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On metrization of the hit-or-miss topology using Alexandroff compactification

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Abstract

When the hit-or-miss topology is employed, the space of all closed subsets of a Hausdorff, locally compact and second countable space (HLCSC) is known to be Hausdorff, compact and second countable, thus metrizable. This paper investigates metrics on this space by using Alexandroff compactification technique, with a more general metrization procedure developed. A note concerning the necessity of the condition HLCSC on E is included. With the constructed metric, we investigate a hyperspace Birkhoff ergodic theorem to explore the connection between orbital behaviors of hyperspace dynamical systems and Choquet capacities of random closed sets. Moreover, relations between the hit-or-miss topology and other hyperspace topologies or metrics such as the Vietoris topology, Hausdorff metric and Hausdorff–Buseman metric are also given.

Keywords: Hit-or-miss topology and metrization; Alexandroff compactification; Embedding; Hausdorff metric; Choquet capacity; Hyperspace dynamics; Hyperspace Birkhoff ergodic theorem

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1. Introduction

In his pioneering work [17], Matheron showed that the space of all closed subsets of a Hausdorff, locally compact and second countable (i.e., a space admits a countable topological base) space (HLCSC), equipped with the hit-or-miss topology, is Hausdorff, compact and second countable (HCSC), and hence metrizable [11]. However, he did not pursue the metrization of that space. Viewing random closed sets as bona fide random elements, we need to have the space of closed sets as a separable and complete metric space for stochastic analysis, such as defining the concept of convergence in probability of a sequence of random closed sets. Without a metric, the convergence in probability is defined in a cumbersome way [30, pp. 420–422; 20, p. 92]. In another case, while the study of dynamics has been a central part of mathematics and its applications since the middle of the 20th century when scientists from all related disciplines realized the power and beauty of the geometric and qualitative techniques developed during this period for nonlinear systems, much less research on the hyperspace dynamics is seen in the literature (including ours) [29,1,23,38], largely due to the high complexity and difficulty of hyperspace dynamics and particularly the lack of an appropriate hyperspace topology that is compact and metrizable for non-compact underlying spaces, e.g., finite-dimensional manifolds or HLCSC underlying spaces.

HLCSC spaces as a wide class of topological spaces are complex, and these spaces could become structurally invisuable when constructing metrics for their hyperspaces, even in some simplest cases. For instance, denote by E the union of finitely (or countably) many discrete open unit disks in R^2. E is a HLCSC subspace of R^2. It is invisuable to give a metric on the space of all closed subsets of E. An Alexandroff compactification (i.e., one-point compactification) of this union can be a quotient space, obtained by compressing these disks into narrower and narrower ellipses, adhering the boundaries of these ellipses to a fixed point (so each ellipse becomes an ellipsoid in R^3), and placing these ellipsoids as a cluster of separated ellipsoids (except an arbitrarily small neighborhood of the adhered point) appropriately. The required metric is then the Hausdorff metric calculated using the metric of R^3. Another example is the union of finitely (or countably) many discrete torus-like spaces in R^3. Needless to say, the geometry of a general HLCSC space is too complex to be visuable. As such, an appropriate topological treatment that takes into account all underlying HLCSC spaces is critically important in the theory (Sections 3 and 4).

In this paper, a consistent metric for the hit-or-miss topology on any HLCSC space is given. The resulting metric also reveals, despite of various differences, a deep connection among the hit-or-miss topology, Hausdorff metric and Vietoris topology. In Section 2, relations between the hit-or-miss topology and other hyperspace topologies are introduced. In Section 3, the hit-or-miss topology on any HLCSC underlying space E is embedded into the hyperspace of the Alexandroff compactification ωE to explore further relations among the hit-or-miss topology, Hausdorff metric and Vietoris topology, which leads to the desired metric for the hit-or-miss topology. In Section 4, a more general metrization procedure for an arbitrary HLCSC (in particular, HCSC) space is developed. In Section 5, with the constructed metric, a hyperspace Birkhoff ergodic theorem, which explores the relation between orbital behaviors of hyperspace dynamical systems and Choquet capacities of random closed sets through ergodic (probability) measures, is investigated. Section 6 includes some historical notes, a remark concerning the necessity of the assumption HLCSC on E, and an example regarding the relation between the hit-or-miss topology and Hausdorff–Buseman metric.

2. Hit-or-miss topology, other hyperspace topologies and their relations

2.1. Definitions of hyperspace topologies

Throughout this paper, E denotes an arbitrary (non-compact) HLCSC space. Let $\mathscr{F}(E)$, $\mathscr{G}(E)$, and $\mathscr{K}(E)$ denote respectively the sets of all closed, open and compact subsets of E. If there is no ambiguity, these spaces will be abbreviated as $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{K}$ respectively. Notice that $\emptyset \in \mathscr{F}$, $\emptyset \in \mathscr{G}$, and $\emptyset \in \mathscr{K}$. With Matheron's notations [17], define for each $B \subseteq E$ and each $\mathscr{A} \subseteq \mathscr{P}(E)$ (the power set of E),

$$\mathscr{A}^B = \{A \in \mathscr{A} : A \cap B = \emptyset\} \quad \text{and} \quad \mathscr{A}_B = \{A \in \mathscr{A} : A \cap B \neq \emptyset\}.$$

In particular, when $\mathscr{A}$ is replaced by $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{K}$ respectively, three topological spaces are generated as follows.

The hit-or-miss topology τ_f on the set $\mathscr{F}$ (also known as H-topology [13], Choquet–Matheron topology [32], Fell topology [20,22], or weak Vietoris topology [40]) is generated by the subbase

$$\mathscr{F}^K, K \in \mathscr{K}; \quad \mathscr{F}_G, G \in \mathscr{G} \tag{1}$$

(i.e., miss compact sets, or hit open sets (in particular, miss compact sets, or hit interiors of compact sets)).

The hit-or-miss topology τ_g on the set $\mathscr{G}$ is generated by the subbase

$$\mathscr{G}_K, K \in \mathscr{K}; \quad \mathscr{G}^G, G \in \mathscr{G}. \tag{2}$$

The myope topology τ_k on the set $\mathscr{K}$ is generated by the subbase

$$\mathscr{K}^F, F \in \mathscr{F}; \quad \mathscr{K}_G, G \in \mathscr{G}. \tag{3}$$

Moreover, the Hausdorff metric d_{H} on the family of all *non-empty bounded* closed subsets of a metric space (E, d) is defined by [11]

$$d_{\mathrm{H}}(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}. \tag{4}$$

The topology induced by the Hausdorff metric d_{H} is denoted by τ_h.

The Vietoris topology τ_v on $2^E = \mathscr{F} \setminus \{\emptyset\}$, the family of all *non-empty* closed subsets of E, is generated by the base [37,19,11]

$$v(U_1, U_2, \ldots, U_n) = \left\{F \in 2^E | F \subseteq \bigcup_{i=1}^n U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leqslant n\right\}, \tag{5}$$

where $U_1, U_2, \ldots, U_n$ are open subsets of E. 2^E with the Vietoris topology is the exponential space of E [11]. Alternatively, a subbase of the Vietoris topology on 2^E can be obtained from (1) by replacing $\mathscr{F}$ by 2^E and compact subsets K by closed subsets F (i.e., miss closed sets, or hit open sets (in particular, miss closed sets, or hit interiors of closed sets)).

2.2. Relations among hyperspace topologies

If E is compact, τ_f and τ_k are identical; if E is not compact, τ_k is strictly finer than the relative topology induced by τ_f [17]. τ_k is relatively simple since it almost has the same properties as E does. Also, in many aspects, τ_g is a "dual" of τ_f. Hence, the study of τ_f, including the relations among τ_f, τ_h and τ_v, becomes important.

There exist examples of bounded separable metrics spaces E such that τ_v on 2^E and the topology induced by d_H on the set of all non-empty bounded closed subsets of E are incomparable [11], though (1) these two topologies coincide when E is compact; (2) their induced topologies on the family of all non-empty compact subsets of E coincide [2].

