JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 25, 378-387 (1969)

Best Constants in Inequalities Related to Opial's Inequality*

DAVID W. BOYD

California Institute of Technology, Pasadena, California 91109

Submitted by Norman Levinson

In recent years there have been a number of generalizations of Opial's inequality which, in its original form, states that if y is an absolutely continuous function with $y(a) = 0$, and $y' \in L^2(a, b)$, where a and b are finite, then

$$\int_a^b | yy' | \, dx \leqslant \tfrac{1}{2} (b - a)^2 \int_a^b | y' |^2 \, dx, \tag{1}$$

with equality only if $y(x) = c(x - a)$. (See [1] for an extensive bibliography.)

In this paper we wish to point out that the application of some well-known results in operator theory allows one to treat many of these inequalities in a unified way and is particularly appropriate in cases where best constants are desired. The method is restricted to situations in which the right hand side of the inequality involves a Hilbert space norm, and so does not include all types of generalizations of (1).

To begin with, let (a, b) be a finite or infinite interval of real numbers and σ be a measurable function on (a, b) which is positive a.e. We shall denote by L_σ^2 the Hilbert space with inner product given by

$$(f, g) = \int_a^b f(x) \overline{g(x)} \, \sigma(x) \, dx. \tag{2}$$

Let $k(x, t)$ be measurable and nonnegative on $[a, b] \times [a, b]$ and suppose that the operator K is defined by

$$Kf(x) = \int_a^b k(x, t) f(t) \, \sigma(t) \, dt, \tag{3}$$

and is a bounded operator from L_σ^2 into itself. We call k the kernel of K.

The following result is a generalization of (1) which we shall use to obtain more concrete results.

* This work was supported in part by N.S.F. grant G.P. 6111.

THEOREM 1. *Let K be defined as in* (3), *and let $G = (K + K^*)/2$ be the operator with kernel $g(x, t) = [k(x, t) + k(t, x)]/2$. Let ρ denote the norm of G as an operator in L_σ^2. Then, for all $f \in L_\sigma^2$,*

$$\int_a^b | Kf(x) | \cdot | f(x) | \sigma(x)\, dx \leqslant \rho \int_a^b | f(x) |^2 \sigma(x)\, dx, \tag{4}$$

and ρ is the best possible constant.

If G is a compact operator then ρ is an eigenvalue of G, with a corresponding non-negative eigenfunction, and equality holds in (4) *if and only if f is an eigenfunction corresponding to $\pm\rho$. If f is an eigenfunction corresponding to $\pm\rho$, then $| f |$ is an eigenfunction corresponding to ρ.*

Finally, if G is compact and $g(x, t)$ is positive a.e., then ρ is a simple eigenvalue, and $-\rho$ is not an eigenvalue for G.

PROOF. Since k is nonnegative, $| Kf | \leqslant K | f |$, so that

$$\sup\{(| Kf |, | f |) : \| f \| \leqslant 1\} = \sup\{(Kf, f) : f \geqslant 0, \| f \| \leqslant 1\}. \tag{5}$$

and, $(Kf, f) = (f, K^*f) = (K^*f, f)$ if $f \geqslant 0$, so that $(Kf, f) = (Gf, f)$ for $f \geqslant 0$. Hence

$$\begin{aligned} \sup\{(| Kf |, | f |) : \| f \| \leqslant 1\} &= \sup\{(Gf, f) : f \geqslant 0, \| f \| \leqslant 1\} \\ &= \sup\{| (Gf, f) | : \| f \| \leqslant 1\}. \end{aligned} \tag{6}$$

The last relation follows from $g(x, t) \geqslant 0$ a.e.

However, $G = G^*$ so it is well known that

$$\rho = \| G \| = \sup\{| (Gf, f) | : \| f \| \leqslant 1\}. \tag{7}$$

(See [2], p. 230, for example.)

Now, if G is compact ρ or $-\rho$ is an eigenvalue for G (see [2], p. 232) and equality holds in (7) only for corresponding eigenfunctions. If $Gf = \pm\rho f$ with $\| f \| = 1$, then

$$\rho = | (Gf, f) | \leqslant (G | f |, | f |) \leqslant \| G \| = \rho, \tag{8}$$

so that $G | f | = \rho | f |$, since equality holds in (7) for $| f |$. This shows that ρ is an eigenvalue for G, with a nonnegative eigenfunction.

