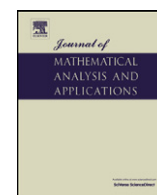




Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa


Estimates of gradient perturbation series

Tomasz Jakubowski^{*,1}, Karol Szczypkowski

Institute of Mathematics, Wrocław University of Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

ARTICLE INFO

Article history:

Received 4 May 2011

Available online 28 November 2011

Submitted by U. Stadtmueller

Keywords:

Transition density

Gradient perturbations

ABSTRACT

We give upper and lower bounds of perturbation series for transition densities, corresponding to additive gradient perturbations satisfying certain space–time integrability conditions.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Perturbation series is one of few explicit methods to construct new semigroups and it is widely used in many areas of mathematics and physics. In this paper we study the perturbation series in the context of gradient perturbations of transition densities on \mathbb{R}^d , $d \in \mathbb{N}_+$. A function $p : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$ is called a transition density if for all $-\infty < s < t < \infty$ and $x, y \in \mathbb{R}^d$ it satisfies the Chapman–Kolmogorov equation,

$$\int_{\mathbb{R}^d} p(s, x, u, z) p(u, z, t, y) dz = p(s, x, t, y), \quad s < u < t. \quad (1)$$

The function p may describe the distribution of particles evolving in space and time. We are interested in adding a drift to the picture. Let $b = (b_i)_{i=1}^d : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ (the drift function). The perturbation series is

$$\tilde{p}(s, x, t, y) = \sum_{n=0}^{\infty} p_n(s, x, t, y), \quad (2)$$

where $p_0(s, x, t, y) = p(s, x, t, y)$ and for $n = 1, 2, \dots$,

$$p_n(s, x, t, y) = \int_s^t \int_{\mathbb{R}^d} p_{n-1}(s, x, u, z) b(u, z) \cdot \nabla_z p(u, z, t, y) dz du. \quad (3)$$

We will focus on the convergence and estimates of the perturbation series. The series (2) is motivated by the perturbation theory of operator semigroups. Namely, if we denote by L the generator of the time-inhomogeneous semigroup with the integral density p , then, heuristically, \tilde{p} stands for the density of the evolution generated by $L + b \cdot \nabla$. This observation and

* Corresponding author.

E-mail addresses: tomasz.jakubowski@pwr.wroc.pl (T. Jakubowski), karol.szczypkowski@pwr.wroc.pl (K. Szczypkowski).

¹ The research was partially supported by ANR-09-BLAN-0084-01 and grants MNiSW N N201 397137 and N N201 422539. Fellowship co-financed by European Union within European Social Fund.

the series (2) were used in the study of gradient perturbations of elliptic operators (e.g. [13,11]) and the fractional Laplacian [5,10]. In such approach the convergence of (2) is delicate and therefore suitable conditions on b need to be assumed. The general philosophy is to state the conditions in terms of the density p in such a way that they fit the iteration procedure (3). This lead to the relative Kato conditions for Schrödinger perturbations in [3] and [9]. We note that there exist probabilistic methods to study Schrödinger perturbations based on multiplicative functionals and Khasminski’s lemma (strengthened in [3]). However gradient perturbations are more delicate and probabilistic methods (e.g. Girsanov transform) are applicable in special situations.

In the present paper we will consider natural conditions (5) and (6) analogous to those used in the case of Schrödinger perturbations in [3] and [9].

Definition 1. Let $\eta \geq 0$ and $Q : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ satisfy

$$Q(r, u) + Q(u, v) \leq Q(r, v), \quad r < u < v. \tag{4}$$

We say that $b \in \mathcal{N}(\eta, Q, p)$ if for all $-\infty < s < t < \infty$ and $x, y \in \mathbb{R}^d$,

$$\int_s^t \int_{\mathbb{R}^d} p(s, x, u, z) |b(u, z)| |\nabla_z p(u, z, t, y)| dz du \leq [\eta + Q(s, t)] p(s, x, t, y). \tag{5}$$

Definition 2. Let $\eta > 0$. We will say that $b \in \mathcal{P}(\eta, p)$ if there exists $h > 0$ such that for all $t - s < h$ and $x, y \in \mathbb{R}^d$,

$$\int_s^t \int_{\mathbb{R}^d} p(s, x, u, z) |b(u, z)| |\nabla_z p(u, z, t, y)| dz du \leq \eta p(s, x, t, y). \tag{6}$$

If η or Q are not specified, by writing $b \in \mathcal{N}(\eta, Q, p)$ we mean that (5) is satisfied for some η and Q .

As a part of Definitions 1 and 2 we will always make the following assumption on the gradient of p : for all $x, y \in \mathbb{R}^d$ and $s < u < t$,

$$\begin{aligned} \nabla_x p(s, x, t, y) \text{ exists, and} \\ \nabla_x p(s, x, t, y) = \int_{\mathbb{R}^d} \nabla_x p(s, x, u, z) p(u, z, t, y) dz, \end{aligned} \tag{7}$$

where the integral is absolutely convergent.

Remark 3. If $b \in \mathcal{P}(\eta, p)$, then $b \in \mathcal{N}(\eta, \beta(t - s), p)$ with $\beta = \frac{\eta}{h}$, where h is taken from Definition 2 (see [3,10]). We will generally state our results for the larger class \mathcal{N} , but occasionally more specific results will be given for \mathcal{P} .

