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A fast high-order finite difference algorithm for pricing American options

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Abstract

We describe an improvement of Han and Wu's algorithm [H. Han, X.Wu, A fast numerical method for the Black–Scholes equation of American options, SIAM J. Numer. Anal. 41 (6) (2003) 2081–2095] for American options. A high-order optimal compact scheme is used to discretise the transformed Black–Scholes PDE under a singularity separating framework. A more accurate free boundary location based on the smooth pasting condition and the use of a non-uniform grid with a modified tridiagonal solver lead to an efficient implementation of the free boundary value problem. Extensive numerical experiments show that the new finite difference algorithm converges rapidly and numerical solutions with good accuracy are obtained. Comparisons with some recently proposed methods for the American options problem are carried out to show the advantage of our numerical method.

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Keywords: American options; Free boundary; Finite difference method; High-order compact scheme; Singularity separating

1. Introduction

Closed form solutions for pricing American options are difficult to obtain and the design of an efficient and accurate numerical pricing algorithm remains a topic of considerable interest among researchers. The American option pricing problem can be posed either as a linear complementarity problem (LCP) or as a free boundary value problem. These two different formulations have led to a number of different methodologies for solving American options and we study some of them in this paper. The first algorithm to value an American option was introduced by Brennan and Schwartz [4] and the convergence of their finite difference method was proved by Jaillet, Lamberton and Lapeyre [12]. Another popular method is the projected successive overrelaxation method (PSOR) [20] but the iterative procedure converges slowly. Algorithms that solve the discrete LCP in linear number of spatial grid points have been suggested in [3,2,6]. Other methods include the front-fixing transformations [21,16], penalty methods [7,16], a method based on operator splitting [11] and the algorithm developed by Han and Wu [9].

In this paper, we describe a new finite difference algorithm for the American options problem. The Black–Scholes PDE is transformed to a standard heat equation and we use an optimal higher-order compact scheme due to Smith [18].

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As shown by Tavella and Randall [19], this transformation leads to the flattening of the eigenvalue distribution of the discretisation matrix. This means that any numerical scheme applied to the heat equation has better stability range than when it is used to discretise the Black–Scholes equation. However, the space variable for the heat equation is unbounded and this infinite domain has to be truncated in order to apply finite difference schemes. The common practice is to choose the computational domain large enough that the error introduced by applying far field boundary conditions at extremity is negligible. This results in quantities of useless computations since many of the grid points are not of interest. For European options, Kangro and Nicolaides [13] derived bounds for the near field error in terms of the maximum error incurred on the boundary. This error estimate is then used to locate a priori the artificial boundary in terms of a given error tolerance. The same procedure was used by Han and Wu [9] for pricing an American call option. We choose here a different approach and use a non-uniform grid that is coarse for the part that is not of interest and refined for places near spot prices and where the early exercise boundary is located in order to decrease the computational cost of large domains. This procedure is easier to implement than the complicated discretisation of the artificial boundary conditions at each time step.

In addition to the above problem, a numerical procedure for solving the American options problem has to take into account the following two additional constraints. First, the Black–Scholes equation leads to a free boundary value problem and since the location of the boundary is unknown, an accurate numerical method for determining the location of the free boundary needs to be combined with the solution process. For this, Han and Wu [9] described a method for locating the free boundary based on some properties of the Black–Scholes PDE. However their method does not take into account the smooth pasting condition that ensures the continuity of the hedge parameter delta. Second, the non-smoothness of the payoff function at the strike price affects the accuracy of the numerical solutions. For the binomial pricing method, Heston and Zhou [10] showed that this non-smoothness prevents the binomial method from achieving its theoretical rate of convergence. Another problem that is a consequence of the kink made by the payoff function is the inaccurate computations of the Greeks. As a remedy, Zhu, Ren and Xu [22] described an approach in a singularity separating framework which consists of computing the difference between the American and European options prices. Their method allows computations of numerical solutions having a higher accuracy.

Our proposed finite difference algorithm for pricing American options will combine these two approaches. To find the free boundary location, we extend the properties showed by Han and Wu [9] to a singularity separated American put problem and the location procedure is augmented with the essential smooth pasting condition. The use of a non-uniform grid allows us to introduce an effective method for readjusting the tridiagonal linear system once the free boundary is located. Computational results show that our algorithm is faster and more accurate than the original method proposed by Han and Wu. Comparisons with some other existing methods are also carried out. To assess the accuracy of all methods, we use the monotonically convergent binomial method proposed by Liesen and Reimer [15] for the option prices and Greeks and use the optimal exercise boundary computed by Chen, Chadam and Stamicar [5]; see also [8].

An outline of this paper follows. In Section 2, we recall some properties of the LCP and the free boundary value problem associated with the pricing of an American option and we briefly review some existing algorithms. In Section 3, we describe the new method and in Section 4, we give the results of some numerical tests and comparisons are made between the different methods.

2. Numerical evaluation of American options

We consider a financial market consisting of a risky asset with price process $\{S_t\}_{t\geq 0}$ and constant volatility $\sigma > 0$ in a risk neutral economy with fixed rate of return $r > 0$. Let the market measure be denoted by $\mathbb{P}$, let $\{W_t\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration. Then, under the equivalent martingale measure $\mathbb{Q}$, the dynamics of the Black–Scholes model is given by

$$\frac{\mathrm{d}S_t}{S_t} = (r - \delta)\,\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where δ denotes the continuous dividend yield. Numerically, the American put problem is posed either as a linear complementarity problem (LCP) [23] of the form

$$\begin{aligned} &V_\tau \geq \mathcal{L}V, \quad V(S,0) = \max(E - S, 0), \\ &V(S,\tau) \geq V(S,0), \quad (V_\tau = \mathcal{L}V) \wedge (V(S,\tau) = V(S,0)), \end{aligned} \tag{1}$$

where T is the expiry time, $\tau = T - t$ and

$$\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta) S \frac{\partial}{\partial S} - r, \tag{2}$$

represents the spatial operator, or it can be posed as a free boundary value problem as

$$\begin{aligned} &V_\tau = \mathcal{L}V, \quad \min\left(E, \frac{rE}{\delta}\right) = S_f(\tau) \le S < \infty, \quad 0 \le \tau \le T, \\ &V(S, 0) = \max(E - S, 0), \\ &V(S_f(\tau), \tau) = E - S_f(\tau), \quad V(S, \tau) = 0 \qquad \text{as } S \to \infty. \end{aligned} \tag{3}$$

