Journal of Computational and Applied Mathematics 235 (2011) 3006–3014

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval

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ARTICLE INFO

Article history:
Received 17 September 2010
Received in revised form 23 December 2010

MSC:
26A18
47H10
54C05

Keywords:
Continuous functions
Convergence theorem
Fixed point
Nondecreasing functions
Rate of convergence

ABSTRACT

In this paper, we propose a new iteration, called the SP-iteration, for approximating a fixed point of continuous functions on an arbitrary interval. Then, a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval is given. We also compare the convergence speed of Mann, Ishikawa, Noor and SP-iterations. It is proved that the SP-iteration is equivalent to and converges faster than the others. Our results extend and improve the corresponding results of Borwein and Borwein [D. Borwein, J. Borwein, Fixed point iterations for real functions, J. Math. Anal. Appl. 157 (1991) 112–126], Qing and Qihou [Y. Qing, L. Qihou, The necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval, J. Math. Anal. Appl. 323 (2006) 1383–1386], Rhoades [B.E. Rhoades, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56 (1976) 741–750], and many others. Moreover, we also present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

1. Introduction

Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. A point $p \in E$ is a *fixed point* of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f. It is known that if E is also bounded, then $F(f)$ is nonempty. The *Mann iteration* (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0, 1], and will be denoted by $M(u_1, \alpha_n, f)$. The *Ishikawa iteration* (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1-\beta_n)s_n + \beta_n f(s_n) \\ s_{n+1} = (1-\alpha_n)s_n + \alpha_n f(t_n) \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0, 1], and will be denoted by $I(s_1, \alpha_n, \beta_n, f)$. The *Noor iteration* (see [3]) is defined by $w_1 \in E$ and

$$\begin{cases} r_n = (1-\gamma_n)w_n + \gamma_n f(w_n) \\ q_n = (1-\beta_n)w_n + \beta_n f(r_n) \\ w_{n+1} = (1-\alpha_n)w_n + \alpha_n f(q_n) \end{cases} \tag{1.3}$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.12.022

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and will be denoted by $N(w_1, \alpha_n, \beta_n, \gamma_n, f)$. Clearly the Mann and Ishikawa iterations are special cases of the Noor iteration.

In 1976, Rhoades [4] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. Further, Borwein and Borwein [5] proved the convergence of the Mann iteration of continuous functions on a bounded closed interval. Recently, Qing and Qihou [6] extended their results to an arbitrary interval and to the Ishikawa iteration and gave some control conditions for the convergence of Ishikawa iteration on an arbitrary interval.

It was shown in [7] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [8] showed that the Mann iteration converges faster than the Ishikawa iteration for this class of operators. Two years later, Qing and Rhoades [9] provided an example to show that the claim in [8] is false.

Motivated by the above results, we propose a new iteration as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n f(x_n) \\ y_n = (1-\beta_n)z_n + \beta_n f(z_n) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n f(y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $x_1 \in E$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$, and it will be denoted by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and is called the SP-iteration.

In this paper, we give a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval. We also prove that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges faster than the others for the class of continuous and nondecreasing functions. Moreover, we present numerical examples for the SP-iteration to compare with the Mann, Ishikawa and Noor iterations.

2. Convergence theorems

In this section, we prove the convergence theorems of the SP-iteration for continuous functions on an arbitrary interval.

Theorem 2.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $x_1 \in E$, let the SP-iteration $\{x_n\}_{n=1}^{\infty}$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$,(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* (iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.
Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof. It is obvious that if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f, then it is bounded. Now, assume that $\{x_n\}_{n=1}^{\infty}$ is bounded. We shall show that $\{x_n\}_{n=1}^{\infty}$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n\to\infty} x_n$, $b = \limsup_{n\to\infty} x_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists δ, $0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \tag{2.1}$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\{x_n\}_{n=1}^{\infty}$ belongs to a bounded closed interval. By continuity of f, we have that $\{f(x_n)\}_{n=1}^{\infty}$ belongs to another bounded closed interval, so $\{f(x_n)\}_{n=1}^{\infty}$ is bounded, and since $z_n = (1-\gamma_n)x_n + \gamma_n f(x_n)$, so $\{z_n\}_{n=1}^{\infty}$ is bounded, and thus $\{f(z_n)\}_{n=1}^{\infty}$ is bounded. Similarly, since $y_n = (1-\beta_n)z_n + \beta_n f(z_n)$, we have $\{y_n\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are bounded. Using (1.4), we have $x_{n+1} - y_n = \alpha_n(f(y_n) - y_n)$, $y_n - z_n = \beta_n(f(z_n) - z_n)$ and $z_n - x_n = \gamma_n(f(x_n) - x_n)$. By (i), (iii) and (iv), we get $|x_{n+1} - y_n| \to 0$, $|y_n - z_n| \to 0$ and $|z_n - x_n| \to 0$. Since $|x_{n+1} - x_n| \leq |x_{n+1} - y_n| + |y_n - z_n| + |z_n - x_n|$, we have $|x_{n+1} - x_n| \to 0$. Thus there exists N such that