The main results of the paper are the following two theorems.

Theorem 1. Let p be a function satisfying (1) and (7) and $b \in \mathcal{N}(\eta, Q, p)$ with $\eta < \frac{1}{2}$. Then the perturbation series (2) converges absolutely, there exists a constant $C > 1$ such that for all $-\infty < s < t < \infty$ and $x, y \in \mathbb{R}^d$,

$$\frac{p(s, x, t, y)}{C^{1+Q(s,t)}} \leq \tilde{p}(s, x, t, y) \leq p(s, x, t, y) \begin{cases} \left(\frac{1}{1-2\eta}\right)^{1+\frac{Q(s,t)}{\eta}}, & \text{if } 0 < \eta < \frac{1}{2}, \\ e^{Q(s,t)}, & \text{if } \eta = 0, \end{cases} \tag{8}$$

and the Chapman–Kolmogorov equation holds for \tilde{p} ,

$$\tilde{p}(s, x, t, y) = \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) \tilde{p}(u, z, t, y) dz, \quad u \in (s, t). \tag{9}$$

Similar results were first obtained in [10] for the density p of the isotropic α -stable process ($1 < \alpha < 2$). The authors considered drift functions b satisfying the following condition (see also [13]),

$$\int_s^t \int_{\mathbb{R}^d} \left(\frac{p(s, x, u, z)}{(u - s)^{1/\alpha}} + \frac{p(u, z, t, y)}{(t - u)^{1/\alpha}} \right) |b(u, z)| dz du \leq \eta + Q(s, t), \tag{10}$$

where $\eta \geq 0$, and Q satisfies (4). We note that the condition (5) is more natural and general than (10), e.g. it allows the drift $|b(u, z)| = p(0, 0, u, z)^{(\alpha-1)/d}$ (see [10, Remark 7, Example 5]).

In this paper we propose a new general method which may be applied to various functions p . As an example in Section 3 we consider the density p of the semigroup generated by $L = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$ ($1 < \beta < \alpha < 2$, $a \geq 0$), and prove that the resulting \tilde{p} is the density of the semigroup corresponding to $L + b \cdot \nabla$. This result is stated in the following theorem (see Section 3 for detailed definitions).

Theorem 2. *Let $1 < \beta < \alpha < 2$ and $p^a(s, x, t, y)$ be the density of the semigroup generated by $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. If $b \in \mathcal{N}(\eta, Q, p^a)$ with $\eta < 1/2$, then there exists a transition density \tilde{p}^a satisfying (8) and such that*

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}^a(s, x, u, z) [\partial_u \phi(u, z) + (\Delta_z^{\alpha/2} + a^\beta \Delta_z^{\beta/2}) \phi(u, z) + b(u, z) \cdot \nabla_z \phi(u, z)] dz du = -\phi(s, x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$.

2. Proofs

Throughout this section we assume that $\eta \geq 0$ and Q is a function satisfying (4). The following lemma is taken from [10] (see also [3]).

Lemma 4. *Let $b \in \mathcal{N}(\eta, Q, p)$. For all $s < u < t$, $x, y \in \mathbb{R}^d$ and $n = 0, 1, 2, \dots$, we have*

$$\sum_{m=0}^n \int_{\mathbb{R}^d} p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz = p_n(s, x, t, y). \quad (11)$$

Lemma 5 is crucial in our consideration. It will allow us to sum the series (2) regardless of the smallness of $Q(s, t)$.

Lemma 5. *Let $s < t$, $k \in \mathbb{N}_+$, $\theta > 0$, and $s = t_0 < t_1 < \dots < t_k = t$, be such that for all $i = 0, 1, \dots, k-1$ and $x, y \in \mathbb{R}^d$,*

$$\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} p(t_i, x, u, z) |b(u, z)| |\nabla_z p(u, z, t_{i+1}, y)| dz du \leq \theta p(t_i, x, t_{i+1}, y). \quad (12)$$

Then for all $x, y \in \mathbb{R}^d$,

$$|p_n(s, x, t, y)| \leq \binom{n+k-1}{k-1} \theta^n p(s, x, t, y).$$

Proof. For $k=1$ the inequality is true for every n by the definition of p_n and induction in n . If $k > 1$ and the statement is true for $k-1$ and all n , then for every $n \in \mathbb{N}$, by Lemma 4 we obtain

$$\begin{aligned} |p_n(s, x, t, y)| &\leq \sum_{m=0}^n \int_{\mathbb{R}^d} |p_m(s, x, t_1, z)| |p_{n-m}(t_1, z, t, y)| dz \\ &\leq \sum_{m=0}^n \int_{\mathbb{R}^d} \theta^m p(s, x, t_1, z) \binom{n-m+k-2}{k-2} \theta^{n-m} p(t_1, z, t, y) dz \\ &\leq \sum_{m=0}^n \binom{n-m+k-2}{k-2} \theta^n p(s, x, t, y) \\ &= \binom{n+k-1}{k-1} \theta^n p(s, x, t, y). \quad \square \end{aligned}$$

