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Difference analogue of the Lemma on the Logarithmic Derivative with applications to difference equations ☆

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Abstract

The Lemma on the Logarithmic Derivative of a meromorphic function has many applications in the study of meromorphic functions and ordinary differential equations. In this paper, a difference analogue of the Logarithmic Derivative Lemma is presented and then applied to prove a number of results on meromorphic solutions of complex difference equations. These results include a difference analogue of the Clunie lemma, as well as other results on the value distribution of solutions.

Keywords: Logarithmic difference; Nevanlinna theory; Difference equation

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1. Introduction

The Lemma on the Logarithmic Derivative states that outside of a possible small exceptional set

$$m\left(r, \frac{f'}{f}\right) = O\big(\log T(r, f) + \log r\big), \tag{1}$$

where $m(r, f)$ denotes the Nevanlinna proximity function and $T(r, f)$ is the characteristic of a meromorphic function f [7]. This is undoubtedly one of the most useful results of Nevanlinna theory, having a vast number of applications in the theory of meromorphic functions and in the theory of ordinary differential equations. For instance, the proofs of the Second Main Theorem of Nevanlinna theory [12] and Yosida's generalization [18] of the Malmquist theorem [9] both rely heavily on the Lemma on the Logarithmic Derivative. One major problem in the study of complex difference equations has so far been the lack of efficient tools, which can play roles similar to that played by relation (1) for differential equations. This has meant that most results have had to be proved separately for each difference equation. This slows down the efforts to construct a coherent theory, and it may be one of the reasons why the theory of meromorphic solutions of complex difference equations is not as developed as the theory of differential equations.

The foundations of the theory of complex difference equations were laid by Nörlund, Julia, Birkhoff, Batchelder and others in the early part of the twentieth century. Later on, Shimomura [14] and Yanagihara [16,17] studied non-linear complex difference equations from the viewpoint of Nevanlinna theory. Recently, there has been a renewed interested in the complex analytic properties of solutions of difference equations. In differential equations, Painlevé and his colleagues identified all equations, out of a large class of second-order ordinary differential equations, that possess the Painlevé property [4,5,13]. Those equations which could not be integrated in terms of known functions or through solutions of linear equations are now known as the Painlevé differential equations. Similarly, Ablowitz, Halburd and Herbst [1] suggested that the growth of meromorphic solutions of difference equations could be used to identify those equations which are of "Painlevé type." In [6] the existence of one finite-order non-rational meromorphic solution was shown to be sufficient to single out a list of difference Painlevé equations from a general class of second-order difference equations, provided that the solution does not satisfy a certain first-order difference Riccati equation or a linear difference equation. The proof of this fact relies on a difference analogue of the Lemma on the Logarithmic Derivative, Theorem 2.1, as well as on its consequences, Theorems 3.1 and 3.2, which were used in [6] without proving them. Findings in [6] suggest that finite-order meromorphic solutions of difference equations have a similar role as meromorphic solutions of differential equations.

The purpose of this paper is to prove a difference analogue of the Lemma on the Logarithmic Derivative, and to apply it to study meromorphic solutions of large classes of difference equations. The difference analogue appears to be in its most useful form when applied to study finite-order meromorphic solutions of difference equations, which is in agreement with the findings in [6]. Applications include, for instance, a difference analogue of the Clunie lemma [3]. The original lemma has proved to be an invaluable tool in the study of non-linear differential equations. The difference analogue gives similar information about the finite-order meromorphic solutions of non-linear difference equations.

2. Difference analogue of the Lemma on the Logarithmic Derivative

Theorem 2.1. *Let f be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^{\delta}}\right) \tag{2}$$

for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

Proof. Let $\xi(x)$ and $\phi(r)$ be positive, nondecreasing, continuous functions defined for $e \leqslant x < \infty$ and $r_0 \leqslant r < \infty$, respectively, where r_0 is such that $T(r+|c|, f) \geqslant e$ for all $r \geqslant r_0$. Then by Borel's lemma [2, Lemma 3.3.1]

$$T\left(r + |c| + \frac{\phi(r)}{\xi(T(r+|c|, f))}, f\right) \leqslant 2T\big(r+|c|, f\big)$$

for all r outside of a set E satisfying

$$\int\limits_{E\cap[r_0,R]} \frac{dr}{\phi(r)} \leqslant \frac{1}{\xi(e)} + \frac{1}{\log 2} \int\limits_{e}^{T(R+|c|,f)} \frac{dx}{x\xi(x)}$$