In case $g(x, t) > 0$ a.e., ρ is simple, for if not there are linearly independent functions f_1, f_2 with $Gf_1 = \rho f_1$, $Gf_2 = \rho f_2$, and it is easily seen that we can choose α and β so that $f = \alpha f_1 + \beta f_2$ is not of constant sign a.e. Since $g(x, t) > 0$ a.e., this means that $| Gf | < G | f |$. But then f and $| f |$ cannot both satisfy $Gf = \rho f$, $G | f | = \rho | f |$ which is a contradiction.

Similarly, if $g(x, t) > 0$ a.e., then $-\rho$ is not an eigenvalue for G. For, if it were, let $Gf = -\rho f$. Then $G\,|f| = \rho\,|f|$. We show first that f must be of constant sign. Otherwise $|-\rho f| = |Gf| < G\,|f| = \rho\,|f|$ which is a contradiction. But if sgn f is the constant α, then

$$Gf = G(\alpha\,|f|) = \alpha G\,|f| = \alpha\rho\,|f| = \rho f,$$

a contradiction.

REMARKS 1. The condition that $g(x, t) > 0$ a.e. is stronger than necessary to insure the simplicity of ρ, and that $-\rho$ is not an eigenvalue; a weaker condition is the following:

Let $S_x = \{t : g(x, t) > 0\}$, and suppose there is a set E of measure zero so that for $x, y \notin E$, $S_x \cap S_y$ is a set of positive measure. Then ρ is simple and $-\rho$ is not an eigenvalue.

For example, if (a, b) is finite and $m(S_x) > (b - a)/2$ for almost every x, then the above condition is satisfied.

2. The existence of a nonnegative eigenfunction corresponding to $\rho = \| G \|$ could have been obtained in a less elementary way by the Krein-Rutman theory of positive operators [3].

The inequality to be discussed in Theorem 2 was proved by Willet in [4], without obtaining the best constant.

THEOREM 2. *Let $y \in C^{n-1}[a, b]$, with $y^{(n-1)}$ absolutely continuous, and $y^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n-1$. Then, there is a constant c_n such that*

$$\int_a^b |yy^{(n)}|\,dx \leqslant c_n(b-a)^n \int_a^b |y^{(n)}|^2\,dx. \tag{9}$$

For n odd $(= 2m + 1)$, the best constant c_n is $\lambda_0/2^n(n-1)!$, where λ_0 is the largest positive eigenvalue of the following $(m+1) \times (m+1)$ positive matrix $A = (a_{ij})$:

$$a_{ij} = \binom{2m}{2i}(2m - 2j + 2i + 1)^{-1} \qquad (i, j = 0, 1, \ldots, m). \tag{10}$$

If $(u_0, \ldots, u_m)^t = u$ is the positive vector (unique up to scalar multiples) with $Au = \lambda_0 u$, then equality holds in (9) only if $y^{(n)}$ is a multiple of f_n, where

$$f_n(x) = \sum_{j=0}^{m} u_j \left(\frac{2x - a - b}{b - a}\right)^{2m-2j}. \tag{11}$$

For n even (= 2m), the best constant c_n is $(\alpha_0{}^n 2^n (n-1)!)^{-1}$, where α_0 is the smallest positive solution of the equation det $B(\alpha) = 0$, and $B(\alpha) = (b_{ij}(\alpha))$ is the $m \times m$ matrix given by

$$b_{ij}(\alpha) = \omega^{2ij}\alpha^{2i} \sum_{k=0}^{2m-2i-1} (-1)^k \frac{g_j^{(k)}(1)}{k} \qquad (i, j = 0, 1,..., m-1). \tag{12}$$

In (12), $g_j(x) = \cosh(\omega^j \alpha x)$, and $\omega = \exp(2\pi i/n)$ $(i = \sqrt{-1})$.