We note that because the free boundary $S_f(\tau)$ is unknown in (3), we need to impose one additional Neumann boundary condition

$$\frac{\partial V}{\partial S}(S_f(\tau), \tau) = -1, \tag{4}$$

which is the requirement that the hedge parameter delta should be continuous at the critical asset price. On the other hand, the free boundary is dealt with abstractly within the LCP formulation (1) so that the optimal exercise price can be computed once the option value has been found. As we will now see, the two different formulations lead to different numerical algorithms for the pricing of American options.

In both formulations, we first need to discretise the problem on a bounded computational domain. Let

$$\begin{aligned} \Omega_{\Delta S} &= \left\{ S_i \in \mathbb{R}_+ : S_i = \widehat{S}_{\min} + i\Delta S, i = 0, 1, \ldots, m, \Delta S = \frac{\widehat{S}_{\max} - \widehat{S}_{\min}}{m} \right\}, \\ \Omega_{\Delta\tau} &= \left\{ \tau_j \in \mathbb{R}_+ : \tau_j = j\Delta\tau, j = 0, 1, \ldots, n, \Delta\tau = \frac{T}{n} \right\}, \end{aligned}$$

be the computational domain where the boundaries $\widehat{S}_{\min}$ and $\widehat{S}_{\max}$ are chosen so as not to introduce huge errors in the computed approximation V_i^j to the solution $V(S_i, \tau_j)$ of an American option. Also let $V^j = [V_1^j, \ldots, V_{m-1}^j]^{\mathrm{T}}$ denote the vector of unknowns at the interior grid points of $\Omega_{\Delta S}$. Then the discretisation of the spatial operator (2) using central difference approximations gives the system tridiagonal matrix A with coefficients α_i, β_i and γ_i given by

$$[\alpha_i, \beta_i, \gamma_i] = \left[\frac{\sigma^2 S_i^2}{2\Delta S^2} - \frac{(r-\delta)S_i}{2\Delta S}, \frac{-\sigma^2 S_i^2}{\Delta S^2} - r, \frac{\sigma^2 S_i^2}{2\Delta S^2} + \frac{(r-\delta)S_i}{2\Delta S} \right].$$

To derive fully discrete systems, we then need to determine the time evolution of the scheme. A popular time discretisation is the weighted average θ-scheme which is given by

$$[I - \theta\Delta\tau A]V^{j+1} = [I + (1-\theta)\Delta\tau A]V^j + g, \tag{5}$$

where

$$g = \left[\alpha_1 \Delta\tau \left((1-\theta)V_0^j + \theta V_0^{j+1}\right), 0, \ldots, 0, \gamma_{m-1}\Delta\tau\left((1-\theta)V_m^j + \theta V_m^{j+1}\right)\right]^{\mathrm{T}},$$

incorporates the boundary conditions into the linear system (5). For $\theta = 0, 1$ and $1/2$, we respectively obtain the explicit, implicit and the Crank–Nicolson schemes.

2.1. Brennan–Schwartz algorithm

This algorithm is based on an implicit discretisation but also works for a Crank–Nicolson scheme and runs in time that varies linearly with the number of spatial discretisations. The idea behind the Brennan–Schwartz algorithm is based upon transforming the tridiagonal system (5) to a lower bidiagonal system and then solving this system while enforcing the American constraint.

2.2. Borici and Lüthi LCP method

It is because of the slow convergence of PSOR iterations that several attempts have been made to solve the LCP problem with algorithms that run in linear time. The Borici and Lüthi algorithm uses the transformation $x = \log(S/E)$ to reformulate the discretised version of the LCP (1) in x-coordinates in terms of the excess vector $u^{j+1} = V^{j+1} - V^0$, the slack vector $s^{j+1} = [I - \theta\Delta\tau A]V^{j+1} - [I + (1-\theta)\Delta\tau A]V^j - g$ and the tridiagonal constant coefficient matrix A. This gives the following LCP:

$$\begin{aligned} &[I - \theta\Delta\tau A]u^{j+1} - s^{j+1} = b^{j+1}, \quad j = 0, 1, \ldots, n-1, \\ &\quad u^{j+1} \geq 0, \quad s^{j+1} \geq 0, \quad \left(s^{j+1}\right)^{\mathrm{T}} u^{j+1} = 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &b^{j+1} = b^0 + [I + (1-\theta)\Delta\tau A]u^j, \\ &b^0 = g + [I + (1-\theta)\Delta\tau A]V^0 - [1 - \theta\Delta\tau A]V^0, \quad \text{and} \quad u^0 = 0. \end{aligned}$$

The algorithm assumes the existence of a continuity and stopping region that leads to an optimal feasible basis $\left(s_1^{j+1}, \ldots, s_{i_f}^{j+1}, u_{i_f+1}^{j+1}, \ldots, u_m^{j+1}\right)^{\mathrm{T}}$. This means that the system (6) can be partitioned as follows:

$$\begin{pmatrix} [I_{11} - \theta\Delta\tau A_{11}] & -\theta\Delta\tau\gamma e_{i_f-1} & \\ -\theta\Delta\tau\alpha e_{i_f-1}^{\mathrm{T}} & 1 - \theta\Delta\tau\beta & -\theta\Delta\tau\gamma e_1^{\mathrm{T}} \\ & -\theta\Delta\tau\alpha e_1 & [I_{33} - \theta\Delta\tau A_{33}] \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_3^{j+1} \end{pmatrix} - \begin{pmatrix} s_1^{j+1} \\ s_2^{j+1} \\ 0 \end{pmatrix} = \begin{pmatrix} b_1^{j+1} \\ b_2^{j+1} \\ b_3^{j+1} \end{pmatrix},$$

where i_f is chosen such that $b_2^{j+1} < 0$ and $[I_{33} - \theta\Delta\tau A_{33}]^{-1} u_3^{j+1} \geq 0$. Then the solution becomes

$$u_3^{j+1,\text{old}} = [I_{33} - \theta\Delta\tau A_{33}]^{-1} b_3^{j+1}, \quad \text{and} \quad s_2^{j+1,\text{old}} = -\left(b_2^{j+1} + \theta\Delta\tau\gamma e_1^{\mathrm{T}} u_3^{j+1,\text{old}}\right).$$