We note that the function Q may be discontinuous. If $\lim_{t \rightarrow t_0^+} Q(s, t) - \lim_{t \rightarrow t_0^-} Q(s, t) > \theta$ for some $t_0 > s$, (12) may not hold for any partition of the interval (s, t) . We will overcome this problem by replacing $Q(s, t)$ in (5) by $\lim_{h \rightarrow 0} Q(s_0, t-h) - Q(s_0, s+h)$ for some $s_0 \leq s$. We will write, as usual,

$$F(t^-) = \lim_{u \uparrow t} F(u), \quad F(s^+) = \lim_{u \downarrow s} F(u).$$

Lemma 6. Let $b \in \mathcal{N}(\eta, Q, p)$ and $s_0 \in \mathbb{R}$. Define $F(u) = Q(s_0, u)$ if $u > s_0$ and $F(u) = 0$ if $u \leq s_0$. Then for all $s_0 \leq s < t < \infty$ and $x, y \in \mathbb{R}^d$,

$$\int_s^t \int_{\mathbb{R}^d} p(s, x, u, z) |b(u, z)| |\nabla_z p(u, z, t, y)| dz du \leq (\eta + F(t^-) - F(s^+)) p(s, x, t, y).$$

Proof. For all $s_0 \leq s < t < \infty$ and $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned} & \int_s^t \int_{\mathbb{R}^d} p(s, x, u, z) |b(u, z)| |\nabla_z p(u, z, t, y)| dz du \\ &= \lim_{h \rightarrow 0^+} \int_{s+h}^{t-h} \int_{\mathbb{R}^d} p(s, x, u, z) |b(u, z)| |\nabla_z p(u, z, t, y)| dz du \\ &\leq \limsup_{h \rightarrow 0^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(s, x, s+h, w_2) \int_{s+h}^{t-h} \int_{\mathbb{R}^d} p(s+h, w_2, u, z) |b(u, z)| \\ &\quad \times |\nabla_z p(u, z, t-h, w_2)| dz du p(t-h, w_1, t, y) dw_1 dw_2 \\ &\leq \lim_{h \rightarrow 0^+} [\eta + Q(s+h, t-h)] p(s, x, t, y) \\ &\leq \lim_{h \rightarrow 0^+} [\eta + F(t-h) - F(s+h)] p(s, x, t, y) \\ &= [\eta + F(t^-) - F(s^+)] p(s, x, t, y), \end{aligned}$$

because $Q(s+h, t-h) \leq Q(s_0, t-h) - Q(s_0, s+h)$. \square

Lemma 7. Let $F : \mathbb{R} \rightarrow [0, \infty)$ be non-decreasing. Let $\theta > 0, s < t$ and $k \in \mathbb{N}_+$ be such that $F(t^-) - F(s^+) \leq k\theta$. Then there are $m \in \mathbb{N}_+, m \leq k, s = t_0 < t_1 < \dots < t_m = t$ such that $F(t_{i+1}^-) - F(t_i^+) \leq \theta$ for $i = 0, \dots, m-1$.

Proof. Let $l \in \mathbb{N}_+$ be the smallest number such that $F(t^-) - F(s^+) \leq l\theta$. If $l = 1$ we take $t_0 = s, t_1 = t$. Otherwise, we define $r_0 = s, r_l = t$ and $r_i = \sup\{u \in (s, t) : F(u) - F(s^+) \leq i\theta\}$ for $i = 1, \dots, l-1$. We note that $F(r_{i+1}^-) - F(s^+) \leq (i+1)\theta$ and $F(r_i^+) - F(s^+) \geq i\theta$ for $i = 0, \dots, l-1$, hence

$$F(r_{i+1}^-) - F(r_i^+) = (F(r_{i+1}^-) - F(s^+)) - (F(r_i^+) - F(s^+)) \leq (i+1)\theta - i\theta = \theta.$$

Now let $m+1$ be the number of the elements of the set $R = \{r_0, \dots, r_l\}$. We put $t_0 = r_0$ and $t_k = \min\{r_i \in R : r_i > t_{k-1}\}$, for $t = 1, \dots, m$. \square

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $s < t$ and $s_0 = s$. Let $F(u) = Q(s_0, u)$ if $u > s_0$, and $F(u) = 0$ if $u \leq s_0$.

We will prove the upper bound of (8) first.

Let $\varepsilon > 0$ and $k \in \mathbb{N}_+$ be such that $(k-1)\varepsilon \leq F(t^-) - F(s^+) \leq k\varepsilon$. By Lemma 7 there are $m \in \mathbb{N}_+, m \leq k$ and $s = t_0 < t_1 < \dots < t_m = t$ such that $F(t_{i+1}^-) - F(t_i^+) \leq \varepsilon$ for $i = 0, \dots, m-1$. By Lemma 6 and Lemma 5 with $\theta = \eta + \varepsilon$, for all $x, y \in \mathbb{R}^d$ we obtain

$$\begin{aligned} \tilde{p}(s, x, t, y) &\leq \sum_{n=0}^{\infty} |p_n(s, x, t, y)| \leq \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} (\eta + \varepsilon)^n p(s, x, t, y) \\ &= \left(\frac{1}{1 - (\eta + \varepsilon)} \right)^m p(s, x, t, y) \leq \left(\frac{1}{1 - (\eta + \varepsilon)} \right)^k p(s, x, t, y) \\ &\leq \left(\frac{1}{1 - (\eta + \varepsilon)} \right)^{1 + \frac{F(t^-) - F(s^+)}{\varepsilon}} p(s, x, t, y). \end{aligned}$$