If $(v_0, ..., v_{m-1})^t = v$ *is the vector (unique up to scalar multiples) with* $B(\alpha_0)\, v = 0$, *then equality holds in* (9) *only if* $y^{(n)}$ *is a multiple of* f_n, *where*

$$f_n(x) = \sum_{j=0}^{m-1} v_j g_i \left(\frac{2x - a - b}{b - a}\right). \tag{13}$$

PROOF. Let $\sigma(x) \equiv 1$, and let $f = y^{(n)} \in L^2$. Define $k(x, t)$ by

$$k(x, t) = \begin{cases} (x-t)^{n-1}/(n-1)!, & a \leqslant t \leqslant x \leqslant b \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Then $y = Kf$, so Theorem 1 shows that if $\rho = \| G \| = \| (K + K^*)/2 \|$, then

$$\int_a^b | yy^{(n)} |\, dx \leqslant \rho \int_a^b | y^{(n)} |^2\, dx. \tag{15}$$

G is a Hilbert-Schmidt operator, and hence compact ([2], p. 147) and $g(x, t)$ is positive except when $x = t$, so by Theorem 1, ρ is a simple eigenvalue of G with a nonnegative eigenfunction.

By a linear change of variable, we may assume that $a = -1$, $b = 1$, and we have $\rho = c(b - a)^n 2^{-n}$, where c is the norm of the following operator:

$$Tf(x) = \frac{1}{2(n-1)!} \left\{ \int_{-1}^{x} (x-t)^{n-1} f(t)\, dt + \int_{x}^{1} (t-x)^{n-1} f(t)\, dt \right\}. \tag{16}$$

We note first that if $f \geqslant 0$ is an eigenfunction of T for c, then f is even. For, let $Rf(x) = f(-x)$. Then $TRf(x) = RTf(x)$, as a change of variables shows. Thus, if $Tf = cf$, then $TRf = RTf = cRf$. But c is a simple eigenvalue so $Rf = \ell f$ for some constant ℓ; and $\ell = 1$ since $\| Rf \| = \| f \|$ and $Rf \geqslant 0$. Thus $f(x) = f(-x)$.

For $n = 2m + 1$, T is of finite rank and Tf is a polynomial of degree at most $n - 1$ for any $f \in L^2$. Thus, if $Tf = cf$,

$$f(x) = u_0 x^{2m} + u_1 x^{2m-2} + \cdots + u_m. \tag{17}$$

Now let $c = \lambda_0/2(n-1)!$, apply T to (17), and equate coefficients of x^{2m-2i} to obtain

$$\lambda_0 u_i = \sum_{j=0}^{m} a_{ij} u_j, \qquad i = 0, 1, \ldots, m, \tag{18}$$

where a_{ij} is given by (10).

This shows that λ_0 must be a positive eigenvalue of A, and since any eigenvector of A leads to an eigenfunction for T, λ_0 must be the largest such eigenvalue. The statement about the case of equality is now clear using Theorem 1.

Now let $n = 2m$. The expression for Tf shows that, for any $f \in L^2$, Tf is differentiable. Thus, if f satisfies,

$$Tf(x) = cf(x), \tag{19}$$

then, by induction $f \in C^\infty(-1, 1)$. Differentiating (19) n times gives

$$cf^{(n)}(x) = f(x). \tag{20}$$

Thus, if $\alpha = c^{-1/n}$ and $\omega = \exp(2\pi i/n)$, the eigenfunction f must have the form

$$f(x) = \sum_{j=0}^{m-1} v_j \cosh(\alpha\omega^j x) = \sum_{j=0}^{m-1} v_j g_j(x), \tag{21}$$

for certain constants v_j. (Here we use $f(x) = f(-x)$.)