$$|x_{n+1} - x_n| < \frac{\delta}{3}, \qquad |z_n - x_n| < \frac{\delta}{3}, \qquad |y_n - z_n| < \frac{\delta}{3} \tag{2.2}$$

for all $n > N$. Since $b = \limsup_{n\to\infty} x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $x_k > m$. For x_k, there exist only two cases:
(A1) $x_k \geq m + \frac{\delta}{3}$, then by (2.2), we have $x_{k+1} - x_k > -\frac{\delta}{3}$, then $x_{k+1} > x_k - \frac{\delta}{3} \geq m$, so $x_{k+1} > m$.
(A2) $m < x_k < m + \frac{\delta}{3}$, then by (2.2), we have $m - \frac{\delta}{3} < z_k < m + \frac{2\delta}{3}$ and $m - \frac{2\delta}{3} < y_k < m + \delta$. So we have $|x_k - m| < \frac{\delta}{3} < \delta$, $|z_k - m| < \frac{2\delta}{3} < \delta$ and $|y_k - m| < \delta$. Using (2.1), we get

$$f(x_k) - x_k > 0, \qquad f(z_k) - z_k > 0, \qquad f(y_k) - y_k > 0. \tag{2.3}$$

By (1.4), we have

$$\begin{aligned} x_{k+1} &= y_k + \alpha_k(f(y_k) - y_k) \\ &= z_k + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\ &= x_k + \gamma_k(f(x_k) - x_k) + \beta_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k). \end{aligned} \tag{2.4}$$

By (2.3), we have $x_{k+1} \geq x_k$. Thus $x_{k+1} > m$.

By (A1) and (A2), we can conclude that $x_{k+1} > m$. By using the above argument, we obtain $x_{k+2} > m, x_{k+3} > m, x_{k+4} > m, \ldots$. Thus we get $x_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n\to\infty} x_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{x_n\}_{n=1}^{\infty}$, there exist only two cases:

(B1) There exists x_M such that $a < x_M < b$.

Then $f(x_M) = x_M$. Thus

$$z_M = (1-\gamma_M)x_M + \gamma_M f(x_M) = x_M,$$
$$y_M = (1-\beta_M)z_M + \beta_M f(z_M) = (1-\beta_M)x_M + \beta_M f(x_M) = x_M,$$
$$x_{M+1} = (1-\alpha_M)y_M + \alpha_M f(y_M) = (1-\alpha_M)x_M + \alpha_M f(x_M) = x_M.$$

Analogously, we have $x_M = x_{M+1} = x_{M+2} = x_{M+3} = \cdots$, so $x_n \to x_M$. It follows that $x_M = a$ and $x_n \to a$, which is a contradiction with the assumption.

(B2) For all n, $x_n \leq a$ or $x_n \geq b$.

Because $b-a > 0$ and $|x_{n+1}-x_n| \to 0$, so there exists N_0 such that $|x_{n+1}-x_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $x_n \leq a$ for all $n > N_0$ or $x_n \geq b$ for all $n > N_0$. If $x_n \leq a$ for $n > N_0$, then $b = \limsup_{n\to\infty} x_n \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n\to\infty} x_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{x_n\}_{n=1}^{\infty}$ is convergent.

Next, we prove that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. Let $x_n \to p$ and suppose $f(p) \neq p$. By continuity of f, we have $\{f(x_n)\}_{n=1}^{\infty}$ is bounded. From $z_n = (1-\gamma_n)x_n + \gamma_n f(x_n)$ and $\gamma_n \to 0$, we obtain $z_n \to p$. Similarly, by $y_n = (1-\beta_n)z_n + \beta_n f(z_n)$ and $\beta_n \to 0$, it follows that $y_n \to p$. Let $h_k = f(y_k) - y_k$, $q_k = f(z_k) - z_k$ and $s_k = f(x_k) - x_k$. By continuity of f, we have $\lim_{k\to\infty} h_k = \lim_{k\to\infty}(f(y_k) - y_k) = f(p) - p \neq 0$, $\lim_{k\to\infty} q_k = \lim_{k\to\infty}(f(z_k) - z_k) = f(p) - p \neq 0$ and $\lim_{k\to\infty} s_k = \lim_{k\to\infty}(f(x_k) - x_k) = f(p) - p \neq 0$. Put $w = f(p) - p$. Then $w \neq 0$. By (2.4), we have

$$\sum_{k=1}^{n-1}(x_{k+1} - x_k) = \sum_{k=1}^{n-1}(\alpha_k h_k + \beta_k q_k + \gamma_k s_k).$$