We put $\varepsilon = \eta$ when $\eta > 0$ and we let $\varepsilon \rightarrow 0$ when $\eta = 0$. The above calculation justifies the last inequality in the statement in the theorem, as well as the change of the order of the integration and the use of Cauchy product in what follows. By Lemma 4,

$$\begin{aligned} \int_{\mathbb{R}^d} \tilde{p}(s, x, u, z) \tilde{p}(u, z, t, y) dz &= \int_{\mathbb{R}^d} \sum_{n=0}^{\infty} \sum_{m=0}^n p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \int_{\mathbb{R}^d} p_m(s, x, u, z) p_{n-m}(u, z, t, y) dz \\ &= \sum_{n=0}^{\infty} p_n(s, x, t, y) = \tilde{p}(s, x, t, y). \end{aligned}$$

We now prove the lower bound. Let $\delta \in (0, \frac{1}{2} - \eta)$.

If $F(t^-) - F(s^+) \leq \frac{1}{2} - \eta - \delta$, then by Lemma 6 and induction

$$|p_n(s, x, t, y)| \leq (\eta + F(t^-) - F(s^+))^n p(s, x, t, y) \leq \left(\frac{1}{2} - \delta\right)^n p(s, x, t, y),$$

and we get

$$\begin{aligned} \tilde{p}(s, x, t, y) &\geq p(s, x, t, y) - \sum_{n=1}^{\infty} |p_n(s, x, t, y)| \geq \left(1 - \sum_{n=1}^{\infty} \left(\frac{1}{2} - \delta\right)^n\right) p(s, x, t, y) \\ &= \left(1 - \frac{\frac{1}{2} - \delta}{1 - \frac{1}{2} + \delta}\right) p(s, x, t, y) = \left(\frac{4\delta}{1 + 2\delta}\right) p(s, x, t, y). \end{aligned} \tag{13}$$

Now in general, we set $k \in \mathbb{N}_+$ such that $(k - 1)(\frac{1}{2} - \eta - \delta) \leq F(t^-) - F(s^+) \leq k(\frac{1}{2} - \eta - \delta)$. By Lemma 7 there are $m \in \mathbb{N}_+$, $m \leq k$ and $s = t_0 < t_1 < \dots < t_m = t$ such that $F(t_{i+1}^-) - F(t_i^+) \leq \frac{1}{2} - \eta - \delta$. By (13) and (9) for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \tilde{p}(s, x, t, y) &\geq \left(\frac{4\delta}{1 + 2\delta}\right)^m p(s, x, t, y) \geq \left(\frac{4\delta}{1 + 2\delta}\right)^k p(s, x, t, y) \\ &\geq \left(\frac{4\delta}{1 + 2\delta}\right)^{1 + \frac{F(t^-) - F(s^+)}{\frac{1}{2} - \eta - \delta}} p(s, x, t, y). \end{aligned}$$

The assertion is true with $C = (4\delta/(1 + 2\delta))^{-\frac{1}{\frac{1}{2} - \eta - \delta}}$. \square

If $b \in \mathcal{P}(\eta, p)$, then the proof is simpler and the estimates are better.

Theorem 3. Let $p(s, x, t, y)$ be a function satisfying (1) and (7) and $b \in \mathcal{P}(\eta, p)$ with $\eta < \frac{1}{2}$. Then for all $-\infty < s < t < \infty, x, y \in \mathbb{R}^d$,

$$\left(\frac{1 - \eta}{1 - 2\eta}\right)^{-1 - \frac{\beta(t-s)}{\eta}} p(s, x, t, y) \leq \tilde{p}(s, x, t, y) \leq \left(\frac{1}{1 - \eta}\right)^{1 + \frac{\beta(t-s)}{\eta}} p(s, x, t, y) \tag{14}$$

and the Chapman–Kolmogorov equation (9) holds.

Proof. We have (9) by Theorem 1 and Remark 3. For the proof of (14) we let $s < t$ and $k \in \mathbb{N}_+$ be such that $(k - 1)h \leq t - s < kh$. By Lemma 5 with $t_i = s + \frac{t-s}{k}i$ and $\theta = \eta$,

$$\begin{aligned} \tilde{p}(s, x, t, y) &\leq \sum_{n=0}^{\infty} |p_n(s, x, t, y)| \leq \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \eta^n p(s, x, t, y) \\ &= \left(\frac{1}{1 - \eta}\right)^k p(s, x, t, y) \leq \left(\frac{1}{1 - \eta}\right)^{1 + \frac{\beta(t-s)}{\eta}} p(s, x, t, y), \end{aligned}$$

where $\beta = \frac{\eta}{h}$.