where $R < \infty$. Therefore, by choosing $\phi(r) = r$ and $\xi(x) = x^{\varepsilon/2}$ with $\varepsilon > 0$, and defining

$$\alpha = 1 + \frac{r}{(r+|c|)T(r+|c|, f)^{\varepsilon/2}}, \tag{3}$$

we have

$$T\big(\alpha\big(r+|c|\big), f\big) = T\left(r + |c| + \frac{\phi(r)}{\xi(T(r+|c|, f))}, f\right) \leqslant 2T\big(r+|c|, f\big) \tag{4}$$

for all r outside of a set E with finite logarithmic measure. Hence, if $f(0) \neq 0, \infty$, the assertion follows by combining (3) and (4) with Lemma 2.3 below. Otherwise we apply Lemma 2.3 with the function $g(z) = z^p f(z)$, where $p \in \mathbb{Z}$ is chosen such that $g(0) \neq 0, \infty$. □

When f is of finite order, the right-hand side of (2) is small compared to $T(r, f)$, and therefore relation (2) is a natural analogue of the Lemma on the Logarithmic Derivative (1). Concerning the sharpness of Theorem 2.1, the finite-order functions $\Gamma(z)$, $\exp(z^n)$ and $\tan(z)$ show that δ in (2) cannot be replaced by any number strictly greater than one.

If f is of infinite order, the quantity $T(r+|c|, f)r^{-\delta}$ may be comparable to $T(r, f)$. For instance, by choosing $f(z) = \exp(\exp(z))$, we have

$$m\left(r, \frac{f(z+1)}{f(z)}\right) = (e-1)T(r, f).$$

Therefore Theorem 2.1 is mostly useful when applied to functions with finite order, although the assertion remains valid for all meromorphic functions. In the finite-order case we can also remove the ε in Theorem 2.1.

Corollary 2.2. *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^\delta}\right) \tag{5}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Proof. Choose any $\delta < 1$ and denote $\delta' = (1+\delta)/2$. Since f is of finite order, we have $T(r+|c|, f) \leqslant r^\rho$ for some $\rho > 0$ and for all r sufficiently large. Therefore, by Theorem 2.1

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^{\delta'-\varepsilon\rho}}\right),$$

where $\varepsilon > 0$. The assertion follows by choosing $\varepsilon = (\delta' - \delta)/\rho$. $\square$

Note that by replacing z by $z+h$, where $h \in \mathbb{C}$, and c by $c-h$ in (5), and using the inequality

$$T\bigl(r, f(z+h)\bigr) \leqslant (1+\varepsilon) T\bigl(r+|h|, f(z)\bigr), \quad \varepsilon > 0,\ r > r_0,$$

see [16] or [1], we immediately have

$$m\left(r, \frac{f(z+c)}{f(z+h)}\right) = o\left(\frac{T(r+|c-h|+|h|, f)}{r^\delta}\right) \tag{6}$$

for all $\delta < 1$ outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.3. *Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let $c \in \mathbb{C}$. Then for all $\alpha > 1$, $\delta < 1$ and $r \geqslant 1$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) \leqslant \frac{K(\alpha,\delta,c)}{r^\delta}\left(T\bigl(\alpha\bigl(r+|c|\bigr), f\bigr) + \log^+ \frac{1}{|f(0)|}\right),$$

where

$$K(\alpha,\delta,c) = \frac{8|c|(3\alpha+1) + 8\alpha(\alpha-1)|c|^\delta}{\delta(1-\delta)(\alpha-1)^2 r^\delta}.$$

Proof. Let $\{a_n\}$ denote the sequence of all zeros of f, and similarly let $\{b_m\}$ be the pole sequence of f, where $\{a_n\}$ and $\{b_m\}$ are listed according to their multiplicities and ordered by increasing modulus. By applying Poisson–Jensen formula with $s = \frac{\alpha+1}{2}(r+|c|)$, see, for instance, [7, Theorem 1.1], we obtain

$$\log\left|\frac{f(z+c)}{f(z)}\right| = \int_0^{2\pi} \log\bigl|f(se^{i\theta})\bigr| \operatorname{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c} - \frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi}$$

$$
\begin{aligned}
&+\sum_{|a_n|<s}\log\left|\frac{s(z+c-a_n)}{s^2-\bar a_n(z+c)}\frac{s^2-\bar a_n z)}{s(z-a_n)}\right| \\
&-\sum_{|b_m|<s}\log\left|\frac{s(z+c-b_m)}{s^2-\bar b_m(z+c)}\frac{s^2-\bar b_m z)}{s(z-b_m)}\right|' \\
=:{}& S_1(z)+S_2(z)-S_3(z). \qquad (7)
\end{aligned}
$$