When E is compact, $\emptyset$ is an isolated point in $\mathscr{F}$ and $\mathscr{F} \setminus \{\emptyset\}$ is compact [17], and in this case the hit-or-miss topology, Hausdorff metric and Vietoris topology are consistent on $2^E = \mathscr{F} \setminus \{\emptyset\}$. However, if E is non-compact, e.g., R^n, τ_h is non-compact and not defined for unbounded closed subsets and the empty set of R^n; τ_v is non-compact, even non-metrizable [19,2].

When E is HLCSC (which is of great interest for the theory and applications of the hit-or-miss topology), $\mathscr{F}$ with τ_f is HCSC, thus (both E and $\mathscr{F}$) metrizable by the Urysohn's metrization theorem [11]. As $\mathscr{F}$ is HCSC, any consistent metric of $\mathscr{F}$ is complete, separable and totally bounded [11].

For relations among the hit-or-miss topology, lower or upper Vietoris topology, Kuratowski or upper Kuratowski topology, we refer to Nogura and Shakhmatov's paper [22].

2.3. The importance of the hit-or-miss topology and its metrization

The importance of the hit-or-miss topology is seen from its close relations to other hyperspace topologies (Section 2.2). In probability theory, the hit-or-miss topology is the foundation of the Choquet Theorem on random sets, which plays a central role in the study of random sets by connecting the probability measure and the Choquet capacity of a random closed set (Choquet capacities are the counterparts of distribution functions of ordinary random variables) [17,40]. The metrization of the hit-or-miss topology is important as a concrete metric provides a more convenient and visuable approach for the study of this topology (and other relevant theories where the convergence of a sequence of closed sets is involved) and the Choquet Theorem as well as its applications. More general, it is convenient to have a concrete separable and complete metric (i.e., Polish) in order to define the "convergence in probability" of random elements.

The hit-or-miss topology was originally introduced by Fell for the construction of the regularized dual space of a C^*-algebra, named as H-topology by Fell [12,13]. This topology has some remarkable properties: (1) It is always compact (but not necessarily Hausdorff), independent of the property of the underlying space E. (2) If E is locally compact, then the H-topology is compact and Hausdorff, (3) If E is Hausdorff and compact, then the H-topology and Michael's "finite topology" (i.e., Vietoris topology) are consistent (Section 2.2 and [13,19]). Notice that the finite topology is Hausdorff if and only if E is regular [19], while for the H-topology, it is the local compactness of E (no axioms of separation assumed) that implies the Hausdorff property of the H-topology [13]. More general, properties of the finite topology parallel those of E more closely than do those of the H-topology. The compactness property makes the H-topology highly useful in various applications, in particular, in convex analysis and in the study of upper semicontinuous functions, random semicontinuous functions and random capacities by probabilists (i.e., sup vague topology, inf vague topology).

In the literature of dynamics, where the underlying spaces are manifolds, the Hausdorff metric and Vietoris topology are currently used primarily for compact manifolds

[1,29,23,38]. The lack of an appropriate metrizable hyperspace topology and a concrete metric for (locally compact) manifolds prevents from further studies of the hyperspace dynamics including hyperspaces of symbolic dynamical systems [39]. The hit-or-miss topology and a consistent metric to be constructed in Section 3 fulfil such a requirement (Section 5).

2.4. The approach for the metrization of the hit-or-miss topology

In Section 3, we will characterize the convergence of a sequence F_n of closed subsets toward a closed subset F of E by embedding E into its Alexandroff compactification ωE. This embedding induces an embedding of $\mathscr{F}(E)$ into the hyperspace $2^{\omega E}$ $(= \mathscr{F}(\omega E) \setminus \{\emptyset\})$ of all non-empty closed subsets of ωE and plays the key role to solve the metrization problem for the hit-or-miss topology of $\mathscr{F}(E)$. Particularly, when $F = \emptyset$, the stated convergence of the sequence F_n of closed subsets toward the empty set $\emptyset$ of E under the hit-or-miss topology, which describes a divergence of F_n in E (i.e., convergent to the infinity), is equivalent to the convergence of the corresponding sequence $F_n \cup \{\omega\}$ (strictly speaking, this latter F_n represents the subset of ωE obtained from the embedding of the former F_n, which is a closed subset of E, into ωE) toward the singleton set $\{\omega\}$ in $2^{\omega E}$. This approach leads to the finding of a surprising fact: The convergence of a sequence F_n of closed subsets of E under the hit-or-miss topology of $\mathscr{F}(E)$ is exactly equivalent to the convergence of the corresponding sequence $F_n \cup \{\omega\}$ of non-empty closed (thus compact) subsets of ωE under the Hausdorff metric $\bar{d}_{\mathrm{H}}$ defined on the space $2^{\omega E}$. Recall that, as ωE is compact, the Hausdorff metric and Vietoris topology are consistent on $2^{\omega E}$. As the result, an explicit metric has been constructed for the hit-or-miss topology of $\mathscr{F}(E)$. The convergence of a sequence of closed subsets toward the empty set of E is a unique characteristic of the hit-or-miss topology (*Half* of the open subsets, i.e., $\mathscr{F}^K,\ K \in \mathscr{K}$, are the neighborhoods of $\emptyset \in \mathscr{F}$!), and the convergence toward a non-compact closed subset of E involves the convergence toward the empty set as a part (see the proof of case 2 in Theorem 1). In addition, the connection between *all* consistent metrics of $\mathscr{F}(E)$ and the metrics of E (in terms of metrics of ωE) is given in the immediate succeeding paragraph of Theorem 2.

3. Embedding of $\mathcal{F}(E)$ into the hyperspace $2^{\omega E}$

Let E be any (non-compact) HLCSC space. Denote by $\omega E = E \cup \{\omega\}$ the Alexandroff compactification of E (though points of $\omega E \setminus E$ and points of E are not explicitly distinguished, it should be clear when a space or a metric is specified). By identifying all the points except ω of ωE with the corresponding points of E, E has been topologically embedded into ωE as an open subspace. In particular, each point of E is also viewed as a point of ωE. Since E and ωE are both metrizable, let d and $\bar{d}$ be their metrics, respectively. Then these two metrics are consistent on E, i.e., in terms of the induced topologies on E, $\bar{d}(\cdot,\cdot)|_{E\times E}$ is equivalent to $d(\cdot,\cdot)$. In other words, the topology induced on E by the restricted metric $\bar{d}(\cdot,\cdot)|_{E\times E}$ is consistent with the original topology of E.

Let $\mathscr{F} = \{F \subseteq E : F \text{ is a closed subset of } E\}$ and let

$$2^{\omega E} = \{F \subseteq \omega E : F \text{ is a non-empty closed subset of } \omega E\}.$$

Notice that $\emptyset \in \mathscr{F}$ but $\emptyset \notin 2^{\omega E}$.

For all $A, B \in 2^{\omega E}$, the Hausdorff metric on $2^{\omega E}$ is well-defined (as any metric on the compactification ωE is bounded, in fact totally bounded and complete [11]) by

$$\bar{d}_{\mathrm{H}}(A,B) = \max\left\{\sup_{a\in A}\bar{d}(a,B), \sup_{b\in B}\bar{d}(b,A)\right\},$$

or equivalently by

$$\bar{d}_{\mathrm{H}}(A,B) = \inf\{\epsilon : S(A,\epsilon) \supseteq B, S(B,\epsilon) \supseteq A\},$$

where $\bar{d}$ is the metric of ωE and in the second definition of $\bar{d}_{\mathrm{H}}$, $S(A,\epsilon) = \{x \in \omega E : \bar{d}(x,A) < \epsilon\}$ is an ϵ-neighborhood of A in the space ωE and likewise $S(B,\epsilon)$ is an ϵ-neighborhood of B. For a singleton set $\{x\}$, $S(\{x\},\epsilon)$ will be abbreviated as $S(x,\epsilon)$.

The procedure to construct $\bar{d}$ is given in the next section. However, if E is a Euclidean space, $\bar{d}$ is ready at hand for use. In fact, if E is the n-dimensional Euclidean space (R^n, d_n), then ωE is homeomorphic to a n-dimensional sphere S_n in the $(n+1)$-dimensional Euclidean space (R^{n+1}, d_{n+1}) and $\bar{d}$ is the spherical distance on S_n (or topologically equivalently, the Euclidean distance d_{n+1} in (R^{n+1}, d_{n+1}) but restricted on the sphere S_n).