Also, if $t - s \leq h$, then

$$\begin{aligned} \tilde{p}(s, x, t, y) &\geq p(s, x, t, y) - \sum_{n=1}^{\infty} |p_n(s, x, t, y)| \\ &\geq \left(1 - \sum_{n=1}^{\infty} \eta^n\right) p(s, x, t, y) = \left(\frac{1 - 2\eta}{1 - \eta}\right) p(s, x, t, y). \end{aligned} \tag{15}$$

Now, for any $s < t$ and $k \in \mathbb{N}_+$ such that $(k - 1)h \leq t - s < kh$, by (15) and (9) we obtain

$$\tilde{p}(s, x, t, y) \geq \left(\frac{1 - 2\eta}{1 - \eta}\right)^k p(s, x, t, y) \geq \left(\frac{1 - 2\eta}{1 - \eta}\right)^{1 + \frac{\beta(t-s)}{\eta}} p(s, x, t, y). \quad \square$$

3. The case of the mixed fractional Laplacian

As an example we consider the transition density of the Brownian motion subordinated by the sum of two independent stable subordinators. Such processes were recently studied in [6]. Let $a \geq 0$, $0 < \beta < \alpha < 2$. Denote $p^a(s, x, t, y) = p^a(t - s, y - x)$, where $-\infty < s < t < \infty$, $x, y \in \mathbb{R}^d$, and

$$p^a(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} e^{-ix \cdot \xi} d\xi, \quad x \in \mathbb{R}^d, t > 0.$$

For $t \leq 0$ we put $p^a(t, x) = 0$. The convolution semigroup $p^a(t, x)$ has $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$ as its infinitesimal generator [1,12,2,6]. In particular, for $f \in C_c^\infty(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} (\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})f(x) &= \lim_{t \rightarrow 0^+} \frac{1}{t} \int_{\mathbb{R}^d} p^a(t, y - x)(f(y) - f(x)) dy \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{|y| > \varepsilon} \left(\frac{\mathcal{A}_{d,-\alpha}}{|y|^{d+\alpha}} + \frac{a^\beta \mathcal{A}_{d,-\beta}}{|y|^{d+\beta}} \right) [f(x+y) - f(x)] dy, \end{aligned}$$

where $\mathcal{A}_{d,\gamma} = \Gamma((d - \gamma)/2) / (2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|)$. Let $\eta_t^a(u)$ be the density function of the sum of the $\alpha/2$ -stable subordinator and a^2 times the $\beta/2$ -stable subordinator. Let $g_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}$ be the d -dimensional Gaussian kernel. Then $p^a(t, x)$ can be expressed as

$$p^a(t, x) = \int_0^\infty g_u(x) \eta_t^a(u) du.$$

Differentiating we obtain

$$\nabla_x p^a(t, x) = -2\pi x p_{(d+2)}^a(t, \tilde{x}), \tag{16}$$

where $\tilde{x} \in \mathbb{R}^{d+2}$ is such that $|\tilde{x}| = |x|$ and $p_{(d+2)}^a$ stands for the function p^a in dimension $d + 2$ (see also [5]). It is crucial here to notice that $p^a(s, x, t, y)$ satisfies (1) and (7) for every $a \geq 0$.

In what follows we assume that $1 < \beta < \alpha < 2$. This restriction emerges naturally for gradient perturbations of stable processes, although some of the results below (Lemma 9 and Remark 11) are true for any $0 < \beta < \alpha < 2$.

We first consider the case of $a = 0$. Then $p^a(t, x) = p^0(t, x)$ simplifies to the transition density of the isotropic α -stable Lévy process. Gradient, or drift, perturbations have been recently intensely studied for this process (see [5,8,7,10,4]). Theorem 2 takes the following form

Proposition 8. *Let $b \in \mathcal{N}(\eta, Q, p^0)$. If $\eta < 1/2$, then there is a positive transition density \tilde{p}^0 such that*

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}^0(s, x, u, z) [\partial_u \phi(u, z) + \Delta^{\alpha/2} \phi(u, z) + b(u, z) \cdot \nabla_z \phi(u, z)] dz du = -\phi(s, x),$$

where $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$.

We note that this result extends Theorem 1 in [10] to the wider class of drift functions from \mathcal{N} . We omit the proof as it is similar to that of [10, Theorem 1]. We also remark that Theorem 1 in the present paper gives estimates for the gradient perturbations \tilde{p}^0 , if $b \in \mathcal{N}$.

Now let $a > 0$. By writing $f(x) \approx g(x)$ we mean that there is a number $0 < C < \infty$ such that for every x we have $C^{-1}f(x) \leq g(x) \leq Cf(x)$. It is known that (see [6])

$$p^a(t, x) \approx (t^{-d/\alpha} \wedge (a^\beta t)^{-d/\beta}) \wedge \left(\frac{t}{|x|^{d+\alpha}} + \frac{a^\beta t}{|x|^{d+\beta}} \right) \tag{17}$$

on $(0, \infty) \times \mathbb{R}^d$, and that the scaling property holds

$$p^a(t, x) = a^{\frac{\beta d}{\alpha - \beta}} p^1\left(a^{\frac{\alpha \beta}{\alpha - \beta}} t, a^{\frac{\beta}{\alpha - \beta}} x\right). \tag{18}$$

To simplify the notation we denote

$$\widehat{p}^a(t, x) = \left(t^{-\frac{1}{\alpha}} \wedge (a^\beta t)^{-\frac{1}{\beta}}\right) p^a(t, x), \quad t > 0, x \in \mathbb{R}^d.$$

Lemma 9. *There exists a constant C such that for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$|\nabla_x p^a(t, x)| \leq C \widehat{p}^a(t, x).$$