Therefore, by denoting

$$
E:=\left\{\varphi\in[0,2\pi):\ \left|\frac{f(re^{i\varphi}+c)}{f(re^{i\varphi})}\right|\geqslant 1\right\},
$$

we have

$$
\begin{aligned}
m\left(r,\frac{f(z+c)}{f(z)}\right)&=\int_E\log\left|\frac{f(re^{i\varphi}+c)}{f(re^{i\varphi})}\right|\frac{d\varphi}{2\pi} \\
&\leqslant\int_0^{2\pi}\left|S_1(re^{i\varphi})\right|+\left|S_2(re^{i\varphi})\right|+\left|S_3(re^{i\varphi})\right|\frac{d\varphi}{2\pi}.
\end{aligned}
$$

We will now proceed to estimate each $\int_0^{2\pi}|S_j(re^{i\varphi})|\frac{d\varphi}{2\pi}$ separately. Since

$$
\begin{aligned}
|S_1|&=\left|\int_0^{2\pi}\log\left|f(se^{i\theta})\right|\operatorname{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-z-c)(se^{i\theta}-z)}\right)\frac{d\theta}{2\pi}\right| \\
&\leqslant\frac{2|c|s}{(s-r-|c|)^2}\int_0^{2\pi}\left|\log\left|f(se^{i\theta})\right|\right|\frac{d\theta}{2\pi}=\frac{2|c|s}{(s-r-|c|)^2}\left(m(s,f)+m\left(s,\frac{1}{f}\right)\right),
\end{aligned}
$$

we have

$$
\int_0^{2\pi}\left|S_1(re^{i\varphi})\right|\frac{d\varphi}{2\pi}\leqslant\frac{4|c|s}{(s-r-|c|)^2}\left(T(s,f)+\log^+\frac{1}{|f(0)|}\right). \qquad (8)
$$

Next we consider the cases $j=2,3$ combined together. First, by denoting $\{q_k\}:=\{a_n\}\cup\{b_m\}$ and using the fact that $|\log x|=\log^+x+\log^+(1/x)$ for all $x>0$, we have

$$
\begin{aligned}
\int_0^{2\pi}\left|S_2(re^{i\varphi})\right|+\left|S_3(re^{i\varphi})\right|\frac{d\varphi}{2\pi}\leqslant{}&\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\theta}-q_k}\right|\frac{d\theta}{2\pi} \\
&+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-q_k}\right|\frac{d\theta}{2\pi} \\
&+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{\bar q_k c}{s^2-\bar q_k(z+c)}\right|\frac{d\theta}{2\pi}
\end{aligned}
$$

$$+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{\bar{q}_k c}{s^2-\bar{q}_k z}\right|\frac{d\theta}{2\pi}. \tag{9}$$

Second, for any $a\in\mathbb{C}$, and for all $\delta<1$,

$$\int_0^{2\pi}\frac{d\theta}{|re^{i\theta}-a|^{\delta}}\leqslant 4\int_0^{\pi/2}\frac{d\theta}{|re^{i\theta}-|a||^{\delta}}\leqslant\frac{2\pi}{1-\delta}\frac{1}{r^{\delta}}$$

since $|re^{i\theta}-|a||\geqslant r\theta\frac{2}{\pi}$ for all $0\leqslant\theta\leqslant\frac{\pi}{2}$. Therefore

$$\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\theta}-a}\right|\frac{d\theta}{2\pi}\leqslant\frac{1}{\delta}\int_0^{2\pi}\log^+\left(1+\left|\frac{c}{re^{i\theta}-a}\right|^{\delta}\right)\frac{d\theta}{2\pi}$$
$$\leqslant\frac{1}{\delta}\int_0^{2\pi}\left|\frac{c}{re^{i\theta}-a}\right|^{\delta}\frac{d\theta}{2\pi}\leqslant\frac{|c|^{\delta}}{\delta(1-\delta)}\frac{1}{r^{\delta}}, \tag{10}$$

and similarly

$$\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-a}\right|\frac{d\theta}{2\pi}\leqslant\frac{|c|^{\delta}}{\delta(1-\delta)}\frac{1}{r^{\delta}}. \tag{11}$$