A convenient sphere is that centered at $y = (0,0,\ldots,0,1) \in R^{n+1}$ with radius 1 (with the north pole at $\omega = (0,0,\ldots,0,2)$). Namely, choose $S_n = \{(x_1,x_2,\ldots,x_n,x_{n+1}) \in R^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 + (x_{n+1}^2 - 1)^2 = 1\}$. With this selection of the ωR^n, the mapping C defined below can be chosen as follows: For any $x = (x_1,x_2,\ldots,x_n) \in R^n$, the singleton set $\{x\} \subseteq R^n$ is mapped to $C(\{x\}) \subseteq R^{n+1}$ where $C(\{x\})$ contains two points: ω and the (unique) intersection of the sphere S_n and the line segment that connects $(x_1,x_2,\ldots,x_n,0)$ and ω (this intersection is a point in R^{n+1}). As such, each subset $C(F)$ of the spherical space S_n (defined below) is conveniently determined using the spherical coordinates. See the example at the end of this paper.

Let $\mathcal{O}_h$ be the topology on the hyperspace $2^{\omega E}$, induced by $\bar{d}_{\mathrm{H}}$. Denote $\mathscr{F}(E)$ by $\mathscr{F}$. Recall that the hit-or-miss topology on $\mathscr{F}$ is denoted by τ_f. Define an embedding

$$C : \mathscr{F} \to 2^{\omega E} \quad \text{by } C(F) = F \cup \{\omega\} \text{ for each } F \in \mathscr{F}.$$

(Note. Points of the $F\,(\subseteq \omega E)$ appearing on the right side of $C(F) = F \cup \{\omega\}$ are the identification points of the $F\,(\subseteq E)$ appearing on the left of $C(F) = F \cup \{\omega\}$ under the Alexandroff compactification. This is a convention in the literature [11].)

Theorem 1. *The mapping* $C : (\mathscr{F}, \tau_f) \to (2^{\omega E}, \mathcal{O}_h)$ *is a topological embedding.*

To prove Theorem 1, the following equivalent characterization of the hit-or-miss topology will be used (Lemma 1, see [17]).

Lemma 1. *A sequence of closed subsets* $\{F_n\}$ *of* E *converges to a closed subset* F *of* E *in* $\mathscr{F}$ *if and only if*

(1) *If an open set* G *hits* F*, then* G *hits all the* F_n*'s except at the most a finite number of* F_n*'s.*
(2) *If a compact set* K *is disjoint of* F*, then it is disjoint of all the* F_n*'s except at the most a finite number of* F_n*'s.*

Proof of Theorem 1. From the definition, C is a one-to-one mapping. By applying Alexandroff's following result: every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism [11] (Observe that such a mapping is a closed mapping, and thus its inverse is also continuous.), it follows that every continuous one-to-one mapping of a compact space into a Hausdorff space is a topological embedding.

The space $(\mathscr{F}, \tau_f)$ is compact (and second countable Hausdorff) [17]. Hence, it suffices to prove that C is a continuous mapping. Since $\mathscr{F}$ is second countable (thus admits a countable topological base at each point), only need to prove that if a sequence of points $\{F_n\}_{n=1}^{\infty}$ of $\mathscr{F}$ is convergent to F in $(\mathscr{F}, \tau_f)$, then $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \to 0$ as $n \to \infty$.

Case 1. $F = \emptyset$, i.e., $C(F) = \{\omega\}$. For any $\epsilon > 0$, $S(\omega, \epsilon)$ is an open neighborhood of ω in ωE (recall that the metric on ωE is $\bar{d}$) and $\omega E \setminus S(\omega, \epsilon)$ is a closed subset of ωE thus compact as ωE is compact. As $F_n \xrightarrow{\tau_f} F$, $\omega E \setminus S(\omega, \epsilon)$ is a compact subset of E (as $\omega \notin (\omega E \setminus S(\omega, \epsilon))$) and $F \cap (\omega E \setminus S(\omega, \epsilon)) = \emptyset$ (in fact, $F = \emptyset$ by assumption), by the condition (2) of Lemma 1, there exists a positive integer m such that for any integer $n \geqslant m$, $F_n \cap (\omega E \setminus S(\omega, \epsilon)) = \emptyset$ and hence $F_n \subseteq S(\omega, \epsilon)$ for all $n \geqslant m$. Therefore, for all $n \geqslant m$, it holds that (recall $C(F) = \{\omega\}$ and $C(F_n) = F_n \cup \{\omega\}$)

$$S(C(F), \epsilon) \supseteq C(F_n) \quad \text{and} \quad S(C(F_n), \epsilon) \supseteq C(F),$$

which implies $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \leqslant \epsilon$ for all $n \geqslant m$. Therefore, $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \to 0$ as $n \to \infty$.

Case 2. $F \neq \emptyset$. Assume $F_n \xrightarrow{\tau_f} F$. Need to show $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \to 0$ as $n \to \infty$, i.e., for any $\epsilon > 0$, $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) < \epsilon$ for all sufficiently large n.

As $F \neq \emptyset$, there is some point $y \in F$. Because of $F \subseteq E$ and $\omega \notin E$, $y \neq \omega$ (recall that all points of E have been identified as the corresponding points of ωE by the embedding of E into ωE). As ωE is a Hausdorff space, its points y and ω can be separated by disjoint open subsets of ωE, and thus these exists an open neighborhood $S(\omega, \eta)$ of ω in ωE satisfying $y \notin S(\omega, \eta)$ (with $0 < \eta < \epsilon$). Let $E_\eta = \omega E \setminus S(\omega, \eta)$ and $F' = E_\eta \cap F$. Then $F' \neq \emptyset$ (as $y \in F'$) and E_η is compact (as E_η is a closed subset of compactification ωE), which implies that $E_\eta \setminus S(F', \eta)$ is a compact subset of E. Notice that F' is also compact as it is the intersection of the compact subset E_η and the closed subset F of E.

Since F is disjoint of the compact subset $E_\eta \setminus S(F', \eta)$, and thus by the condition (2) of Lemma 1, there exists a positive integer m_1 such that for all $n \geqslant m_1, F_n \cap (E_\eta \setminus S(F', \eta)) = \emptyset$. Hence, $(F_n \cap E_\eta) \subseteq S(F', \eta)$ for $n \geqslant m_1$. In addition, $F \supseteq F'$ thus $C(F) \supseteq C(F')$.Therefore,

$$\begin{aligned} S(C(F), \eta) &\supseteq S(C(F'), \eta) = S(F' \cup \{\omega\}, \eta) = S(F', \eta) \cup S(\omega, \eta) \supseteq (F_n \cap E_\eta) \cup S(\omega, \eta) \\ &\supseteq (F_n \cap E_\eta) \cup (F_n \cap S(\omega, \eta)) \cup \{\omega\} = F_n \cap (E_\eta \cup S(\omega, \eta)) \cup \{\omega\} \\ &= (F_n \cap \omega E) \cup \{\omega\} = F_n \cup \{\omega\} = C(F_n), \end{aligned}$$

i.e., $S(C(F), \eta) \supseteq C(F_n)$ for $n \geqslant m_1$.

On the other hand, for any point $x \in F'$, $S(x, \frac{\eta}{2})$ is an open subset of E and hits F. By the condition (1) of Lemma 1, there exists a positive integer $k(x)$ such that for any $n \geqslant k(x)$, $F_n \cap S(x, \frac{\eta}{2}) \neq \emptyset$. Clearly, $\{S(x, \frac{\eta}{2}) : x \in F'\}$ is an open cover of F'. Since F' is compact, there exist finitely many points $x_1, x_2, \ldots, x_l$ of F' such that $\cup_{i=1}^{l} S(x_i, \frac{\eta}{2}) \supseteq F'$. Let $m_2 = \max\{k(x_i) : 1 \leqslant i \leqslant l\}$. Then for any $n \geqslant m_2$, $F_n \cap S(x_i, \frac{\eta}{2}) \neq \emptyset$, $i = 1, 2, \ldots, l$. Hence, for any $n \geqslant m_2$, $S(F_n, \eta) \supseteq F'$. Moreover,

$$F = F \cap (E_\eta \cup S(\omega, \eta)) = (F \cap E_\eta) \cup (F \cap S(\omega, \eta)) \subseteq (F' \cup S(\omega, \eta)) \setminus \{\omega\}.$$

Therefore, for any $n \geqslant m_2$,

$$S(C(F_n), \eta) = S(F_n \cup \{\omega\}, \eta) = S(F_n, \eta) \cup S(\omega, \eta) \supseteq F' \cup S(\omega, \eta) \supseteq F \cup \{\omega\} = C(F),$$

i.e., $S(C(F_n), \eta) \supseteq C(F)$ for $n \geqslant m_2$.