It follows that

$$x_n = x_1 + \sum_{k=1}^{n-1}(\alpha_k h_k + \beta_k q_k + \gamma_k s_k). \tag{2.5}$$

By (ii)–(iv) and $h_k \to w \neq 0$, we have that $\sum_{k=1}^{\infty} \alpha_k h_k$ is divergent, $\sum_{k=1}^{\infty} \beta_k q_k$ and $\sum_{k=1}^{\infty} \gamma_k s_k$ are convergent. This implies by (2.5) that $\{x_n\}_{n=1}^{\infty}$ is divergent, which is a contradiction with $x_n \to p$. Thus $f(p) = p$, that is $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f. □

The following result give a convergence theorem for the Noor iteration, for continuous functions on a closed interval of a real line under some control conditions which are weaker than those in Theorem 2.1.

Theorem 2.2. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $w_1 \in E$, let the Noor iteration $\{w_n\}_{n=1}^{\infty}$ be defined by (1.3), where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, (iii) $\lim_{n\to\infty} \beta_n = 0$ and (iv) $\lim_{n\to\infty} \gamma_n = 0$.*

Then $\{w_n\}_{n=1}^{\infty}$ is bounded if and only if $\{w_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof. It is clear that if $\{w_n\}_{n=1}^{\infty}$ converges to a fixed point of f, so it is bounded. Now, assume that $\{w_n\}_{n=1}^{\infty}$ is bounded. We shall show that $\{w_n\}_{n=1}^{\infty}$ is convergent. To show this, suppose not. Then there exist $a, b \in \mathbb{R}$, $a = \liminf_{n\to\infty} w_n$, $b = \limsup_{n\to\infty} w_n$ and $a < b$. First, we show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Because $f(x)$ is a continuous function, there exists δ, $0 < \delta < b - a$, such that

$$f(x) - x > 0 \quad \text{for } |x - m| \leq \delta. \tag{2.6}$$

Since $\{w_n\}_{n=1}^{\infty}$ is bounded, $\{w_n\}_{n=1}^{\infty}$ belongs to a bounded closed interval. By continuity of f, we have that $\{f(w_n)\}_{n=1}^{\infty}$ belongs to another bounded closed interval, so $\{f(w_n)\}_{n=1}^{\infty}$ is bounded, and since $r_n = (1-\gamma_n)w_n + \gamma_n f(w_n)$, so $\{r_n\}_{n=1}^{\infty}$ is bounded, and thus $\{f(r_n)\}_{n=1}^{\infty}$ is bounded. Similarly, since $q_n = (1-\beta_n)w_n + \beta_n f(r_n)$, we have $\{q_n\}_{n=1}^{\infty}$ and $\{f(q_n)\}_{n=1}^{\infty}$ are bounded. Using (1.3), we have $w_{n+1} - w_n = \alpha_n(f(q_n) - w_n)$, $q_n - w_n = \beta_n(f(r_n) - w_n)$ and $r_n - w_n = \gamma_n(f(w_n) - w_n)$. By (i), (iii) and (iv), we get $|w_{n+1} - w_n| \to 0$, $|q_n - w_n| \to 0$ and $|r_n - w_n| \to 0$. Thus there exists N such that

$$|w_{n+1} - w_n| < \frac{\delta}{3}, \qquad |q_n - w_n| < \frac{\delta}{3}, \qquad |r_n - w_n| < \frac{\delta}{3} \tag{2.7}$$

for all $n > N$. Since $b = \limsup_{n\to\infty} w_n > m$, there exists $k_1 > N$ such that $w_{n_{k_1}} > m$. Let $k = n_{k_1}$, then $w_k > m$. For w_k, there exist only two cases:

(A1) $w_k \geq m + \frac{\delta}{3}$, then by (2.7), we have $w_{k+1} - w_k > -\frac{\delta}{3}$, then $w_{k+1} > w_k - \frac{\delta}{3} \geq m$, so $w_{k+1} > m$.

(A2) $m < w_k < m+\frac{\delta}{3}$, then by (2.7), we have $m-\frac{\delta}{3} < q_k < m+\frac{2\delta}{3}$ and $m-\frac{\delta}{3} < r_k < m+\frac{2\delta}{3}$. So we have $|w_k-m| < \frac{\delta}{3} < \delta$, $|q_k - m| < \frac{2\delta}{3} < \delta$ and $|r_k - m| < \frac{2\delta}{3} < \delta$. Using (2.6), we get

$$f(w_k) - w_k > 0, \qquad f(q_k) - q_k > 0, \qquad f(r_k) - r_k > 0. \tag{2.8}$$

By (1.3), we have

$$\begin{aligned} w_{k+1} &= w_k + \alpha_k(f(q_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - w_k) \\ &= w_k + \alpha_k(f(q_k) - q_k) + \alpha_k\beta_k(f(r_k) - r_k) + \alpha_k\beta_k\gamma_k(f(w_k) - w_k). \end{aligned}$$

By (2.8), we have $w_{k+1} \geq w_k$. Thus $w_{k+1} > m$.