Third, since for all a such that $|a|<s$,

$$\left|\frac{a}{s^2-\bar{a}z}\right|\leqslant\frac{1}{s-r},$$

we have

$$\int_0^{2\pi}\log^+\left|1+\frac{\bar{a}c}{s^2-\bar{a}(z+c)}\right|\frac{d\theta}{2\pi}\leqslant\frac{|c|}{s-r-|c|} \tag{12}$$

and

$$\int_0^{2\pi}\log^+\left|1-\frac{\bar{a}c}{s^2-\bar{a}z}\right|\frac{d\theta}{2\pi}\leqslant\frac{|c|}{s-r}. \tag{13}$$

Finally, by combining inequalities (8)–(13), we obtain

$$m\left(r,\frac{f(z+c)}{f(z)}\right)\leqslant\left(\frac{2|c|}{s-r-|c|}+\frac{2|c|^{\delta}}{\delta(1-\delta)}\frac{1}{r^{\delta}}\right)\left(n(s,f)+n\left(s,\frac{1}{f}\right)\right)$$
$$+\frac{4|c|s}{(s-r-|c|)^2}\left(T(s,f)+\log^+\frac{1}{|f(0)|}\right).$$

Therefore, using the fact that

$$n(s,f)+n\left(s,\frac{1}{f}\right)\leqslant\frac{4\alpha}{\alpha-1}\left(T\big(\alpha(r+|c|),f\big)+\log^+\frac{1}{|f(0)|}\right),$$

see [7, p. 37], and $s = \frac{\alpha+1}{2}(r+|c|)$, we conclude

$$\begin{aligned} m\left(r, \frac{f(z+c)}{f(z)}\right) &\leqslant \left(\frac{8|c|(3\alpha+1)}{(\alpha-1)^2(r+|c|)} + \frac{8\alpha|c|^\delta}{\delta(1-\delta)(\alpha-1)r^\delta}\right)\left(T\big(\alpha\big(r+|c|\big), f\big) + \log^+\frac{1}{|f(0)|}\right) \\ &\leqslant \frac{8|c|(3\alpha+1)+8\alpha(\alpha-1)|c|^\delta}{\delta(1-\delta)(\alpha-1)^2r^\delta}\left(T\big(\alpha\big(r+|c|\big), f\big) + \log^+\frac{1}{|f(0)|}\right). \qquad \square \end{aligned}$$

3. Difference analogues of the Clunie and Mohon'ko lemmas

The Lemma on the Logarithmic Derivative is an integral part of the proof of the Second Main Theorem, one of the deepest results of Nevanlinna theory. In addition, logarithmic derivative estimates are crucial for applications to complex differential equations. Similarly, Theorem 2.1 enables an efficient study of complex analytic properties of finite-order meromorphic solutions of difference equations. We are concerned with functions which are polynomials in $f(z+c_j)$, where $c_j \in \mathbb{C}$, with coefficients $a_\lambda(z)$ such that

$$T(r, a_\lambda) = o\big(T(r, f)\big)$$

except possibly for a set of r having finite logarithmic measure. Such functions will be called *difference polynomials in* $f(z)$. We also denote

$$|c| := \max\big\{|c_j|\big\}.$$

The following theorem is analogous to the Clunie lemma [3], which has numerous applications in the study of complex differential equations, and beyond.

Theorem 3.1. *Let $f(z)$ be a non-constant meromorphic solution of*

$$f(z)^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and let $\delta < 1$ and $\varepsilon > 0$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m\big(r, P(z, f)\big) = o\left(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^\delta}\right) + o\big(T(r, f)\big)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Proof. We follow the reasoning behind the original Clunie lemma, see, for instance, [7,8], just replacing the Lemma on the Logarithmic Derivative with Theorem 2.1. First of all,

$$m(r, P) = \int_{E_1} \log^+\big|P(re^{i\theta}, f)\big| \frac{d\theta}{2\pi} + \int_{E_2} \log^+\big|P(re^{i\theta}, f)\big| \frac{d\theta}{2\pi}, \tag{14}$$

where $E_1 = \{\theta \in [0, 2\pi]\colon |f(re^{i\theta})| < 1\}$, and E_2 is the complement of E_1. Now, by denoting $P(z, f) = \sum_\lambda a_\lambda(z) F_\lambda(z, f)$, we have