Finally, let $m = \max\{m_1, m_2\}$. Then for any $n \geqslant m$, it holds that

$$S(C(F), \eta) \supseteq C(F_n) \quad \text{and} \quad S(C(F_n), \eta) \supseteq C(F).$$

This implies $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \leqslant \eta < \epsilon$ for any $n \geqslant m$. Therefore $\bar{d}_{\mathrm{H}}(C(F_n), C(F)) \to 0$ as $n \to \infty$. □

Now, define a metric on $\mathscr{F} = \mathscr{F}(E)$ by

$$\rho : \mathscr{F} \times \mathscr{F} \to R^+ \quad \text{by } \rho(A, B) = \bar{d}_{\mathrm{H}}(C(A), C(B)) \text{ for all } A, B \in \mathscr{F}.$$

Because the embedding C is a one-to-one mapping, the defined ρ is a metric on $\mathscr{F}$. By Theorem 1, the following theorem has been established.

Theorem 2. *The topology on $\mathscr{F}$ induced by the metric ρ coincides with the hit-or-miss topology. The embedding $C : (\mathscr{F}, \tau_f) \to (2^{\omega E}, \mathcal{O}_h)$ is an isometric topological embedding under the metrics ρ and $\bar{d}_{\mathrm{H}}$.*

As $\mathscr{F}$ is a HCSC space, any consistent metric v of $\mathscr{F}$ holds the properties of a compact metric space, e.g., v is separable and complete (thus Polish) and totally bounded [11]. Further, as $\mathscr{F}_1 = \{\{x\} : x \in E\} \cup \{\emptyset\}$ is a compact subspace of $\mathscr{F}$, the restriction $v|_{\mathscr{F}_1}$ of v on $\mathscr{F}_1$ holds these properties, too. On the other hand, ωE and $\mathscr{F}_1$ are homeomorphic by $\bar{\bar{h}}$ ($\bar{\bar{h}}$ is defined in the conjugacy diagram of Section 5.3) when E is HLCSC. Therefore, $v|_{\mathscr{F}_1}$ is necessarily consistent with a metric $\bar{d}$ of ωE, i.e., $v|_{\mathscr{F}_1}(\{x\}, \{y\}) = \bar{d}(x, y)$ for $x, y \in E$ and $v|_{\mathscr{F}_1}(\{x\}, \emptyset) = \bar{d}(x, \omega)$ for $x \in E$ (again, on the right hand sides of these two equalities, x and y represent their identification points in ωE). When these metrics of ωE are restricted on E, the relation between $v|_{\mathscr{F}_1}$ and metrics of E is obtained from the above first equality.

4. Procedure to construct a metric for any Hausdorff, locally compact and second countable space

The procedure developed in this section can be used to construct a metric for any HLCSC space, e.g., for E and ωE respectively. Of course, the procedure is also valid for $\mathscr{F}(E)$ though this is not necessary as the defined metric ρ on $\mathscr{F}(E)$ is given by the metric $\bar{d}$ on ωE and the Hausdorff metric $\bar{d}_{\mathrm{H}}$ on the hyperspace $2^{\omega E}$. Particularly, when E is the n-dimensional Euclidean space R^n (or some simple HLCSC subspaces of R^n), there is no need to apply this procedure as $\bar{d}$ in this special case is the spherical distance on S_n (or topologically equivalently, the Euclidean distance d_{n+1} in (R^{n+1}, d_{n+1})), which has been discussed in Section 3.

The metrization procedure provided here has been developed from [11]. This procedure can be used to construct the metrics for many metrizable spaces, including all HLCSC spaces. This method itself provides a sort of metrization theorem and was developed by a lot of mathematicians including Urysohn. Note that any HLCSC space is metrizable by Urysohn's metrization theorem. Every locally compact Hausdorff space is a Tychonoff space and any HLCSC space is, of course, a T_3 space with a countable base. Thus, the Urysohn's metrization theorem applies. Notice also that, unless for some simple spaces like R^n, different HLCSC spaces have varying and complicated topological bases and thus it is probably impossible to develop a better procedure that could result in explicit metrics for all such spaces.

A topological space is perfectly normal if it is T_4 (i.e. T_1 and normal) and every closed subset is a G_δ set in the space. Every metrizable space is perfectly normal. Locally compact

Hausdorff spaces may not be perfectly normal (even may not be T_4). Every HLCSC space is perfectly normal since it is metrizable.

Let E be a HLCSC space. E itself is metrizable. Let us now take E as an example and see how to construct a compatible metric on this space. The method works for any T_3 space with a countable base and in particular for obtaining a metric $\bar{d}$ of ωE.

Let U_i $(i \in N)$ be a countable base of E (for ωE, a countable base consists of a countable base of E and a countable topological base at point ω; for $\mathscr{F}$, a countable base is given in [17]). First, for each $i \in N$, construct a continuous function f_i on E such that

$$0 \leqslant f_i \leqslant 1 \quad \text{and} \quad E \setminus U_i = f^{-1}(0),$$

where $E \setminus U_i$ is a zero-set and U_i is a cozero set. Since the space E is perfectly normal, for each $i \in N$, write $E \setminus U_i = \cap_{j=1}^{\infty} V_j$, where V_j $(j \in N)$ are open in E. Let i be fixed. For each j, by the construction in the proof of Urysohn's lemma [11], one can construct a continuous function $g_j^{(i)}$ on E such that

$$0 \leqslant g_j^{(i)} \leqslant 1, \ \ g_j^{(i)}(x) = 0, \ \text{if } x \in E \setminus U_i \quad \text{and} \quad g_j^{(i)}(x) = 1, \ \text{if } x \in E \setminus V_j.$$

Let

$$f_i(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} g_j^{(i)}(x), \quad x \in E.$$

The function f_i is continuous on E and $0 \leqslant f_i \leqslant 1$ for every $x \in E$. Clearly, $f_i(x) = 0$ if $x \in E \setminus U_i$. If $x_0 \notin E \setminus U_i$, then $x_0 \notin V_{j_0}$ for some index j_0 and, consequently, $g_{j_0}^{(i)}(x_0) = 1$. But then $f_i(x_0) \neq 0$. Thus $E \setminus U_i = f_i^{-1}(0)$. The set $\{f_i : i \in N\}$ is a regular (or partitioning) collection of functions on E, i.e. every function in the collection is continuous and the functions in the collection separate each point x of E from each closed subset W of E not containing x, i.e. there exists a function f_{i_0} in the collection such that $f_{i_0}(x) \notin cl_{[0,1]}(f_{i_0}(W))$ (closure of $f_{i_0}(W)$ in $[0, 1]$).

Next, consider the diagonal mapping [11]

$$f = \mathop{\triangle}_{i=1}^{\infty} f_i : E \to \prod_{i=1}^{\infty} I_i,$$

where $\prod_{i=1}^{\infty} I_i$ is the Cartesian product of countably many closed unit intervals with the Tychonoff topology, i.e. the Hilbert cube, and $f(x) = \{f_i(x)\}$. It is easy to check that f is a homeomorphic embedding.

Finally, the metric on E is defined by

$$\varrho(x_1, x_2) = d(f(x_1), f(x_2)) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_1(f_i(x_1), f_i(x_2)),$$

where d_1 is the usual metric on $[0, 1]$ and d is the metric on the Hilbert cube.