Proof. By scaling we may assume that $a = 1$. By (16) and (17) we have

$$|\nabla_x p^1(t, x)| \approx |x| \left(t^{-\frac{d+2}{\alpha}} \wedge t^{-\frac{d+2}{\beta}} \wedge \left(\frac{t}{|x|^{d+2+\alpha}} \vee \frac{t}{|x|^{d+2+\beta}} \right) \right). \tag{19}$$

We claim that the right-hand side of (19) equals

$$|x| \left(t^{-\frac{2}{\alpha}} \wedge t^{-\frac{2}{\beta}} \wedge \frac{1}{|x|^2} \right) \left(t^{-\frac{d}{\alpha}} \wedge t^{-\frac{d}{\beta}} \wedge \left(\frac{t}{|x|^{d+\alpha}} \vee \frac{t}{|x|^{d+\beta}} \right) \right). \tag{20}$$

Indeed, the inequality

$$\left(\frac{t}{|x|^{d+\alpha}} \vee \frac{t}{|x|^{d+\beta}} \right) \leq \left(t^{-\frac{d}{\alpha}} \wedge t^{-\frac{d}{\beta}} \right),$$

holds if and only if $|x|^\alpha \geq t$, $|x|^\beta \geq t$, $|x|^{\beta \frac{d+\alpha}{d+\beta}} \geq t$ and $|x|^{\alpha \frac{d+\beta}{d+\alpha}} \geq t$. But $\beta \leq \beta \frac{d+\alpha}{d+\beta} \leq \alpha$ and $\beta \leq \alpha \frac{d+\beta}{d+\alpha} \leq \alpha$, so these are equivalent to $|x|^\alpha \geq t$ and $|x|^\beta \geq t$, regardless of the dimension d . This proves the claim.

We now notice that

$$|x| \left(t^{-\frac{2}{\alpha}} \wedge t^{-\frac{2}{\beta}} \wedge \frac{1}{|x|^2} \right) \leq \left(t^{-\frac{1}{\alpha}} \wedge t^{-\frac{1}{\beta}} \right),$$

which ends the proof. \square

Now we prove Theorem 2.

Proof of Theorem 2. We note that by Lemma 9 and (17) p^a satisfies

$$\frac{\partial}{\partial x_i} \int_s^\infty \int_{\mathbb{R}^d} p^a(s, x, r, z) \psi(r, z) dz dr = \int_s^\infty \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} p^a(s, x, r, z) \psi(r, z) dz dr,$$

for any $\psi : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $|\psi(s, x)| \leq c p^a(s, x, t_0, y_0)$ for some $c > 0$, $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$ and all $(s, x) \in \mathbb{R} \times \mathbb{R}^d$. Moreover, by (17) for any $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ we can take $\psi(s, x) = (\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}) \phi(s, x)$. Thus, the proof may be carried out as the proof of Theorem 1 in [10]. \square

Next we will show some properties of the function $p^a(t, x)$ useful when dealing with conditions (6) or (5).

Lemma 10 (3P). *There exists a constant C such that for all $0 < u, r < \infty$ and $x, y \in \mathbb{R}^d$ we have*

$$\widehat{p}^a(u, x) \wedge \widehat{p}^a(r, y) \leq C \widehat{p}^a(u + r, x + y). \tag{21}$$

Proof. By (18) it suffices to consider only $a = 1$. We first notice that

$$\begin{aligned} \left(u^{-\frac{d+1}{\alpha}} \wedge u^{-\frac{d+1}{\beta}} \right) \wedge \left(r^{-\frac{d+1}{\alpha}} \wedge r^{-\frac{d+1}{\beta}} \right) &\leq c \left((u+r)^{-\frac{d+1}{\alpha}} \wedge (u+r)^{-\frac{d+1}{\beta}} \right) \\ &= c \left((u+r)^{-\frac{1}{\alpha}} \wedge (u+r)^{-\frac{1}{\beta}} \right) \left((u+r)^{-\frac{d}{\alpha}} \wedge (u+r)^{-\frac{d}{\beta}} \right). \end{aligned} \tag{22}$$

Since

$$\begin{aligned} \left(u^{1-\frac{1}{\alpha}} \wedge u^{1-\frac{1}{\beta}} \right) &\leq \left((u+r)^{1-\frac{1}{\alpha}} \wedge (u+r)^{1-\frac{1}{\beta}} \right), \\ \left(r^{1-\frac{1}{\alpha}} \wedge r^{1-\frac{1}{\beta}} \right) &\leq \left((u+r)^{1-\frac{1}{\alpha}} \wedge (u+r)^{1-\frac{1}{\beta}} \right), \end{aligned}$$