$$\left|a_\lambda(re^{i\theta}) F_\lambda(re^{i\theta}, f)\right| \leqslant \left|a_\lambda(re^{i\theta})\right| \left|\frac{f(re^{i\theta} + c_1)}{f(re^{i\theta})}\right|^{l_1} \cdots \left|\frac{f(re^{i\theta} + c_\nu)}{f(re^{i\theta})}\right|^{l_\nu}$$

whenever $\theta \in E_1$. Therefore for each λ we obtain

$$\int_{E_1} \log^+ \left|a_\lambda(re^{i\theta}) F_\lambda(re^{i\theta}, f)\right| \frac{d\theta}{2\pi} \leqslant m(r, a_\lambda) + O\left(\sum_{j=1}^{\nu} m\left(r, \frac{f(z + c_j)}{f(z)}\right)\right),$$

and so, by Theorem 2.1,

$$\int_{E_1} \log^+ \left|P(re^{i\theta}, f)\right| \frac{d\theta}{2\pi} = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right) + o\big(T(r, f)\big) \tag{15}$$

outside of an exceptional set with finite logarithmic measure.

Similarly on E_2, by denoting $Q(z, f) = \sum_\gamma b_\gamma(z) G_\gamma(z, f)$, we obtain

$$\begin{aligned} \left|P(z, f)\right| &= \left|\frac{1}{f(z)^n} \sum_\gamma b_\gamma(z) f(z)^{l_0} f(z + c_1)^{l_1} \cdots f(z + c_\mu)^{l_\mu}\right| \\ &\leqslant \sum_\gamma \left|b_\gamma(z)\right| \left|\frac{f(re^{i\theta} + c_1)}{f(re^{i\theta})}\right|^{l_1} \cdots \left|\frac{f(re^{i\theta} + c_\mu)}{f(re^{i\theta})}\right|^{l_\mu} \end{aligned}$$

since $\sum_{j=0}^{\mu} l_j \leqslant n$ by assumption. Therefore by Theorem 2.1 again,

$$\int_{E_2} \log^+ \left|P(re^{i\theta}, f)\right| \frac{d\theta}{2\pi} = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right) + o\big(T(r, f)\big). \tag{16}$$

The assertion follows by combining (14)–(16). □

Similarly as Theorem 3.1 can be used to obtain information about the pole distribution of meromorphic solutions of difference equations, the next result is concerned with distribution of *slowly moving targets* a such that $T(r, a) = o(T(r, f))$ outside of a possible exceptional set of finite logarithmic measure. In particular, constant functions are always slowly moving. The following theorem is an analogue of a result due to A.Z. Mohon'ko and V.D. Mohon'ko [11] on differential equations.

Theorem 3.2. *Let $f(z)$ be a non-constant meromorphic solution of*

$$P(z, f) = 0, \tag{17}$$

where $P(z, f)$ is difference polynomial in $f(z)$, and let $\delta < 1$ and $\varepsilon > 0$. If $P(z, a) \not\equiv 0$ for a slowly moving target a, then

$$m\left(r, \frac{1}{f - a}\right) = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right) + o\big(T(r, f)\big)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Proof. By substituting $f = g + a$ into (17) we obtain

$$Q(z, g) + D(z) = 0, \tag{18}$$

where $Q(z, g) = \sum_\gamma b_\gamma(z) G_\gamma(z, f)$ is a difference polynomial in g such that all of its terms are at least of degree one, and $T(r, D) = o(T(r, g))$ outside a set of finite logarithmic measure. Also $D \not\equiv 0$, since a does not satisfy (17). Next we compute $m(r, 1/g)$. To this end, note that the integral to be evaluated vanishes on the part of $|z| = r$ where $|g| > 1$. It is therefore sufficient to consider only the case $|g| \leqslant 1$. But then,

$$\begin{aligned} \left|\frac{Q(z,g)}{g}\right| &= \frac{1}{|g|}\left|\sum_\gamma b_\gamma(z) g(z)^{l_0} g(z+c_1)^{l_1} \dots g(z+c_\nu)^{l_\nu}\right| \\ &\leqslant \sum_\gamma \left|b_\gamma(z)\right| \left|\frac{g(z+c_1)}{g(z)}\right|^{l_1} \dots \left|\frac{g(z+c_\nu)}{g(z)}^{l_\nu}\right| \end{aligned}$$

