/2$ such that if $G_1, G_2 \in \mathrm{PL}_l(N \times I^{n-1}; (M \times I^{n-1}) \cup (N \times \partial I^{n-1}))$ and $d(G_1, G_2) < \delta$, then there is a PL level preserving $\epsilon/2$-homeomorphism $\Phi: N \times I^{n-1} \times I \circlearrowleft$ such that $\Phi_0 = G_2^{-1} G_1$ and $\Phi_1 = 1$. By Theorem 7, there is a positive number $\eta < \epsilon/2$ such that if $h: M \times I^{n-1} \hookrightarrow N \times I^{n-1}$ is PL and within η of $F \mid M \times I^{n-1}$, then h extends to a PL $H: N \times I^{n-1} \circlearrowleft$, a $\delta/2$-approximation to $F_1 \mid N \times I^{n-1}$. Let $H_i: N \times I_i^n \circlearrowleft$ be a PL $\eta/2$-approximation to F_i. Let $I^{n-1} \times I$ denote a PL collar of I^{n-1} in I_2^n, I^{n-1} identified with $I^{n-1} \times \{0\}$, so small, that $\pi_{1,0} H_2 \pi_{0,1}$ is within $\eta/2$ of $H_2 \mid N \times I^{n-1}$. Then, $H_1 \mid M \times I^{n-1}$ and $\pi_{1,0} H_2 \pi_{0,1} \mid M \times I^{n-1}$ are within η of $f \mid M \times I^{n-1}$, so extend to PL $\delta/2$-approximations G_1 and G_2 to $F_1 \mid N \times I^{n-1}$. (There is no reason why G_1 cannot be taken to be $H_1 \mid N \times I^{n-1}$.) Let $h: M \times I^n \hookrightarrow N \times I^n$ be defined by $h(x, t) = H_1(x, t)$ if $t \in I_1^n$, $h(x, y, s) = \pi_{0,s} G_2 \pi_{s,0} \Phi(x, y, s)$ if $(y, s) \in I^{n-1} \times I$, and $h(x, t) = H_2(x, t)$ if $t \in I_2^n - (I^{n-1} \times I)$. Since $G_2 \mid M \times I^{n-1} = \pi_{1,0} H_2 \pi_{0,1}$, this definition is consistent and $d(f, h) < \epsilon$.

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