we conclude that

$$\begin{aligned}
 & \left(\frac{u^{1-\frac{1}{\alpha}} \wedge u^{1-\frac{1}{\beta}}}{|x|^{d+\alpha}} + \frac{u^{1-\frac{1}{\alpha}} \wedge u^{1-\frac{1}{\beta}}}{|x|^{d+\beta}} \right) \wedge \left(\frac{r^{1-\frac{1}{\alpha}} \wedge r^{1-\frac{1}{\beta}}}{|y|^{d+\alpha}} + \frac{r^{1-\frac{1}{\alpha}} \wedge r^{1-\frac{1}{\beta}}}{|y|^{d+\beta}} \right) \\
 & \leq ((u+r)^{1-\frac{1}{\alpha}} \wedge (u+r)^{1-\frac{1}{\beta}}) \left(\left(\frac{1}{|x|^{d+\alpha}} + \frac{1}{|x|^{d+\beta}} \right) \wedge \left(\frac{1}{|y|^{d+\alpha}} + \frac{1}{|y|^{d+\beta}} \right) \right) \\
 & = ((u+r)^{-\frac{1}{\alpha}} \wedge (u+r)^{-\frac{1}{\beta}}) \left(\left(\frac{u+r}{|x|^{d+\alpha}} \wedge \frac{u+r}{|y|^{d+\alpha}} \right) + \left(\frac{u+r}{|x|^{d+\beta}} \wedge \frac{u+r}{|y|^{d+\beta}} \right) \right) \\
 & \leq c((u+r)^{-\frac{1}{\alpha}} \wedge (u+r)^{-\frac{1}{\beta}}) \left(\frac{u+r}{|x+y|^{d+\alpha}} + \frac{u+r}{|x+y|^{d+\beta}} \right). \tag{23}
 \end{aligned}$$

Finally, by (22) and (23),

$$\begin{aligned}
 \widehat{p}^1(u, x) \wedge \widehat{p}^1(r, y) & \approx (u^{-\frac{d+1}{\alpha}} \wedge u^{-\frac{d+1}{\beta}}) \wedge \left(\frac{u^{1-\frac{1}{\alpha}} \wedge u^{1-\frac{1}{\beta}}}{|x|^{d+\alpha}} + \frac{u^{1-\frac{1}{\alpha}} \wedge u^{1-\frac{1}{\beta}}}{|x|^{d+\beta}} \right) \\
 & \quad \wedge (r^{-\frac{d+1}{\alpha}} \wedge r^{-\frac{d+1}{\beta}}) \wedge \left(\frac{r^{1-\frac{1}{\alpha}} \wedge r^{1-\frac{1}{\beta}}}{|y|^{d+\alpha}} + \frac{r^{1-\frac{1}{\alpha}} \wedge r^{1-\frac{1}{\beta}}}{|y|^{d+\beta}} \right) \\
 & \leq c((u+r)^{-\frac{1}{\alpha}} \wedge (u+r)^{-\frac{1}{\beta}}) \left(((u+r)^{-\frac{d}{\alpha}} \wedge (u+r)^{-\frac{d}{\beta}}) \wedge \left(\frac{u+r}{|x+y|^{d+\alpha}} + \frac{u+r}{|x+y|^{d+\beta}} \right) \right) \\
 & \approx \widehat{p}^1(u+r, x+y). \quad \square
 \end{aligned}$$

We note that for $a = 0$, Lemma 10 reduces to Lemma 4 from [10].

Remark 11. By exactly the same proof the inequality (21) is true for $1 \leq \beta < \alpha < 2$. For that range of α and β it implies another 3P-type inequality: for all $0 < u, r < \infty$ and $x, y \in \mathbb{R}^d$

$$p^a(u, x) \wedge p^a(r, y) \leq Cp^a(u+r, x+y). \tag{24}$$

However, (24) holds for any $0 < \beta < \alpha < 2$ by a proof simpler than that of Lemma 10 (see [5, Theorem 4]). The details are left to the reader.

Corollary 12. There exists a constant C such that for all $0 < u, r < \infty$ and $x, y \in \mathbb{R}^d$ we have

$$p^a(u, x) \widehat{p}^a(r, y) \leq Cp^a(u+r, x+y) (\widehat{p}^a(u, x) + \widehat{p}^a(r, y)).$$

Proof. For any $a, b \geq 0$ we have $ab = (a \wedge b)(a \vee b)$ and $(a \vee b) \leq (a+b)$. We rewrite the right-hand side, use Lemma 10 and apply the inequality $\frac{1}{(u^{-1/\alpha} \wedge u^{-1/\beta})} ((u+r)^{-1/\alpha} \wedge (u+r)^{-1/\beta}) \leq 1$,

$$\frac{1}{(u^{-1/\alpha} \wedge u^{-1/\beta})} \widehat{p}^a(u, x) \widehat{p}^a(r, y) \leq \frac{C}{(u^{-1/\alpha} \wedge u^{-1/\beta})} \widehat{p}^a(u+r, x+y) (\widehat{p}^a(u+r) + \widehat{p}^a(u+r)). \quad \square$$

Remark 13. Lemma 9 and Corollary 12 give the existence of a constant C , such that for all $s < t$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
 & \int_s^t \int_{\mathbb{R}^d} p^a(s, x, t, y) |b(u, z)| |\nabla_z p^a(u, z, t, y)| dz du \\
 & \leq C \left(\int_s^t \int_{\mathbb{R}^d} (\widehat{p}^a(s, x, u, z) + \widehat{p}^a(u, z, t, y)) |b(u, z)| dz du \right) p^a(s, x, t, y), \tag{25}
 \end{aligned}$$

where $\widehat{p}^a(s, x, t, y) = \widehat{p}^a(t-s, y-x)$, $s < t$, $x, y \in \mathbb{R}^d$. The inequality (25) may be used for verifying that $b \in \mathcal{N}(\eta, Q, p^a)$.

We complete this section with two examples.