5. Hyperspace dynamics, Choquet capacity and hyperspace ergodic theorem

5.1. Hyperspace dynamical systems

A nonlinear dynamical system may behave as forever at rest, forever expanding (unbounded systems), periodic motion, quasi-periodic motion or chaotic motion (e.g., topological transitivity, dense periodic points, and sensitivity on initial conditions [8,9]). Dynamics is concerned with describing for the majority of systems how the majority of orbits evolve, particularly as time goes to infinity, with the emphasis on asymptotic behavior especially in the presence of nontrivial recurrence, and understanding when and in which sense this behavior is robust under small modifications of the system [36].

Hyperspace dynamical systems as particular systems play important roles in the theory of dynamics because (1) A dynamical system often can be considered as a subsystem of its induced hyperspace dynamical system [38]. (2) When studying the iterative properties of a subset of the original system, the natural domain is obviously the induced hyperspace system. Applications of dynamics are diverse and complex, but also have varying levels (e.g., invariant sets, limit sets) requiring the consideration of hyperspace systems. Hence, the study of hyperspace systems is not only useful for determining various dynamical properties of the given systems, but also theoretically important for exploring more complicated dynamical properties of these hyperspaces themselves.

5.2. Induced hyperspace maps and hyperspace dynamical systems

Let (E,f) be a dynamical system with a metric d on E. The hyperspace map $2^f : 2^E \to 2^E$ induced by f is defined as follows: For every $F \in 2^E$,

$$2^f(F) = f(F) \quad [\text{implying } (2^f)^n(F) = f^n(F), \text{ where } f^n(F) = f(f^{n-1}(F))]. \tag{6}$$

When E is a compact metric space, the topology on 2^E can be chosen as τ_h, τ_v or the subspace topology of τ_f (recall that these three topologies are compact and consistent when E is compact). The continuity of f implies that f is a perfect mapping (i.e., f is a closed mapping and all fibers $f^{-1}(x)$ are compact subsets of E [11]). Hence, $f(F)$ is always a closed (in fact compact) subset of E and therefore 2^f is well-defined. Clearly, 2^E employed with such a topology is a compact metric space as E is a compact metric space (of course a HLCSC space). From Mill's results, 2^f is a continuous hyperspace map [35] (also see [19,1]). Hence, the induced hyperspace dynamical system $(2^E, 2^f)$ of (E,f) is also a compact dynamical system.

A dynamical system (E,f) and its induced hyperspace dynamical system $(2^E, 2^f)$ are closely related by $2^f(F) = f(F)$, which implies $(2^f)^n(F) = f^n(F)$, i.e., the orbit of point F under 2^f in the hyperspace system is consistent with the orbit of set F under f in the original system.

When (E,f) is a compact dynamical system, it is conjugate to a compact subsystem of $(2^E, 2^f)$ through the embedding $h : E \to 2^E$ defined by $h(x) = \{x\}$ for $x \in E$ [38], and further this embedding h is in fact a topological conjugacy between (E,f) and $(h(E), 2^f|_{h(E)})$ indicated by $(h \circ f)(x) = h(f(x)) = \{f(x)\} = f(\{x\}) = 2^f(\{x\}) = (2^f \circ h)(x)$. In general, when two dynamical systems (X,f) and (Y,g) are topologically conjugate, i.e., there exists a homeomorphism $k : X \to Y$ satisfying the conjugacy relation $k \circ f = g \circ k$, not only the underlying structures on X and Y are topologically equivalent, but also their dynamical properties such as convergence and chaotic status in these systems under f and g respectively, are consistent. This consistency is characterized by the above conjugacy relation.

However, if E is a (non-compact) locally compact manifold (of course, f is always assumed to be a closed mapping to ensure the definition of 2^f), $(2^E, 2^f)$ is non-compact and unbounded closed subsets of E are not included if the Hausdorff metric is employed; $(2^E, 2^f)$ is non-compact and non-metrizable if the Vietoris topology is employed.

5.3. Locally compact underlying spaces

To resolve above problem, we will consider, for the first time in the literature of dynamics, the hit-or-miss topology of $\mathscr{F}(E)$ whose metric ρ is given in Section 3. We first introduce all involved spaces, embeddings and conjugacy relations as follows.

Recall that $f : E \to E$ is the given mapping and $2^f : \mathscr{F}(E) \to \mathscr{F}(E)$, defined by $2^f(F) = f(F)$, is the induced hyperspace map of f ($2^f(\emptyset)$ not defined yet). It can be proved that $h : E \to \mathscr{F}(E)$ defined by $h(x) = \{x\}$ is an embedding when E is HLCSC (see Remark 1).

For convenience, write $\omega E = \{x_\omega : x \in E\} \cup \{\omega\}$ (so each point x of E and the corresponding point x_ω of ωE are explicitly distinguished). Namely, $i : E \to \omega E$ defined by $i(x) = x_\omega$ is an embedding. Let $C : \mathscr{F}(E) \to 2^{\omega E}$ be the embedding defined by $C(F) = i(F) \cup \{\omega\}$ (except the explicit representation for points of ωE, this is the embedding defined in Section 3).

Let $\bar{f} : \omega E \to \omega E$ be the mapping defined by $\bar{f}(x_\omega) = f(x)_\omega$ ($f(x) \in E$ implies $f(x)_\omega \in \omega E$; $\bar{f}(\omega)$ will be defined later). Let $2^{\bar{f}} : 2^{\omega E} \to 2^{\omega E}$, defined by $2^{\bar{f}}(F') = i(f(F)) \cup \{\omega\}$ for $F' \in 2^{\omega E}$ where F' is written as $i(F) \cup \{\omega\}$ for $F \in \mathscr{F}(E)$, be the induced hyperspace map of $\bar{f}$. Define $\bar{h} : \omega E \to 2^{\omega E}$ by $\bar{h}(x_\omega) = \{x_\omega\}$ and $\bar{h}(\omega) = \{\omega\}$. Then $\bar{h}$ is an embedding [35] (another proof is similar to that for h). Another embedding is $\bar{\bar{h}} : \omega E \to \mathscr{F}(E)$ defined by $\bar{\bar{h}}(x_\omega) = \{x\}$ for $x_\omega \in \omega E \setminus \{\omega\}$ and $\bar{\bar{h}}(\omega) = \emptyset$, which leads to another way to embed ωE into $2^{\omega E}$ (i.e., $C \circ \bar{\bar{h}}$).

The following conjugacy relations can be proved (straightforward): $i \circ f = \bar{f} \circ i$, i.e., (E, f) is conjugate to the subsystem $(\{x_\omega : x \in E\}, \bar{f}|_{\{x_\omega : x \in E\}})$ of $(\omega E, \bar{f})$; $C \circ 2^f = 2^{\bar{f}} \circ C$, i.e., $(\mathscr{F}(E), 2^f)$ is conjugate to a subsystem of $(2^{\omega E}, 2^{\bar{f}})$, i.e., $(\mathscr{F}(E), 2^f)$ is conjugate to the compact subsystem $(C(\mathscr{F}(E)), 2^{\bar{f}}|_{C(\mathscr{F}(E))})$ of $(2^{\omega E}, 2^{\bar{f}})$ – recall that $C(\mathscr{F}(E))$ ($\subseteq 2^{\omega E}$) consists of all closed subsets of ωE that contain ω; $h \circ f = 2^f \circ h$, i.e., (E, f) is conjugate to the subsystem $(\{\{x\} : x \in E\}, 2^f|_{\{\{x\} : x \in E\}})$ of $(\mathscr{F}(E), 2^f)$; $\bar{h} \circ \bar{f} = 2^{\bar{f}} \circ \bar{h}$, i.e., $(\omega E, \bar{h})$ is conjugate to the subsystem $(\{\{y\} : y \in \omega E\}, 2^{\bar{f}}|_{\{\{y\} : y \in \omega E\}})$ of $(2^{\omega E}, 2^{\bar{f}})$. The following conjugacy diagram [though playing a similar role, this is not a traditional commutative diagram because different subsystems of $2^{\omega E}$ are reached: $C(\mathscr{F}(E)) \cap \bar{h}(\omega E) = \{\{\omega\}\}$ and $(C \circ \bar{\bar{h}})(\omega E) \subseteq C(\mathscr{F}(E))$] describes these conjugacy relations in which the metrics of these systems are also explicitly shown:

$$\begin{array}{llcll}
(E, d, f) & & \xrightarrow{i} & (\omega E, \bar{d}, \bar{f}) & \\
h \downarrow & \bar{\bar{h}} & \swarrow & \downarrow & \bar{h} \\
(\mathscr{F}(E), \rho, 2^f) & & \underset{C}{\rightarrow} & (2^{\omega E}, \bar{d}_{\mathrm{H}}, 2^{\bar{f}}). &
\end{array}$$