Furthermore, differentiating (19) k times and setting $x = 0$, we have

$$cf^{(k)}(0) = \begin{cases} \dfrac{1}{(n-k-1)!} \displaystyle\int_0^1 t^{n-k-1} f(t)\, dt, & k \text{ even} \\ 0, & k \text{ odd}. \end{cases} \tag{22}$$

Conversely, any function satifying (20) and (22) can be shown to satisfy (19). Now, if we substitute (21) into (22) with $k = 2i$, and use the following formula (obtained by integration by parts)

$$g_j^{(2i)}(0) - \frac{\alpha^n}{(n-2i-1)!} \int_0^1 t^{n-2i-1} g_j(t)\, dt = \omega^{2ij} \alpha^{2i} \sum_{k=0}^{n-2i-1} \frac{g_j^{(k)}(1)}{k!} (-1)^k$$

$$= b_{ij}(\alpha), \tag{23}$$

we have

$$\sum_{j=0}^{m-1} b_{ij}(\alpha)\, v_j = 0. \tag{24}$$

Thus, for nontrivial solutions $(v_0, \ldots, v_{m-1})$, we must have $\det B(\alpha) = 0$.

Because of the equivalence of the problems (19) and (20)-(22), positive solutions of det $B(\alpha) = 0$ do exist and the smallest such solution α_0 leads to the largest eigenvalue of T. The case of equality is easily handled.

REMARKS 3. Willet gave the inequality (9) showing that $c_n \leqslant \frac{1}{2}$ in [4]. In [5], Das improved the estimate to

$$c_n \leqslant \frac{(n/2n-1)^{1/2}}{2n!} \tag{25}$$

but by analysis of various applications of Schwarz inequality used in his proof, proved that the inequality is strict except when $n = 1$. The constant given by Das can be seen to be the Hilbert-Schmidt norm of G which always dominates the true norm ([2], p. 150).

Using Theorem 2, for $n = 2$, the equation det $B(\alpha) = 0$ reduces to

$$\alpha \sinh \alpha = \cosh \alpha, \tag{26}$$

with the approximate solution $\alpha_0 = 1.1997$. This leads to

$$c_2 = \frac{1}{4\alpha_0^2} = .1737. \tag{27}$$

(The estimate (25) gives $c_2 < .2041$.) For $(a, b) = (-1, 1)$, the corresponding eigenfunction for G is

$$f_2(x) = \cosh \alpha_0 x. \tag{28}$$

For $n = 3$, the matrix A of Theorem 2 is 2×2 and its eigenvalues are $5 \pm 3\sqrt{5}/15$, so $\lambda_0 = 5 + 3\sqrt{5}/15$, and

$$c_3 = \frac{5 + 3\sqrt{5}}{240} = .04878. \tag{29}$$

(The estimate (25) gives $c_3 < .06455$.) For $(a, b) = (-1, 1)$ the corresponding eigenfunction is

$$f_3(x) = 5x^2 + \sqrt{5}. \tag{30}$$

Although Theorem 2 specifies the best constants in (9) exactly, it does not give much indication as to their order of magnitude. The next results gives an improved estimate for c_n and shows that it is asymptotically exact.

THEOREM 3. *Let c_n be the best constant in inequality* (9). *Then* $c_n = b_n/2n!$, *where*

$$\frac{1}{2} < b_n \leqslant \left(\frac{n}{4n-2} + \binom{2n}{n}^{-1}\right)^{1/2}, \tag{31}$$

so $b_n \to \frac{1}{2}$ *as* $n \to \infty$.

PROOF. From the proof of Theorem 2, $c_n = b_n/2n!$, where b_n is the norm of the following operator T_n in $L^2(-1, 1)$.

$$T_n f(x) = \frac{n}{2^n} \int_{-1}^{1} | x - t |^{n-1} f(t)\, dt. \tag{32}$$

Also, the norm is an eigenvalue corresponding to an *even* eigenfunction, and if f is even we have

$$T_n f(x) = \frac{n}{2^{n+1}} \int_{-1}^{1} (| x - t |^{n-1} + | x + t |^{n-1}) f(t)\, dt = U_n f(x). \tag{33}$$