Example 1. Let $\gamma > 1$. Recall that $b(u, z) = b(z)$ belongs to the Kato class $\mathcal{K}_d^{\gamma-1}$ if

$$\limsup_{\varepsilon \rightarrow 0} \int_{\substack{x \in \mathbb{R}^d \\ |z-x| < \varepsilon}} |b(z)| |z-x|^{\gamma-(d+1)} dz = 0.$$

Note that $\mathcal{K}_d^{\beta-1} \subset \mathcal{K}_d^{\alpha-1}$. We have

$$\widehat{p}^a(t, x) = (t^{-\frac{1}{\alpha}} \wedge (a^\beta t)^{-\frac{1}{\beta}}) p^a(t, x) \leq c(t^{-\frac{1}{\alpha}} p^0(t, x) + (a^\beta t)^{-\frac{1}{\beta}} p_{(\beta)}(a^\beta t, x)),$$

where c is a constant independent of $t > 0$ and $x \in \mathbb{R}^d$, $p_{(\beta)}$ is the density function of the isotropic β -stable Lévy process. Thus, by Remark 13 and Example 1 in [10] we obtain that if $b \in \mathcal{K}_d^{\beta-1}$, then $b \in \mathcal{P}(\eta, p^a)$ for any $\eta > 0$.

Next, we give an example of the function b from $\mathcal{K}_d^{\alpha-1} \setminus \mathcal{K}_d^{\beta-1}$, belonging to $\mathcal{P}(\eta, p^1)$ for any $\eta > 0$. It seems that in fact $\mathcal{K}_d^{\alpha-1} \subset \mathcal{P}(\eta, p^1)$ but the proof calls for more delicate argument than the one given below.

Example 2. Let $b(u, z) = b(z)$ be such that $|b(z)| = |z|^{1-\alpha+\varepsilon}$, for some $0 < \varepsilon < \alpha - \beta$. Then $b \in \mathcal{K}_d^{\alpha-1}$ and $b \notin \mathcal{K}_d^{\beta-1}$. By integrating (17) we get

$$\int_0^t \widehat{p}^1(u, x) du \approx \left(\frac{t^{2-\frac{1}{\alpha}} \wedge t^{2-\frac{1}{\beta}}}{|x|^{d+\alpha}} + \frac{t^{2-\frac{1}{\alpha}} \wedge t^{2-\frac{1}{\beta}}}{|x|^{d+\beta}} \right) \wedge (|x|^{\alpha-(d+1)} \wedge |x|^{\beta-(d+1)}) \leq |x|^{\alpha-(d+1)}. \quad (26)$$

Let $\varepsilon > 0$ and $0 < \delta < 1$. We split the integral in (25) with $a = 1$ into two: over $A = \{z \in \mathbb{R}^d: |z| < \delta\}$ and $B = \mathbb{R}^d \setminus A$. We choose δ small enough that by (26) the integral over A is less than $\varepsilon/2$. When integrating over B , we use $|z|^{1-\alpha+\varepsilon} \leq \delta^{1-\alpha+\varepsilon}$ and take $h > 0$ such that for $t-s < h$ the integral does not exceed $\varepsilon/2$. We have just shown that $b \in \mathcal{P}(\eta, p^1)$ for any $\eta > 0$.

Acknowledgment

We would like to thank Krzysztof Bogdan for many helpful comments on this paper.

References

- [1] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag, New York, 1975.
- [2] K. Bogdan, T. Byczkowski, Potential theory for the α -stable Schrödinger operator on bounded Lipschitz domains, *Studia Math.* 133 (1) (1999) 53–92.
- [3] K. Bogdan, W. Hansen, T. Jakubowski, Time-dependent Schrödinger perturbations of transition densities, *Studia Math.* 189 (3) (2008) 235–254.
- [4] K. Bogdan, T. Jakubowski, Estimates of the Green function for the fractional Laplacian perturbed by gradient, *Potential Anal.*, doi:10.1007/s11118-011-9237-x, in press.
- [5] K. Bogdan, T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, *Comm. Math. Phys.* 271 (1) (2007) 179–198.
- [6] Z.-Q. Chen, P. Kim, R. Song, Dirichlet heat kernel estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$, *Illinois J. Math.*, in press.
- [7] T. Jakubowski, The estimates of the mean first exit time from a ball for the α -stable Ornstein–Uhlenbeck processes, *Stochastic Process. Appl.* 117 (10) (2007) 1540–1560.
- [8] T. Jakubowski, On Harnack inequality for α -stable Ornstein–Uhlenbeck processes, *Math. Z.* 258 (3) (2008) 609–628.
- [9] T. Jakubowski, On combinatorics of Schrödinger perturbations, *Potential Anal.* 31 (1) (2009) 45–55.
- [10] T. Jakubowski, K. Szczypkowski, Time-dependent gradient perturbations of fractional Laplacian, *J. Evol. Equ.* 10 (2) (2010) 319–339.
- [11] V. Liskevich, Y. Semenov, Estimates for fundamental solutions of second-order parabolic equations, *J. Lond. Math. Soc.* (2) 62 (2) (2000) 521–543.
- [12] K. Yosida, *Functional Analysis, Classics in Mathematics*, Springer-Verlag, Berlin, 1995.
- [13] Q.S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$, *Manuscripta Math.* 93 (3) (1997) 381–390.