If $s_2^{j+1,\text{old}} \geq 0$, then the LCP is solved for this time level; otherwise we do another sweep decreasing i_f by one grid node and the new solutions for u and s are found in an efficient manner by updating the current solution until the LCP is solved. For further details about the algorithm, we refer the reader to [3,2].

2.3. Penalty methods

For the American option problem, Forsyth and Vetzal [7] showed that the addition of a penalty term

$$\lambda_i^{j+1} = \frac{1}{\epsilon\,\Delta\tau} \max\left(V_i^0 - V_i^{j+1}, 0\right),$$

to the Black–Scholes inequality, gives

$$\frac{\partial V_i}{\partial\tau} = \mathcal{L}V_i + \lambda_i. \tag{7}$$

In [16], Nielsen, Skavhaug and Tveito proposed using

$$\lambda^{j+1} = \frac{\epsilon r E}{V^{j+1} - V^0 + \epsilon},$$

where ϵ is related to the tolerance error in the solution. Discretisation of Eq. (7) with an implicit treatment of the penalty term leads to a non-linear system for which a penalty iterative procedure

$$[I - \theta\Delta\tau A + P]V^{j+1} = [I + (1-\theta)\Delta\tau A]V^j + PV^0 + g,$$

with the matrix P defined as

$$P_{i,i} = \begin{cases} 0, & \text{if } V_i^{j+1} > V_i^0, \\ \dfrac{1}{\epsilon}, & \text{otherwise,} \end{cases}$$

is used in [7]. In [16], the resulting non-linear system is solved using a Newton iteration method. On the other hand, Ikonen and Toivanen [11] proposed a different technique known as operator splitting, for explicitly treating the penalty term in (7). The algorithm first solves

$$[I - \theta \Delta\tau A]\hat{V}^{j+1} = [I + (1-\theta)\Delta\tau A]V^j + g + \lambda^j,$$

for $\hat{V}^{j+1}$, and then uses the two-step formula

$$V^{j+1} = \max\left(V^0, \hat{V}^{j+1} + \Delta\tau\lambda^j\right), \quad \text{and} \quad \lambda^{j+1} = \lambda^j + \frac{1}{\Delta\tau}\left(\hat{V}^{j+1} - V^{j+1}\right),$$

to obtain the new option price V^{j+1} and new update λ^{j+1} for the penalty term.

2.4. *A front-fixing transformation*

We have already seen a coordinate transformation that turns the Black–Scholes PDE into a constant coefficient PDE. After dividing S, $S_f(\tau)$, $V(S,\tau)$ by E to obtain normalised variable and functions, Wu and Kwok [21] proposed another transformation $S = e^y S_f(\tau)$ which turns the unknown free boundary of the American option into a known fixed boundary and the American problem is posed as

$$\begin{aligned}
&\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2} + \left[r - \delta - \frac{\sigma^2}{2} + \frac{S'_f(\tau)}{S_f(\tau)}\right]\frac{\partial V}{\partial y} - rV,\\
&V(y,0) = 0,\ y \in (0,\infty),\\
&V(0,\tau) = 1 - S_f(\tau), \quad \frac{\partial V(0,\tau)}{\partial y} = -S_f(\tau),\\
&\lim_{y\to\infty} V(y,\tau) = 0,
\end{aligned}$$

and the condition

$$-\frac{\sigma^2}{2}\frac{\partial^2 V(0,\tau)}{\partial y^2} - \left(\delta + \frac{\sigma^2}{2}\right)S_f(\tau) + r = 0,$$

at $y = 0$ is used to fix the boundary conditions for a central difference discretisation with a leapfrog scheme. Also a two-step predictor–corrector is required since the scheme has three time levels. For more details about the front-fixing algorithm, we refer the reader to [21].

2.5. *Finite difference method of Han and Wu*

Using the transformation $x = \log(S/E)$ and $\tau = \sigma^2(T-t)/2$ with

$$u(x,\tau) = e^{\hat{\alpha}x + \hat{\beta}\tau} V(S,t)/E, \tag{8}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are defined as

$$\hat{\alpha} = \frac{1}{2}\left(\frac{2(r-\delta)}{\sigma^2} - 1\right), \quad \hat{\beta} = \frac{1}{4}\left(\frac{2(r-\delta)}{\sigma^2} - 1\right)^2 + \frac{2r}{\sigma^2},$$

the free boundary problem (3) is transformed to

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x_f(\tau) \le x < \infty, \quad 0 \le \tau \le \frac{\sigma^2 T}{2},\\
&u(x,0) = h(x,0), \quad x_f(\tau) \le x < \infty,\\
&u\left(x_f(\tau),\tau\right) = h\left(x_f(\tau),\tau\right), \quad h(x,\tau) = e^{\hat{\alpha}x + \hat{\beta}\tau}\max(1 - e^x, 0),\\
&u(x,\tau) \to 0 \quad \text{as} \quad x \to \infty.
\end{aligned}\tag{9}$$