It remains to study the continuity of the mappings f and $\bar{f}$, and the continuity of the hyperspace maps 2^f and the restriction of $2^{\bar{f}}$ on the subsystem $(C(\mathscr{F}(E)), 2^{\bar{f}}|_{C(\mathscr{F}(E))})$. This can be achieved through the study of $(2^{\omega E}, 2^{\bar{f}})$. In fact, to study any of these systems, particularly $(\mathscr{F}(E), 2^f)$, it suffices to investigate $(2^{\omega E}, 2^{\bar{f}})$ from this conjugacy diagram as each system is eventually embedded into $(2^{\omega E}, 2^{\bar{f}})$. On the other hand, $(2^{\omega E}, 2^{\bar{f}})$ as the induced hyperspace system of $(\omega E, \bar{f})$ is completely determined by $(\omega E, \bar{f})$. Clearly, the convergence in $(\omega E, \bar{f})$ is sequential since ωE is second countable.

Case 1. If $\lim_{x_\omega \to \omega} \bar{f}(x_\omega) = a_\omega \in \omega E \setminus \{\omega\}$, then $\bar{f}$ can be continuously extended to ωE by letting $\bar{f}(\omega) = a_\omega$. As ωE is compact and $\bar{f}$ is continuous, $2^{\bar{f}}$ is continuous

[35,19,1]. Consequently, 2^f is continuous on $\mathscr{F}(E)$ when defining $2^f(\emptyset) = \{a\}$; $(\mathscr{F}(E), 2^f)$ and the subsystem $(C(\mathscr{F}(E)), 2^{\bar{f}}|_{C(\mathscr{F}(E))})$ of $(2^{\omega E}, 2^{\bar{f}})$ are conjugate.

Case 2. If $\lim_{x_\omega \to \omega} \bar{f}(x_\omega) = \omega$, define $\bar{f}(\omega) = \omega$. Then $\bar{f}$ is continuous on ωE. Again, $2^{\bar{f}}$ is continuous. Consequently, 2^f is continuous on $\mathscr{F}(E)$ when defining $2^f(\emptyset) = \emptyset$. Remaining discussions are similar to that for Case 1.

Case 3. However, if $\bar{f}$ does not hold any of the above asymptotic behaviors, i.e., $\bar{f}(x_\omega)$ does not converge to a point of ωE as $x_\omega \to \omega$, it can not be continuously extended to ωE. For convenience, we define $\bar{f}(\omega) = \omega$, implying a definition of $2^f(\emptyset) = \emptyset$. Then $\bar{f}$ is continuous at each $x_\omega \in \omega E \setminus \{\omega\}$ and discontinuous at ω. As such, $2^{\bar{f}}|_{C(\mathscr{F}(E))}$ is continuous everywhere except at $\{\omega\}$ (so 2^f is not continuous at $\emptyset$). Therefore, the subsystem $(C(\mathscr{F}(E)), 2^{\bar{f}}|_{C(\mathscr{F}(E))})$, which is conjugate to $(\mathscr{F}(E), 2^f)$, is a system with a discontinuity at $\{\omega\}$, thus essentially a locally compact system.

5.4. Ergodic dynamical systems

The ergodic theory goes back to Boltzmann's ergodic hypothesis: equality of time averages and space averages for systems in statistical mechanics. Poincaré observed that the preservation of a finite invariant measure forces strong conclusions about recurrence, which are encapsulated in his Recurrence Theorem [14]. The systematic development of ergodic theory as a mathematical subject started around 1930 by von Neumann (functional-analytic viewpoint). Other early major contributors were Birkhoff, Hopf, Koopman, Halmos, and Kakutani. Kolmogorov's entropy theory around 1958 provided a critical probabilistic and later geometric and combinational approaches. It built upon Shannon's seminal development of information theory, which was given an appropriate mathematical treatment by Khinchin. Following Kolmogorov, Sinai and Rokhlin developed the entropy theory from the probabilistic viewpoint (weak isomorphism theorem). Later, Ornstein proved the isomorphism of Bernoulli shifts of equal entropy via combinatorial constructions.

A measure-preserving dynamical system is defined as a probability space $(X, \mathscr{B}, T, \mu)$, where X is a topological space, $\mathscr{B}$ is the Borel σ-field over X, $T : X \to X$ is a measurable transformation preserving the measure μ (i.e., μ is T-invariant), i.e., every measurable set $A \subseteq X$ satisfies $\mu(T^{-1}A) = \mu(A)$. The system is ergodic (i.e., μ is T-ergodic) if the only measurable sets invariant under T have a measure of 0 or 1.

In the ergodic theory of dynamical system, instead of a single trajectory (i.e., a sequence $f^n(x)$), all trajectories are considered simultaneously, weighted with a probability measure. Here is the well-known Birkhoff Ergodic Theorem: Let $(X, \mathscr{B}, \mu)$ be a measure space. If $f : X \to X$ is a measure-preserving transformation for the measure μ and if $g : X \to R$ is a μ-integrable function, then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g \circ f^j(x)$ converges μ-almost everywhere to an integrable function $g^* : X \to R$. Also, g^* is f invariant wherever it is defined, i.e., $g^* \circ f(x) = g^*(x)$ for μ-almost all x. Also, (i) if $\mu(X) < \infty$, then $\int_X g^*(x)\,\mathrm{d}\mu(x) = \int_X g(x)\,\mathrm{d}\mu(x)$ and (ii) if μ is an ergodic measure for f, then g^* is a constant μ-almost everywhere [4,26,25].

5.5. Hyperspace ergodic dynamical systems

We can now investigate a Birkhoff Ergodic Theorem for hyperspace systems. Notice that the Brikhoff Ergodic Theorem holds for a pair of functions only if these functions satisfy the required conditions. Namely, the existence of an invariant measure for the first function is an assumption and the integrability of the second function is also an assumption in the theorem. These assumptions are not guaranteed by the theorem itself. Of course, this theorem is valid for the maps defined in Section 5.3, including $2^{\bar{f}}$, 2^{f} and $2^{\bar{f}}|_{C(\mathscr{F}(E))}$. More importantly, we have stronger results for Cases 1 and 2 because of the compactness of $2^{\omega E}$ and the compactness of $C(\mathscr{F}(E))$ which is implied by the compactness of $\mathscr{F}(E)$. In fact, for a compact metric space X and *every* continuous transformation $T : X \to X$, *there always exist* T-invariant and T-ergodic (probability) measures on the Borel σ-field of X (referred as the existence theorems of invariant and ergodic (probability) measures) [24].

Let E be a HLCSC space and $f : E \to E$ a continuous (and closed) mapping. Recall the notations introduced in Section 5.3. Assume that f has a (finite or infinite) limit at infinity, i.e., $\lim_{x_\omega \to \omega} \bar{f}(x_\omega) = a_\omega$ or ω, where a_ω is a point of $\omega E \setminus \{\omega\}$ (namely, Case 1 or Case 2 of Section 5.3). Then the induced hyperspace map $2^{\bar{f}} : 2^{\omega E} \to 2^{\omega E}$ is continuous, and $2^{\bar{f}}$ will be chosen as the measure-preserving transformation. From the stated existence theorems of invariant and ergodic (probability) measures, there always exists a $2^{\bar{f}}$-invariant (probability) measure λ defined on the Borel σ-field of the hyperspace $2^{\omega E}$. Let $G : 2^{\omega E} \to [0, 1]$ be the Choquet capacity associated with λ (recall that G characterizes the probability law λ of its associated random closed set $\mathscr{S}$, which takes closed subsets of ωE as its values, and is uniquely determined through the Choquet Theorem in terms of λ [6,17,20,21,40]). Now, G is measurable and integrable with respect to λ as (1) G is u.s.c. on $\mathscr{K}(\omega E) = 2^{\omega E} \cup \{\emptyset\}$ (also l.s.c. on $\mathscr{O}(\omega E)$) [17], and (2) $0 \leqslant G \leqslant 1$. As such, the Birkhoff Ergodic Theorem holds for $2^{\bar{f}}$ (*which is interested in hyperspace dynamics*), λ and G (*which are interested in probability theory, particularly in random set theory*), i.e., a connection between dynamics and probability is established as follows:

$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} G \circ (2^{\bar{f}})^j(F)$ converges λ-almost everywhere to an integrable function $G^* : 2^{\omega E} \to R$. Also, G^* is $2^{\bar{f}}$ invariant wherever it is defined, i.e., $G^* \circ 2^{\bar{f}}(F) = G^*(F)$ for λ-almost all F. Also, (i) as $\lambda(2^{\omega E}) = 1$, $\int_{2^{\omega E}} G^*(F)\, d\lambda(F) = \int_{2^{\omega E}} G(F)\, d\lambda(F)$ and (ii) if λ is an ergodic measure for $2^{\bar{f}}$, G^* is a constant λ-almost everywhere.