Hence, $b_n \leqslant ||| U_n |||$, where $||| U_n |||$ denotes the Hilbert-Schmidt norm of U_n. That is,

$$b_n \leqslant \frac{n}{2^{n+1}} \left\{ 2 \int_{-1}^{1} dx \int_{-1}^{1} (| x - t |^{2n-2} + | x^2 - t^2 |^{n-1})\, dt \right\}^{1/2}, \tag{34}$$

which gives the estimate (31). The value of the first integral in (34) is $2^{2n}/n(2n - 1)$. The second integral is computed as follows:

$$\int_0^1 dt \int_0^1 | x^2 - t^2 |^{n-1} dx = 2 \int_0^1 dt \int_0^t (t^2 - x^2)^{n-1} dx$$

$$= 2 \int_0^1 dt \int_0^1 t^{2n-1}(1 - v^2)^{n-1} dv \qquad (\text{setting } x = tv)$$

$$= \frac{1}{n} \int_0^{1/2} 2^{2n-1} u^{n-1}(1 - u)^{n-1} du$$

$$(\text{setting } v = 1 - 2u)$$

$$= 2^{2n-2} \frac{B(n, n)}{n}. \tag{35}$$

Hence, from (34),

$$b_n \leqslant \{ n^2 2^{-2n-1} [2^{2n} n^{-1} (2n - 1)^{-1} + 2^{2n} B(n, n)\, n^{-1}] \}^{1/2},$$

which is the upper bound in (31).

To show that $b_n > \frac{1}{2}$, note that for any $f \in L^2(-1, 1)$, $(T_n f, f)/(f, f) \leqslant b_n$. Choose $f(x) = (1 + x)^{n-1} + (1 - x)^{n-1}$, and we have

$$(f, f) = 2 \int_{-1}^{1} (1 + x)^{2n-2} + 4 \int_0^1 (1 - x^2)^{n-1} dx$$

$$= 2^{2n} [(2n - 1)^{-1} + B(n, n)], \tag{36}$$

by a calculation similar to (35). And,

$$(Tf,f) = n2^{-n}\int_{-1}^{1}[(1+x)^{n-1}+(1-x)^{n-1}]\,dx\int_{-1}^{1}|x-t|^{n-1}$$
$$\times[(1+t)^{n-1}+(1-t)^{n-1}]\,dt$$
$$> n2^{-n}\Big\{2\int_0^1 dx\int_0^1[(x+t)^{n-1}+|x-t|^{n-1}](1+x)^{n-1}(1+t)^{n-1}\,dt$$
$$+4\int_0^1 dx\int_0^x (x+t)^{n-1}(1+x)^{n-1}(1-t)^{n-1}\,dt\Big\}. \qquad (37)$$

In the first integral, we use $(1+x)(1+t)\geqslant 2(x+t)$, and in the second, $(1+x)(1-t)\geqslant 2(x-t)$ to obtain

$$(Tf,f) > n2^{-n}\Big\{2^n\int_0^1 dx\int_0^1[(x+t)^{2n-2}+|x^2-t^2|^{n-1}]\,dt$$
$$+2^{n+1}\int_0^1 dx\int_0^x (x^2-t^2)^{n-1}\,dt\Big\}$$
$$= n2^{-n}\{2^{3n-1}n^{-1}(2n-1)^{-1}+2^{3n-1}B(n,n)\,n^{-1}\} = \frac{(f,f)}{2}. \qquad (38)$$

Thus, $b_n > \frac{1}{2}$. Also, the right member of (31) decreases to $\frac{1}{2}$ as $n\to\infty$, so $b_n\to\frac{1}{2}$ as $n\to\infty$.

REMARKS 4. Theorem 3 shows that the estimate (25) is of the correct order of magnitude but is asymptotically in error by a factor of $\sqrt{2}$.

REMARKS 5. For n odd, the best constants c_n can be approximated arbitrarily closely by using any of the standard methods for computing the dominant eigenvalue of a positive matrix. For example, the power method may be used, (see [6], p. 187). Starting with an arbitrary positive vector u, the sequence $A^n u/|A^n u|$ converges to the eigenvector corresponding to the dominant eigenvalue. Here $|A^n u|$ is conveniently the ℓ'-norm of $A^n u$.