For the heat solution $u_c(x,\tau)$ of an American call transformed problem with upper boundary condition $u_c\left(\hat{x}_{f_c}(\tau),\tau\right) = h_c\left(\hat{x}_{f_c}(\tau),\tau\right)$ at any $\hat{x}_{f_c}(\tau) > x_{f_c}(\tau)$, Han and Wu [9] proved that based on the strong maximum principle for parabolic equations, the following inequality holds:

$$u_c(x,\tau) < h_c(x,\tau), \quad x_{f_c}(\tau) < x < \hat{x}_{f_c}(\tau), \tag{10}$$

where $h_c(x,\tau)$ represents the transformed payoff for a call option. On the basis of the put–call symmetry

$$S_f(\tau, r=a, \delta=b) = \frac{E^2}{S_{f_c}(\tau, r=b, \delta=a)},$$
$$V(S,\tau,r=a,\delta=b) = \frac{S}{E} V_c\left(\frac{E^2}{S},\tau,r=b,\delta=a\right),$$

and using (10) with the transformation relation (8) we have

$$u(x,\tau) < h(x,\tau), \quad \hat{x}_f(\tau) < x < x_f(\tau), \tag{11}$$

and this can be used as a test condition to locate the free boundary for an American put option.

For the boundary condition at the other end, it is easy to obtain an artificial boundary condition (also known as a transparent boundary condition [1, p. 110])

$$\frac{\partial u(b,\tau)}{\partial x} = \frac{-1}{\sqrt{\pi}} \int_0^\tau \frac{\partial u(b,\lambda)}{\partial \lambda} \frac{\mathrm{d}\lambda}{\sqrt{\tau-\lambda}}, \tag{12}$$

based on the fundamental solution of the heat equation. Discretising (12), we obtain

$$u_{m+1}^{j+1} = u_{m-1}^{j+1} - \frac{4\Delta x}{\sqrt{\pi \Delta\tau}} \sum_{l=1}^{j+1} \frac{u_m^l - u_m^{l-1}}{\sqrt{j+1-l} + \sqrt{j+2-l}},$$

and following [9], it is easy to use an implicit discretisation for the heat equation at the mth spatial node to eliminate the fictitious boundary value u_{m+1}^{j+1}. This value is then incorporated into the linear system through the boundary vector g and the algorithm then proceeds quite similarly to that of Brennan and Schwartz [4] since it has the same effect of transforming the tridiagonal linear system to a lower bidiagonal one but uses (11) to locate the free boundary.

3. A new finite difference scheme

We now present a new scheme which is an improvement of the method proposed by Han and Wu [9]. It is well known that the singularity that exists at the strike price in the payoff function decreases the accuracy of the solution. Zhu et al. [23,22] used a singularity separating method which computes the difference between an American option and a European option. Since both options satisfy the linear Black–Scholes equation, the difference u_{D} will also satisfy the transformed heat equation and with the American and European payoffs being similar, the initial condition for u_{D} will be zero. The problem set-up in a singularity separating framework is then given by

$$\begin{aligned}
&\frac{\partial u_{\mathrm{D}}}{\partial \tau} = \frac{\partial^2 u_{\mathrm{D}}}{\partial x^2}, \quad x_f(\tau) \le x < \infty,\\
&u_{\mathrm{D}}(x,0) = 0, \quad x_f(0) \le x < \infty,\\
&u_{\mathrm{D}}(x_f(\tau),\tau) = h(x_f(\tau),\tau) - u_{\mathrm{E}}(x_f(\tau),\tau), \quad 0 \le \tau \le \tau_{\max},\\
&u_{\mathrm{D}}(x,\tau) \to 0 \quad \text{as} \quad x \to \infty,
\end{aligned} \tag{13}$$

where u_{E} is the transformed value of a European put option. We note that the transformed value of the American put option which is given by $u = u_{\mathrm{D}} + u_{\mathrm{E}}$ is made up of a numerical part u_{D} and an analytical part u_{E}. With the singularity at the strike price removed, Zhu, Ren and Xu [22] argued that the numerical solution to problem (13) is much smoother than the solution to problem (9) and is also more accurate since u_{D} is smaller than u.

The new finite difference algorithm uses an optimal compact scheme [18] instead of the Crank–Nicolson scheme used in [9]. In the following we describe the derivation of the optimal compact scheme for the heat equation by matching up to fourth-order spatial moments. We first consider the scheme

$$\frac{D\left[u^{j+1}\right]}{\Delta\tau}-\left(\frac{1}{2}\theta-\lambda\right)D_{xx}\left[u^{j+1}\right]=\frac{D\left[u^{j}\right]}{\Delta\tau}+\left(\frac{1}{2}\theta+\lambda\right)D_{xx}\left[u^{j}\right], \tag{14}$$

with two degrees of freedom θ and λ and where

$$D\left[u^j\right]=\frac{1}{3}\left(u_{i-1}^j+u_i^j+u_{i+1}^j\right)-\frac{b_i}{3}D_{xx}\left[u^j\right],\quad \text{and}$$
$$D_{xx}\left[u^j\right]=\frac{2}{(x_{i+1}-x_{i-1})}\left(\frac{u_{i+1}^j-u_i^j}{(x_{i+1}-x_i)}-\frac{u_i^j-u_{i-1}^j}{(x_i-x_{i-1})}\right),$$

represent the approximations at the centroid $x_G=(x_{i-1}+x_i+x_{i+1})/3$ and b_j denotes the local mean square spacing for the x_i grid and is given by

$$b_i=\frac{1}{3}\left((x_i-x_{i-1})^2+(x_i-x_{i-1})(x_{i+1}-x_i)+(x_{i+1}-x_i)^2\right).$$