To interpret this hyperspace ergodic result, we follow [26], where a similar interpretation is given for the original Birkhoff Ergodic Theorem. If λ is ergodic for $2^{\bar{f}}$, the value $\int_{2^{\omega E}} G(F)\, d\lambda(F)$ is the space average of the capacity G; the value $G^*(P)$ is the time average of G along the orbit $\{P, 2^{\bar{f}}(P), (2^{\bar{f}})^2(P), \ldots\}$ of P. Thus the time average of G along all orbits of $2^{\bar{f}}$ equals to the space average of G. This fact indicates that almost all orbits of $2^{\bar{f}}$ for the ergodic measure λ are dense in the support of λ. The consequence is that the time average of the Choquet capacity G along all orbits of $2^{\bar{f}}$ equals to the space average of G, which indicates that almost all orbits of $2^{\bar{f}}$ for the ergodic measure λ are dense in the support of λ.

Notice that G can be determined by Choquet theorem as long as $2^{\bar{f}}$ and an ergodic measure λ for $2^{\bar{f}}$ are known. For the general form of the hyperspace ergodic result, we do not have to choose G as that λ-integrable capacity function. In other words, G can be replaced by any λ-integrable function. Also, λ is not necessarily chosen as a probability measure. However, it would lose the connection between a capacity and the hyperspace map if G and λ are not selected in the specified way.

When above system $(2^{\omega E}, 2^{\bar{f}})$ is replaced by its compact subsystem $(C(\mathscr{F}(E)), 2^{\bar{f}}|_{C(\mathscr{F}(E))})$, which is conjugate to $(\mathscr{F}(E), 2^{f})$, the expected result for $(\mathscr{F}(E), 2^{f})$ is established.

If E itself is compact, simply replace $2^{\omega E}$ by $2^E = \mathscr{F} \setminus \{\emptyset\}$, $\bar{f}$ by f, $2^{\bar{f}}$ by 2^f (no need to define $2^f(\emptyset)$ as $\emptyset$ is an isolated point of $\mathscr{F}$ when E is compact). Then the above hyperspace Birkhoff ergodic result remains valid for the system $(2^E, 2^f)$.

The beauty of the hit-or-miss topology is that the hyperspace $\mathscr{F}(E)$ with this topology is always HCSC (thus a compact metric space) as long as E is HLCSC. This ensures the existence of invariant and ergodic (probability) measures for every continuous map $T : \mathscr{F} \to \mathscr{F}$ on the Borel σ-field of $\mathscr{F}$. In contrast, for a non-compact underlying space, when the Hausdorff metric is employed, the hyperspace is non-compact and does not include unbounded closed subsets; when the Vietoris topology is employed, the hyperspace is non-compact and non-metrizable; and thus the existence of invariant and ergodic (probability) measures for some continuous functions is questionable.

6. Historical notes on hyperspace and other remarks

The hit-or-miss topology belongs to the hyperspace theory. The study of hyperspaces began in the early 1900s with the work of Hausdorff, Vietoris, Hahn and Kuratowski [11]. Subsequent contributions were made by Borsuk, Mazurkiewicz, Wojdyslawski during 1920s and 1930s [5,18,41], Kelley during 1940s [15], and Michael and Segal during 1950s [19,31]. Later contributors include Fell, Transue and Duda during 1960s [13,34,10] and Matheron during 1970s [17]. Afterwards, several significant results were proved, including the Curtis–Schori–West hyperspace theorem on infinite-dimensional topology (2^E with the Vietoris topology is homeomorphic to the Hilbert cube whenever E is a nondegenerate Peano continuum (connected and locally connected compact metrizable space [7,41])). For researches before 1995, many publications can be found in Stoyan's book [33]. For publications after (and before) 1995, we refer to Molchanov's book [20] and Nguyen's book [21].

The hit-or-miss topology is a foundation of random sets. For the study of random sets and their associated capacities and probabilities, Stoyan (in [33]) pointed out that (1) The phrase "region ... whose shape depends on chance" was found in Kolmogoroff's book of 1933 [16], (2) Foundations of modern theory of random (closed) sets were laid by Choquet (1953/1954), Matheron (1967, 1975), (3) A more general theory was given by Kendall (1974) which derives from Davison's work (1960).

Remark 1. If E is HLCSC, then $\mathscr{F} = \mathscr{F}(E)$ employed with the hit-or-miss topology is metrizable and a metric is given via Theorems 1 and 2. We are now concerning a necessary condition of the metrizability of $\mathscr{F}$. Let us consider the mapping $h : E \to \mathscr{F}$ defined by $h(x) = \{x\}$ for $x \in E$. If $\mathscr{F}$ is metrizable, it is Hausdorff. Recall that $\mathscr{F}$ is always compact, independent of the topological properties of the underlying space E (Section 2.3). Hence, $\mathscr{F}$ is also second countable by the Urysohn's metrization theorem [11]. Now, $h(E)$ as a subspace of $\mathscr{F}$ holds the following properties: (1) $h(E)$ is Hausdorff and second countable since the Hausdorff property and second countability are hereditary with respect to all subspaces [11], (2) $h(E)$ is locally compact because $h(E) \cup \{\emptyset\}$ is a closed subspace (thus a compact subspace) of $\mathscr{F}$, $h(E)$ is an open subspace of the compact Hausdorff space $h(E) \cup \{\emptyset\}$, and local compactness is hereditary with respect to open subspaces. Therefore, if $\mathscr{F}$ is metrizable, then $h(E)$ is HLCSC. Clearly, if h is a homeomorphic embedding, then E is necessarily HLCSC. Conversely, when E is HLCSC, it can be proved that h is a homeomorphic embedding by using Lemma 1. Moreover, we can investigate the three assumptions on E individually as follows.

(1) Hausdorff property of E: Let E be any non-Hausdorff space with finitely many points. Then $\mathscr{F}$ is a finite Hausdorff space (recall that local compactness of E implies Hausdorff property of $\mathscr{F}$ [12]), thus metrizable. However, the Hausdorff property of E is usually assumed.
(2) Local compactness of E: In Fell's proof, the local compactness of E is crucial to ensure the Hausdorff property of $\mathscr{F}$.
(3) Second countability of E: If E is not second countable, i.e., any base of E consists of uncountably many open subsets, say $G_\alpha,\ \alpha \in \Lambda$. Hence, the subbase of $\mathscr{F}$ would include $\mathscr{F}_{G_\alpha},\ G_\alpha \in \alpha$. For two different G_{α_1} and G_{α_2}, $\mathscr{F}_{G_{\alpha_1}}$ and $\mathscr{F}_{G_{\alpha_2}}$ are different. Recall the fact that $\mathscr{F}$ has a countable base when it is metrizable.

Remark 2. We will give an example to compare the metric ρ of this paper and the Hausdorff–Buseman metric ρ_{HB}. Let E be a HLCSC space and let d be a consistent metric of E. For non-empty closed subsets A and B of E, the Hausdorff–Buseman metric ρ_{HB} is defined by (see (B.1) of [20])

$$\rho_{\mathrm{HB}}(A,B) = \sup_{x\in E} \mathrm{e}^{-d(0,x)}|d(x,A) - d(x,B)|.$$

The metric between each non-empty closed subset and the empty set $\emptyset$ of E ($\emptyset$ belongs to $\mathscr{F}$) has not been defined by ρ_{HB}. The example to be constructed below shows that this metric ρ_{HB} is inconsistent with the hit-or-miss topology on $\mathscr{F}$ even if E is the n-dimensional Euclidean space R^n.