since $\sum_{j=0}^{\nu} l_j \geqslant 1$ for all γ. Therefore, by Eq. (18) and Theorem 2.1,

$$\begin{aligned} m\left(r, \frac{1}{g}\right) &\leqslant m\left(r, \frac{D}{g}\right) + m\left(r, \frac{1}{D}\right) = m\left(r, \frac{Q(z,g)}{g}\right) + m\left(r, \frac{1}{D}\right) \\ &= o\left(\frac{T(r+|c|, g)^{1+\varepsilon}}{r^\delta}\right) + o\big(T(r, g)\big) \end{aligned}$$

outside of a set of r-values with at most finite logarithmic measure. Since $g = f - a$ the assertion follows. □

Theorems 3.1 and 3.2, like Theorem 2.1, are particularly useful when applied to functions having finite order. The following two corollaries on the Nevanlinna deficiency illustrate this fact.

Corollary 3.3. *Let $f(z)$ be a non-constant finite-order meromorphic solution of*

$$f(z)^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are difference polynomials in $f(z)$, and let $\delta < 1$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m\big(r, P(z, f)\big) = o\left(\frac{T(r+|c|, f)}{r^\delta}\right) + o\big(T(r, f)\big) \tag{19}$$

for all r outside of a possible exceptional set with finite logarithmic measure. Moreover, the Nevanlinna deficiency satisfies

$$\delta(\infty, P) := \liminf_{r\to\infty} \frac{m(r, P)}{T(r, P)} = 0. \tag{20}$$

Proof. Equation (19) follows by combining the proof of Theorem 3.1 with Corollary 2.2, and so we are left with Eq. (20). By a well-known result due to Valiron [15] and A.Z. Mohon'ko [10], we have

$$T(r, P) = \deg(P) T(r, f) + o\big(T(r, f)\big) \tag{21}$$

outside of a possible exceptional set of finite logarithmic measure. In addition, [8, Lemma 1.1.2] yields that if $T(r,g)=o(T(r,f))$ outside of an exceptional set of finite logarithmic measure, then $T(r,g)=o(T(r^{1+\varepsilon},f))$ for any $\varepsilon>0$ and for all r sufficiently large. Thus, by applying (19) together with (21) and [8, Lemma 1.1.2], we have

$$m(r,P)=o\left(\frac{T(r^{1+\varepsilon},P)}{r^{\delta}}\right)+o\big(T(r^{1+\varepsilon},P)\big)$$

for all sufficiently large r. Therefore, since P is of finite order,

$$m(r,P)\leqslant r^{\rho(1+2\varepsilon)-\delta}, \tag{22}$$

where ρ is the order of P and $\delta<1$. Also, there is a sequence $r_n\to\infty$ as $n\to\infty$, such that

$$T(r_n,P)\geqslant r_n^{\rho-\varepsilon} \tag{23}$$

for all r_n large enough. The assertion follows by combining (22) and (23) where ε and δ are chosen such that $\varepsilon(2\rho+1)<\delta<1$, and by letting $n\to\infty$. □

Corollary 3.4. *Let $f(z)$ be a non-constant finite-order meromorphic solution of*

$$P(z,f)=0,$$

where $P(z,f)$ is difference polynomial in $f(z)$, and let $\delta<1$. If $P(z,a)\not\equiv 0$ for a slowly moving target a, then

$$m\left(r,\frac{1}{f-a}\right)=o\left(\frac{T(r+|c|,f)}{r^{\delta}}\right)+o\big(T(r,f)\big)$$

for all r outside of a possible exceptional set with finite logarithmic measure. Moreover, the Nevanlinna deficiency satisfies

$$\delta(a,f):=\liminf_{r\to\infty}\frac{m(r,1/(f-a))}{T(r,f)}=0.$$

We omit the proof since it would be almost identical to that of Corollary 3.3.

4. Conclusion

In this paper we have presented a difference analogue of the Lemma on the Logarithmic Derivative. This result has potentially a large number of applications in the study of difference equations. Many ideas and methods from the theory of differential equations may now be utilized together with Theorem 2.1 to obtain information about meromorphic solutions of difference equations. Section 3 provides a number of examples in this direction. The analogue of the Clunie lemma, Theorem 3.1, may be used to ensure that finite-order meromorphic solutions of certain non-linear difference equations have a large number of poles. Similarly, Theorem 3.2 provides an easy way of telling when a finite-order meromorphic solution of a difference equation does not have any deficient values.

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