The term θ and λ are to be determined as in [18] such that the exact multiplier r matches the numerical multiplier R. These can be easily derived using symbolic computations in Mathematica. For the exact solution over a single time step $\Delta\tau$ of the heat equation, a single Fourier component is

$$u=r(k,\Delta\tau)e^{\imath k(x-x_G)},\quad \text{where } r(k,\Delta\tau)=e^{-k^2\Delta\tau},$$

and k is the wavenumber. For the numerical solution, the power-law time evolution is $u_i^j=R^j e^{\imath k(x_i-x_G)}$ and replacing u_i^j in (14) gives an explicit formula for R. Matching up to the fourth-order spatial moments gives $\theta=1$ and $\lambda=\frac{b_j}{12\Delta\tau}$. With these two optimal parameters, the high-order accuracy of the scheme is shown by the truncation error

$$\left(-\frac{1}{6}u^{0,3}+\frac{1}{4}u^{2,2}-\frac{b_i}{72}u^{2,3}\right)k^2-\frac{k\Gamma_i}{6}u^{3,2}+\left(-\frac{1}{3}u^{3,1}+\frac{1}{30}u^{5,0}\right)\Gamma_i-\frac{b_i^2}{144}u^{4,1},$$

where

$$u^{i,j}=\frac{\partial^{i+j}u}{\partial x^i\partial\tau^j},$$
$$\Gamma_i=\frac{1}{27}(-x_{i-1}-x_i+2x_{i+1})(x_{i-1}-2x_i+x_{i+1})(-2x_{i-1}+x_i+x_{i+1}).$$

Indeed for a uniform grid, this truncation error simplifies to $\mathcal{O}(\Delta\tau^2+\Delta\tau\Delta x^4+\Delta x^4)$ which matches the truncation error of the fourth-order Crandall–Douglas scheme for the heat equation. For the stability condition of (14), we use Von Neumann analysis. Assuming that

$$\Delta x_{\max}=\max(x_{i+1}-x_i),\quad i=0,1,\ldots,m-1,$$
$$\lambda=\frac{\Delta\tau}{\Delta x_{\max}^2},$$

and using the power-law time evolution then the bounded amplification factor

$$|R|=\left|\frac{1-3\lambda}{1+3\lambda}\right|\le 1,\quad \forall\lambda\ge 0,$$

shows that the scheme is unconditionally stable. This allows us to use a non-uniform grid that we choose coarse on the part extended to incorporate the far field boundary conditions. This considerably decreases the cost of computations in comparison to methods that use uniform grids. Also, it avoids the complicated implementation of artificial boundary conditions in Han and Wu's method. We notice that for best accuracy, this artificial boundary location should be eventually chosen large and thus results in a large computational domain. On the other hand, we require a fine grid on the part of the computational domain where the unknown free boundary and spot prices are found. Moreover, since the free boundary is monotonically decreasing for an American put option, we may further restrict the computational domain by setting the lower boundary as the asymptotic limit of the perpetual free boundary [17,14] which is known analytically to be

$$S_f(\infty) = \frac{\rho E}{\rho - 1}, \quad \text{where } \rho = \frac{-(r-\delta-0.5\sigma^2) - \sqrt{(r-\delta-0.5\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}.$$

The compact scheme (14) with the two optimal parameters leads to the tridiagonal linear system

$$\begin{pmatrix} \beta(1) & \gamma(1) & & & \\ \alpha(2) & \beta(2) & \gamma(2) & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha(m-2) & \beta(m-2) & \gamma(m-2) \\ & & & \alpha(m-1) & \beta(m-1) \end{pmatrix} \begin{pmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{m-2}^{j+1} \\ u_{m-1}^{j+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-2} \\ b_{m-1} \end{pmatrix}, \tag{15}$$

and this linear system can be transformed to a bidiagonal one by means of row operations by the following algorithm:

for $i = m-2 : -1 : 1$
  Compute $c(i) = \frac{\gamma(i)}{\beta(i+1)}$, $\beta(i) = \beta(i) - c(i)\alpha(i+1)$ and $b_i = b_i - c(i)b_{i+1}$.
end

For the free boundary location under the singularity separated problem, condition (11) becomes

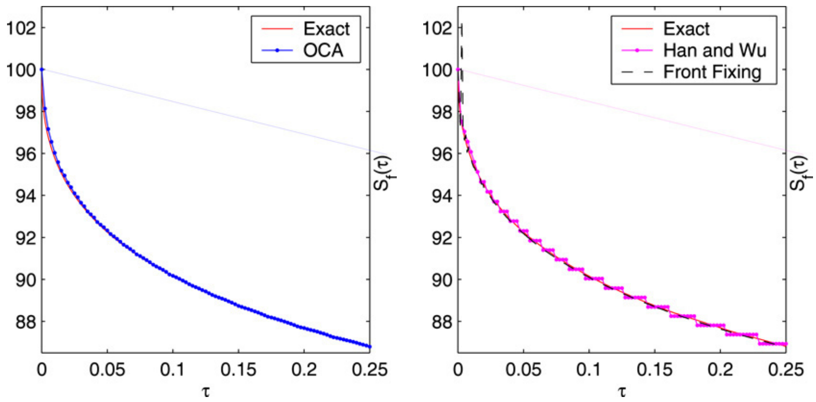
$$u_{\mathrm{D}}(x_f(\tau), \tau) < h(x_f(\tau), \tau) - u_{\mathrm{E}}(x_f(\tau), \tau). \tag{16}$$

Remark. The condition (16) can be regarded as analogous to the ideas used in the Brennan and Schwartz algorithm which enforces the maximum payoff constraint and in the Borici and Lüthi method which uses the test condition that if the slack vector is positive, then the discrete LCP is solved. Similarly, on the basis of the coercivity property of the Black–Scholes operator, Dempster, Hutton and Richards [6] formulated the American option problem as an abstract linear programme that is solved using the same optimal feasible basis as in the Borici and Lüthi method together with a revised simplex method. In general, these different methods use similar principles for the location of the free boundary. However, it is important to point out that these methods do not make use of the smooth pasting condition (4).