In fact, when $E = R^n$, the Hausdorff–Buseman metric can be written as

$$\rho_{\mathrm{HB}}(A,B) = \sup_{x\in E} \mathrm{e}^{-d_n(0,x)}|d_n(x,A) - d_n(x,B)|,$$

where d_n is the Euclidean metric of E, i.e., $d_n(x,y) = \|x-y\|$. The factor $\mathrm{e}^{-d_n(0,x)}$ used to decline $|d_n(x,A) - d_n(x,B)|$ is only valid for large x of E (i.e., $d_n(0,x)$ is large). However, the supremum may be achieved at an x near 0. Noting that every metric space has a bounded metric which is equivalent to the original metric [11], the role of $\mathrm{e}^{-d_n(0,x)}$ is lost when such a metric is employed.

Example 1. For simplicity, let $E = R$. Consider two closed subsets $\{a\}$ and $\{-a\}$ of E, i.e., two singleton sets of E. Here, $a > 0$. Let S_1 be the one-dimensional sphere (i.e., a circle) centered at $(0,1)$ with radius 1 ($\omega = (0,2)$ is the north pole). The equation for S_1 is $x^2 + (y-1)^2 = 1$. S_1 is an Alexandroff compactification ωE of E. The metric on S_1 is simply the Euclidean metric d_2 of R^2 restricted on S_1 (or using the arc length between two points of S_1). The Hausdorff metric between two non-empty closed subsets of S_1 is denoted by $\bar{d}_{\mathrm{H}}$, which is calculated based on the metric d_2. With the notations used previously in this paper, $C(\{a\}) = \left\{\left(\frac{4a}{4+a^2}, \frac{2a^2}{4+a^2}\right), \omega\right\} \subseteq S_1$ and $C(\{-a\}) = \left\{\left(-\frac{4a}{4+a^2}, \frac{2a^2}{4+a^2}\right), \omega\right\} \subseteq S_1$, where $\left(\frac{4a}{4+a^2}, \frac{2a^2}{4+a^2}\right)$ is the intersection of S_1 and the line segment connecting $(a,0)$ and ω (equation: $y = 2 - \frac{2}{a}x$), and $\left(-\frac{4a}{4+a^2}, \frac{2a^2}{4+a^2}\right)$ is the intersection of S_1 and the line segment connecting $(-a,0)$ and ω (equation: $y = 2 + \frac{2}{a}x$). If $a \geqslant \frac{2}{\sqrt{3}}$,

$$\rho(\{a\},\{-a\}) = \bar{d}_{\mathrm{H}}(C(\{a\}),C(\{-a\})) = d_2\left(\left(\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right),\omega\right) = \frac{4}{\sqrt{4+a^2}}$$

(or the arc length between $\left(\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right)$ and ω if the arc length metric is taken for S_1); if $0 < a < \frac{2}{\sqrt{3}}$,

$$\rho(\{a\},\{-a\}) = \bar{d}_{\mathrm{H}}(C(\{a\}),C(\{-a\})) = d_2\left(\left(\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right),\left(-\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right)\right) = \frac{8a}{4+a^2}$$

(or the arc length between $\left(\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right)$ and $\left(-\frac{4a}{4+a^2},\frac{2a^2}{4+a^2}\right)$ if the arc length metric is taken for S_1).

The sequence $F_n = \{(-1)^n n\}$ is convergent to $\emptyset$, as both $\{2n\}$ and $\{-(2n+1)\}$ are convergent to $\emptyset$ under the hit-or-miss topology, which is implied by Matheron's Theorem 1-2-2 of [17] (or observe that for every compact subset K of E, this sequence $\{(-1)^n n\}$ is eventually (for sufficiently large n) contained in $\mathscr{F}^K$ and $\{\mathscr{F}^K : K \subseteq E$ and K is compact$\}$ is a neighborhood system of $\emptyset$). Under the metric ρ defined in this paper, $\{(-1)^n n\}$ is convergent to $\emptyset$ because $\rho(\{(-1)^n n\},\emptyset) = \bar{d}_{\mathrm{H}}(C(\{(-1)^n n\}),C(\emptyset)) = \bar{d}_{\mathrm{H}}(\{((-1)^n\frac{4n}{4+n^2},\frac{2n^2}{4+n^2}),\omega\}, \{\omega\}) = d_2(((-1)^n\frac{4n}{4+n^2},\frac{2n^2}{4+n^2}),\omega) = \frac{4}{\sqrt{4+n^2}} \to 0\ (n\to\infty)$.

On the other hand, the Hausdorff–Buseman metric is

$$\rho_{\mathrm{HB}}(\{a\},\{-a\}) = \sup_{x\in E} \mathrm{e}^{-d_1(0,x)}|d_1(x,\{a\}) - d_1(x,\{-a\})|$$

or written in the following form:

$$\rho_{\mathrm{HB}}(\{a\},\{-a\}) = \sup_{x\in E} \mathrm{e}^{-|x|}||x-a| - |x+a||.$$

As the function $\mathrm{e}^{-|x|}||x-a|-|x+a||$ is even and it has value 0 at $x=0$, the Hausdorff–Buseman metric can be simplified to the following:

$$\rho_{\mathrm{HB}}(\{a\},\{-a\}) = \sup_{x>0} \mathrm{e}^{-x}(x+a-|x-a|).$$

Let $k(x) = \mathrm{e}^{-x}(x+a-|x-a|)$. Then $k(x)\to 0$ as $x\to+\infty$. For $x>a$, $k(x) = 2a\mathrm{e}^{-x}$ and $k'(x)<0$. For $0<x\leqslant a$ (assuming $a>1$), $k(x) = 2x\mathrm{e}^{-x}$, $k'(1)=0$ and $k''(1)<0$. Hence, $\rho_{\mathrm{HB}}(\{a\},\{-a\}) = \max\{k(1),k(a)\} = \max\{2\mathrm{e}^{-1},2a\mathrm{e}^{-a}\} = 2\mathrm{e}^{-1}$ if $a>1$.

Therefore, even if ρ_{HB} could be extended to the empty set, it is a consequence of $2\mathrm{e}^{-1} = \rho_{\mathrm{HB}}(\{n\},\{-n\}) \leqslant \rho_{\mathrm{HB}}(\{n\},\emptyset) + \rho_{\mathrm{HB}}(\emptyset,\{-n\})$ that $\{n\}$ or $\{-n\}$ is not convergent under ρ_{HB} and therefore the Hausdorff–Buseman metric is not consistent with the hit-or-miss topology.

[(a) ρ_{HB} is Hausdorff metric originated, calculated based on a metric of the original (HLCSC) space E (the calculation involves *all points* of E). The metric ρ of this paper is associated with a Hausdorff metric, but this Hausdorff metric is calculated based on a metric $\bar{d}$ of the larger space ωE. It should be indicated that, as ωE is compact, any consistent metric $\bar{d}$ of ωE holds particular properties of a compact metric space (refer to the immediate succeeding paragraph of Theorem 2). (b) Some authors may exclude the empty set from $\mathscr{F}$, e.g., [3]. In such a case, the convergence $F_n\to\emptyset$ becomes a divergence (or a convergence toward the infinity) because $\mathscr{F} \setminus \{\emptyset\}$ is non-compact when E is locally compact but not compact [17].]

Acknowledgments

After this paper was submitted, one referee pointed out to us that, in the special case of the Euclidean space R^n, a metric (the stereographical metric) compatible with the hit-or-miss topology was given by Professors Rockafellar and Wets in the Lecture Notes in Mathematics [27] (related to Section 3). Later, we also found a brief description of this important work on R^n in their book Variational Analysis [28]. We thank the referee for pointing out to us this previous significant work on the case R^n. We also thank Professor Hung T. Nguyen for calling our attention to the Hausdorff–Buseman metric.

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