The following table gives some values for $b_n = 2n!\,c_n$, computed in this way, in comparison with the upper bounds (31) and (25)

n	b_n	(31)	(25)
1	1.000000	1.000000	1.000000
3	.585410	.591608	.774597
5	.529633	.530798	.745356
7	.518742	.519155	.733799
9	.514290	.514516	.727607
11	.511622	.511768	.723747
13	.509800	.509902	.721110
15	.508472	.508548	.719195

FURTHER EXAMPLES

As another example of the use of Theorem 1, we show how it can be applied to inequalities of the following type

$$\int_a^b | y'(x)\, y(x)^p \,|\, w(x)\, dx \leqslant c \int_a^b | y'(x) |^{p+1} \sigma(x)\, dx. \tag{39}$$

In [1], Wong and the author showed that, with some differentiability assumptions on the non-negative functions w and σ, one could obtain best possible constants in (39) as eigenvalues of boundary value problems for certain differential equations. This required certain *ad hoc* assumptions as to the existence of solutions to these problems.

The case $p = 1$ fits into our framework. More generally, let us consider inequalities of the form

$$\int_a^b | y^{(n)}(x)\, y(x) \,|\, w(x)\, dx \leqslant c \int_a^b | y^{(n)}(x) |^2 \sigma(x)\, dx, \tag{40}$$

where $y^{(k)}(a) = 0$ $(k = 0, 1,..., n - 1)$, σ is positive a.e., and w is non-negative a.e. Let $f = y^{(n)}$, and define $k(x, t)$ by

$$k(x, t) = \begin{cases} w(x)\,(x - t)^{n-1}/\sigma(x)\,\sigma(t)\,(n - 1)!, & a \leqslant t \leqslant x \leqslant b \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

Then $Kf(x) = w(x)\, y(x)/\sigma(x)$, and Theorem 1 shows that the best constant in (40) is the norm of $G = (K + K^*)/2$ as an operator in L_σ^2.

If G is compact, our results improve. The simplest way to insure this would be to make K a Hilbert-Schmidt operator, which requires that

$$\int_a^b \frac{w(x)^2}{\sigma(x)}\, dx \int_a^x \frac{(x - t)^{2n-2}}{\sigma(t)}\, dt < \infty. \tag{42}$$

For example, if $\sigma(x) \geqslant \gamma > 0$ on $[a, b]$, and $w \in L^2[a, b]$, then (42) holds.

When G is compact, we need to solve the following eigenvalue problem:

$$\rho 2(n - 1)!\, \sigma(x) f(x) = w(x) \int_a^x (x - t)^{n-1} f(t)\, dt + \int_x^b w(t)\,(x - t)^{n-1} f(t)\, dt. \tag{43}$$

With appropriate differentiability assumptions on σ and w, this can be reduced to a boundary value problem for a linear differential equation. For $n = 1$, assume σ, $w \in C^1[a, b]$, and let $\mu = (2\rho)^{-1}$. Then, differentiating (43) once, and defining $u(x) = \int_a^x f(x)\,dt$, one has

$$\frac{d}{dx}(\sigma(x)\,u(x)) = \mu w'(x)\,u(x), \tag{44}$$

with boundary conditions obtained by setting $x = b$ in (35). That is

$$\sigma(b)\,u'(b) = \mu w(b)\,u(b), \qquad u(a) = 0. \tag{45}$$

This is precisely the equation presented in [1], but here we do not need to assume the existence of a solution with $u'(x) > 0$ in $[a, b]$. This assumption is replaced by the assumption that K be compact in L_σ^2, or more concretely by $\sigma(x) \geqslant \gamma > 0$ in $[a, b]$.

References

1. D. W. Boyd and J. S. W. Wong. An extension of Opial's inequality. *J. Math. Anal. Appl.* **19** (1967), 100–102.
2. F. Riesz and B. Sz.-Nagy. "Functional Analysis," (English translation by L. F. Boron). Ungar, New York, 1955.
3. M. G. Krein and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspehi Mat. Nauk* (*N.S.*) **23** (1948), 3-95 and *Amer. Math. Soc. Transl.* No. 26 (1950).
4. D. Willet. The existence-uniqueness theorem for an *n*th order linear ordinary differential equation. *Amer. Math. Monthly* **75** (1968), 174–178.
5. K. M. Das. An inequality similar to Opial's inequality. *Proc. Amer. Math. Soc.* (to appear).
6. A. S. Householder. "The Theory of Matrices in Numerical Analysis." Blaisdell, New York, 1964.