In our scheme, this smooth pasting condition is utilised for a more accurate approximation of the unknown free boundary location. Once a first guess of the critical asset price S_{i_f} is obtained by condition (16), we compute the first spatial derivative of two more points in the continuity region, that is, we evaluate the delta at S_{i_f+1} and S_{i_f+2} using central difference approximations and extrapolate to obtain the accurate free boundary location x_f where condition (4) is satisfied. The use of a non-uniform grid allows a grid manipulation so that the accurate free boundary x_f becomes a grid node of our computational domain for this time level. To achieve this, we need to compute both of the values $u(x_f, \tau_{j+1})$ and $u(x_f, \tau_j)$ from the free boundary condition of (13) and then calculate the HOC coefficients $\alpha(i_f)$, $\beta(i_f)$ and $\gamma(i_f)$ to obtain the linear equation involving these two values. We note that $\alpha(i_f+1)$, $\beta(i_f+1)$ and $\gamma(i_f+1)$ will remain as in (15). The new linear equation is then adjusted to allow for previous row operations and the remaining option prices in the continuity region are computed. The following describes the new algorithm.

Optimal Compact Algorithm (OCA):

- Construct the non-uniform grid in S-coordinate with $\widehat{S}_{\min} = S_f(\infty)$ and compute the non-uniform log transformed x-coordinate.
- Obtain the tridiagonal linear system with optimal discretisation (14).
- Perform row operations to obtain a bidiagonal linear system.
- At each time step j: Compute h^j.
- Then locate the free boundary with condition (16).
- Compute the Δ and extrapolate to obtain an accurate free boundary x_f.
- Reconstruct the linear equation at the new grid node and perform the necessary row operations.
- Compute the heat solution u_{D} and set $V = Ee^{-\hat{\alpha}x - \hat{\beta}\tau}(u_{\mathrm{D}} + u_{\mathrm{E}})$.



(a) OCA vs. exact. (b) Front fixing, Han and Wu vs. exact.

Fig. 1. Optimal exercise boundary curve for $T = 0.5, \sigma = 0.2, r = 0.05$.

4. Numerical experiments

We consider the case of an American put option for a wide range of financial parameters. As benchmark, we use the monotonically convergent method of Leisen and Reimer [15] with 15001 steps for assessing both the option values and the hedging parameters. For short maturity $T = 0.5$, we construct the computational domain with 200 time steps, $(\widehat{S}_{\min}, \widehat{S}_{\max}, x_{\min}, x_{\max}) = (0, 200, -1, 1)$ and 400 spatial steps. For longer maturity, $T = 3.0$, we use 600 time steps, $(\widehat{S}_{\min}, \widehat{S}_{\max}, x_{\min}, x_{\max}) = (0, 300, -1.2, 1.2)$ and 600 spatial steps for the asset price discretisations and 480 spatial steps for the log transformed coordinate.

The results given in Tables 1 and 2 show that the new method using a non-uniform grid is faster and more accurate than all other methods. The optimal coefficients of the compact scheme and the partly coarse non-uniform grid allow an efficient implementation of the far field boundary conditions. For $T = 3.0$ and $\sigma = 0.4$, we see that most schemes fail to yield solutions with good accuracy because the chosen upper bound $\widehat{S}_{\max} = 300$ is not sufficiently high. Increasing this value to 400 will make all the schemes more accurate but this is done at the expense of a higher computational cost since now, discretisation is performed over a larger computational domain. This is not a problem for our method, that has optimal compactness, or for the method of Han and Wu [9], that uses accurate artificial boundary conditions.

For all the short maturity options, the computed solutions are not very smooth and the error caused by the kink at the strike price is not dampened very quickly. The numerical results for this test case clearly illustrate the advantage of using a singularity separating method where the new method has much better accuracy than all other algorithms.

We also notice the huge advantage of the new method compared to the PSOR algorithm. It also outperforms other algorithms that have computational time varying linearly on the number of spatial grid points such as the Borici and Lüthi method, the Brennan and Schwartz algorithm or the operator splitting algorithm. For the Brennan and Schwartz algorithm, we see that the case $\theta = 0.5$ (Brennan and Schwartz2) gives better accuracy than the case $\theta = 1$ (Brennan and Schwartz1) at no extra CPU timings. For the penalty methods, although the number of iterations depends on the volatility parameters, they perform well in comparison to the PSOR algorithm for which the number of iterations increases considerably for both increasing time and volatility. The penalty method of Nielsen, Skavhaug and Tveito [16] (Penalty2) is less accurate and also less efficient than the one proposed by Forsyth and Vetzal [7] (Penalty1) since the number of iterations in [16] also depends on the parameter ϵ.

Computations of the Greeks for short and long maturities show that most methods give very good accuracies for both the delta and the gamma values. These values are less accurate for algorithms based on a uniform log transformed coordinate since the S grid becomes non-uniform and interpolation has to be used.

In addition to being fast at computing accurate option prices and hedging values, the new algorithm also makes readily available the free boundary curve for the whole duration of the American option contract. To compare the accuracy of this curve, we plot the free boundary computed by the algorithm OCA against the optimal exercise curve given by the solution to an ODE problem of Chen, Chadam and Stamicar [5]; see also [8]. We also plot in Fig. 1 the free boundary curve of the Han and Wu algorithm. We see that the OCA exercise curve is very accurate while that

Table 1
American put and Greeks for $T = 0.5$

Option param	Asset Price	Brennan Schwartz1	Brennan Schwartz2	CN PSOR	Borici Lüthi	Penalty1	Penalty2	Operator Splitting	Front Kwok	Han Wu	OCA	True value
$r = 0.05$	80	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
$\sigma = 0.20$	90	10.6626	10.6653	10.6652	10.6656	10.6651	10.6737	10.6656	10.6701	10.6656	10.6660	10.6661
$\delta = 0.00$	100	4.6495	4.6546	4.6544	4.6546	4.6542	4.6663	4.6548	4.6685	4.6546	4.6555	4.6557
	110	1.6645	1.6673	1.6671	1.6674	1.6670	1.6768	1.6674	1.6786	1.6674	1.6679	1.6680
	120	0.4974	0.4973	0.4972	0.4977	0.4972	0.5034	0.4974	0.5052	0.4977	0.4975	0.4976
RMS		0.0018	3.9160-4	4.9712-4	3.1073-4	5.3303-4	0.0054	3.2026-4	0.0066	3.0598-4	6.6267-5	
CPU/Iter		1.92	1.92	12.75/2887	2.05	4.05/228	4.19/253	1.8910	0.25	1.28	0.73	
$r = 0.07$	80	21.8653	21.8705	21.8704	21.8709	21.8701	21.8699	21.8709	21.8726	21.8709	21.8709	21.8709
$\sigma = 0.40$	90	15.2202	15.2291	15.2289	15.2292	15.2285	15.2265	15.2294	15.2320	15.2292	15.2296	15.2297
$\delta = 0.03$	100	10.2279	10.2380	10.2377	10.2380	10.2374	10.2346	10.2383	10.2442	10.2380	10.2385	10.2387
	110	6.6686	6.6774	6.6770	6.6774	6.6768	6.6749	6.6776	6.6841	6.6774	6.6778	6.6780
	120	4.2412	4.2470	4.2466	4.2474	4.2466	4.2465	4.2472	4.2544	4.2474	4.2474	4.2476
RMS		0.0029	2.1228-4	3.2755-4	1.5351-4	3.9254-4	9.0011-4	1.3391-4	0.0020	1.5277-4	5.8014-5	
CPU/Iter		1.92	1.92	50.09/11599	2.77	4.30/258	5.53/337	1.89	0.25	1.48	0.99	
$r = 0.10$	80	20.2557	20.2575	20.2575	20.2581	20.2573	20.2598	20.2579	20.2572	20.2581	20.2578	20.2578
$\sigma = 0.30$	90	12.5913	12.5973	12.5972	12.5975	12.5969	12.6009	12.5977	12.5966	12.5975	12.5979	15.5980
$\delta = 0.05$	100	7.2685	7.2762	7.2761	7.2762	7.2757	7.2769	7.2766	7.2752	7.2762	7.2769	7.2770
	110	3.9163	3.9223	3.9220	3.9224	3.9219	3.9260	3.9225	3.9224	3.9224	3.9228	3.9230
	120	1.9875	1.9903	1.9899	1.9907	1.9900	1.9946	1.9904	1.9917	1.9907	1.9906	1.9907
RMS		0.0025	2.6062-4	3.8480-4	1.9619-4	4.3334-4	0.0015	1.6882-4	4.9045-4	1.9640-4	5.9049-5	
CPU/Iter		1.92	1.92	26.34/6055	2.25	4.16/242	4.88/296	1.89	0.25	1.37	0.84	
$r = 0.07$	80	−0.7506	−0.7501	−0.7502	−0.7489	−0.7502	−0.7503	−0.7501	−0.7475	−0.7489	−0.7501	−0.7501
$\sigma = 0.40$	90	−0.5794	−0.5791	−0.5791	−0.5781	−0.5791	−0.5793	−0.5791	−0.5769	−0.5781	−0.5791	−0.5791
$\delta = 0.03$	100	−0.4229	−0.4229	−0.4230	−0.4222	−0.4229	−0.4229	−0.4230	−0.4214	−0.4222	−0.4230	−0.4229
Delta	110	−0.2941	−0.2943	−0.2944	−0.2928	−0.2943	−0.2942	−0.2943	−0.2933	−0.2938	−0.2943	−0.2943
Values	120	−0.1965	−0.1968	−0.1968	−0.1965	−0.1968	−0.1966	−0.1968	−0.1963	−0.1965	−0.1968	−0.1968
RMS		4.7737-4	2.0988-5	3.3017-5	0.0011	3.7859-5	2.9804-4	2.8672-5	0.0024	0.0011	1.5227-5	
$r = 0.03$	80	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172	0.0172
$\sigma = 0.40$	90	0.0167	0.0166	0.0166	0.0166	0.0166	0.0167	0.0166	0.0166	0.0166	0.0166	0.0166
$\delta = 0.07$	100	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144	0.0144
Gamma	110	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113	0.0113
Values	120	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083	0.0083
RMS		1.5210-4	9.8084-6	1.0681-5	3.7028-6	1.2986-5	9.8388-5	8.6067-6	1.2983-4	3.7043-6	4.5777-6	

Table 2
American put and Greeks for $T = 3$

Option param	Asset Price	Brennan Schwartz1	Brennan Schwartz2	CN PSOR	Borici Lüthi	Penalty1	Penalty2	Operator Splitting	Front Kwok	Han Wu	OCA	True value
$r = 0.05$	80	20.2785	20.2793	20.2793	20.2803	20.2793	20.2928	20.2795	20.2825	20.2803	20.2798	20.2797
$\sigma = 0.20$	90	13.3047	13.3072	13.3070	13.3074	13.3071	13.3379	13.3074	13.3117	13.3075	13.3076	13.3075
$\delta = 0.00$	100	8.7070	8.7102	8.7099	8.7103	8.7100	8.7434	8.7104	8.7135	8.7103	8.7106	8.7106
	110	5.6791	5.6822	5.6817	5.6823	5.6820	5.7121	5.6824	5.6867	5.6823	5.6825	5.6825
	120	3.6935	3.6961	3.6955	3.6964	3.6960	3.7209	3.6963	3.7001	3.6965	3.6964	3.6964
RMS		0.0011	1.2115-4	2.9342-4	8.9344-5	1.7647-4	0.0102	5.2848-5	0.0014	8.5070-5	2.2291-5	
CPU/Iter		10.05	10.05	121.09/18685	9.45	25.67/642	25.38/693	11.91	4.23	4.72	2.20	
$r = 0.07$	80	28.8951	28.9014	28.9010	28.9037	28.9012	28.9088	28.9018	28.9062	28.9045	28.9045	28.9044
$\sigma = 0.40$	90	24.4347	24.4422	24.4416	24.4463	24.4419	24.4492	24.4426	24.4497	24.4479	24.4481	24.4482
$\delta = 0.03$	100	20.7741	20.7823	20.7816	20.7895	20.7820	20.7887	20.7827	20.7951	20.7927	20.7930	20.7932
	110	17.7444	17.7530	17.7521	17.7650	17.7527	17.7587	17.7534	17.7726	17.7704	17.7708	17.7713
	120	15.2184	15.2271	15.2259	15.2458	15.2267	15.2322	15.2274	15.2567	15.2548	15.2552	15.2560
RMS		0.0057	0.0040	0.0042	0.0014	0.0041	0.0016	0.0040	3.1646-4	1.7034-4	1.1045-4	
CPU/Iter		10.19	10.19	783.67/121867	13.56	26.17/684	30.31/829	11.91	4.23	5.7340	2.91	
$r = 0.10$	80	23.0748	23.0777	23.0775	23.0781	23.0776	23.0932	23.0781	23.0790	23.0781	23.0780	23.0777
$\sigma = 0.30$	90	17.7206	17.7250	17.7246	17.7251	17.7248	17.7430	17.7253	17.7252	17.7251	17.7252	17.7250
$\delta = 0.05$	100	13.7150	13.7201	13.7196	13.7201	13.7199	13.7381	13.7204	13.7212	13.7202	13.7203	13.7203
	110	10.6825	10.6877	10.6871	10.6879	10.6875	10.7046	10.6880	10.6883	10.6880	10.6881	10.6881
	120	8.3664	8.3715	8.3706	8.3720	8.3712	8.3868	8.3717	8.3723	8.3721	8.3720	8.3721
RMS		0.0014	1.0788-4	2.7448-4	4.5259-5	1.6344-4	0.0045	6.5593-5	1.7278-4	4.3122-5	2.6980-5	
CPU/Iter		10.16	10.16	363.86/56373	10.84	26.02/660	28.05/767	11.91	4.23	5.11	2.45	
$r = 0.05$	80	−0.8539	−0.8536	−0.8537	−0.8523	−0.8537	−0.8507	−0.8536	−0.8508	−0.8523	−0.8537	−0.8536
$\sigma = 0.20$	90	−0.5621	−0.5619	−0.5619	−0.5610	−0.5619	−0.5612	−0.5619	−0.5600	−0.5610	−0.5619	−0.5619
$\delta = 0.00$	100	−0.3707	−0.3706	−0.3706	−0.3700	−0.3706	−0.3708	−0.3706	−0.3694	−0.3700	−0.3706	−0.3706
Delta	110	−0.2436	−0.2436	−0.2437	−0.2432	−0.2436	−0.2441	−0.2436	−0.2429	−0.2432	−0.2436	−0.2436
Values	120	−0.1593	−0.1594	−0.1594	−0.1591	−0.1594	−0.1599	−0.1594	−0.1589	−0.1591	−0.1594	−0.1594
RMS		1.7767-4	3.5985-5	5.3602-5	0.0011	4.1728-5	0.0017	3.5225-5	0.0021	0.0011	5.1945-5	
$r = 0.05$	80	0.0361	0.0361	0.0361	0.0361	0.0361	0.0358	0.0361	0.0361	0.0361	0.0361	0.0361
$\sigma = 0.20$	90	0.0234	0.0234	0.0234	0.0234	0.0234	0.0232	0.0234	0.0234	0.0234	0.0234	0.0234
$\delta = 0.00$	100	0.0155	0.0155	0.0155	0.0155	0.0155	0.0154	0.0155	0.0155	0.0155	0.0155	0.0155
Gamma	110	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103
Values	120	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068
RMS		8.0699-5	8.3531-5	8.2879-5	7.5969-6	8.2683-5	8.8254-4	8.4576-5	1.6459-5	7.1678-6	1.3155-5	

Table 3
Percentage change in the option values compared to a uniform grid $\Delta S = 0.5$.

Spot price	$\Delta S = 5$	$\Delta S = 10$	$\Delta S = 20$	$\Delta S = 40$
80	3.5e–7	4.1e–6	5.9e–5	6.9e–4
90	1.7e–6	2.0e–5	3.0e–4	3.7e–3
100	3.9e–6	4.5e–5	6.7e–4	9.4e–3
110	6.9e–6	7.6e–5	1.2e–3	1.9e–2
120	1.0e–5	1.1e–4	1.8e–5	3.4e–2

of Han and Wu is not. In particular, unless the free boundary exactly falls on a computational grid node, the algorithm of Han and Wu will set the free boundary at $\hat{x}_f(\tau)$ less than the true free boundary $x_f(\tau)$ and by condition (11), it will violate the American constraint over the interval $\hat{x}_f(\tau) < x < x_f(\tau)$. This is depicted by the sawtooth free boundary curve in Fig. 1. Similar arguments also apply to algorithms such as Brennan and Schwartz and the Borici and Lüthi method that have practically the same procedure of locating the free boundary. Another method that directly gives the free boundary location as part of the solution process is the front-fixing algorithm. However, the infinite speed of the optimal exercise curve near the strike price seems to cause instability in the leapfrog scheme used so that some oscillations are seen to occur at that place. This seems to affect the accuracy of the method for short maturity options.

Finally we show in Table 3 the effects of applying the far field boundary condition using a non-uniform coarse grid. Here the region englobing the spot prices and the early exercise curve is refined uniformly using $\Delta S = 0.5$. We show that even using a very coarse grid for the extended part to incorporate the far field boundary conditions, only a mild percentage change is observed in the option values at the spot prices compared to using a refined uniform grid over the whole computational domain. This is explained by the optimal compactness of the chosen scheme.

5. Conclusion

We have described a new finite difference scheme for the fast pricing of American options. We have shown that the scheme computes very quickly the American option price, the essential hedging parameters and the early exercise curve. It is the optimal compactness of the scheme used and the non-uniformity of the computational domain that allows the use of an efficient procedure for accurately locating the free boundary. All these enable the new scheme to outperform all existing finite difference algorithms for the pricing of American options. Extension of this methodology to higher dimensional pricing of options with the American feature can be studied.

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