By (A1) and (A2), we can conclude that $w_{k+1} > m$. By using the above argument, we obtain $w_{k+2} > m$, $w_{k+3} > m$, $w_{k+4} > m, \ldots$. Thus we get $w_n > m$, for all $n > k = n_{k_1}$. So $a = \liminf_{n\to\infty} w_n \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

For $\{w_n\}_{n=1}^{\infty}$, there exist only two cases:

(B1) There exists w_M such that $a < w_M < b$.

Then $f(w_M) = w_M$. Thus

$$\begin{aligned} r_M &= (1-\gamma_M)w_M + \gamma_M f(w_M) = w_M, \\ q_M &= (1-\beta_M)w_M + \beta_M f(r_M) = (1-\beta_M)w_M + \beta_M f(w_M) = w_M, \\ w_{M+1} &= (1-\alpha_M)w_M + \alpha_M f(q_M) = (1-\alpha_M)w_M + \alpha_M f(w_M) = w_M. \end{aligned}$$

Analogously, we have $w_M = w_{M+1} = w_{M+2} = w_{M+3} = \cdots$, so $w_n \to w_M$. It follows that $w_M = a$ and $w_n \to a$, which is a contradiction with the assumption.

(B2) For all n, $w_n \leq a$ or $w_n \geq b$.

Because $b - a > 0$ and $|w_{n+1} - w_n| \to 0$, so there exists N_0 such that $|w_{n+1} - w_n| < \frac{b-a}{3}$, for all $n > N_0$. It implies that either $w_n \leq a$ for all $n > N_0$ or $w_n \geq b$ for all $n > N_0$. If $w_n \leq a$ for $n > N_0$, then $b = \limsup_{n\to\infty} w_n \leq a$, which is a contradiction with $a < b$. If $w_n \geq b$ for $n > N_0$, so we have $a = \liminf_{n\to\infty} w_n \geq b$, which is a contradiction with $a < b$.

Hence, we have $\{w_n\}_{n=1}^{\infty}$ is convergent.

Next, we prove that $\{w_n\}_{n=1}^{\infty}$ converges to a fixed point of f. Let $w_n \to p$ and suppose $f(p) \neq p$. By continuity of f, we have $\{f(w_n)\}_{n=1}^{\infty}$ is bounded. Since $r_n = (1-\gamma_n)w_n + \gamma_n f(w_n)$ and $\gamma_n \to 0$, it follows that $r_n \to p$. From $q_n = (1-\beta_n)w_n + \beta_n f(r_n)$ and $\beta_n \to 0$, we obtain $q_n \to p$. Let $h_k = f(q_k) - w_k$. By continuity of f, we have $\lim_{k\to\infty} h_k = \lim_{k\to\infty}(f(q_k) - w_k) = f(p) - p \neq 0$. Let $d = f(p) - p$. Then $d \neq 0$. Using (1.3), we get $w_{k+1} = w_k + \alpha_k(f(q_k) - w_k)$. It follows that

$$w_n = w_1 + \sum_{k=1}^{n-1} \alpha_k h_k. \tag{2.9}$$

By $h_k \to d \neq 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, it implies by (2.9) that $\{w_n\}_{n=1}^{\infty}$ is divergent, which is a contradiction with $w_n \to p$. Thus $f(p) = p$, that is $\{w_n\}_{n=1}^{\infty}$ converges to a fixed point of f. □

The following three corollaries are obtained directly by Theorem 2.1.

Corollary 2.3. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n f(x_n) \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n f(y_n), \end{cases} \tag{2.10}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof. By putting $\gamma_n = 0$ for all $n \geq 1$ in Theorem 2.1, then we obtain the required result directly from Theorem 2.1. □

Corollary 2.4 ([6]). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $u_1 \in E$, let the Mann iteration $\{u_n\}_{n=1}^{\infty}$ be defined by (1.1), where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{u_n\}_{n=1}^{\infty}$ is bounded if and only if $\{u_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof. By putting $\gamma_n = 0$ and $\beta_n = 0$ for all $n \geq 1$ in Theorem 2.1, we obtain the desired result. □

Corollary 2.5. *Let $f : [a, b] \to [a, b]$ be a continuous function. For $x_1 \in [a, b]$, let the SP-iteration $\{x_n\}_{n=1}^{\infty}$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* (iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.
Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

The following result is obtained direclty from Theorem 2.2.

Corollary 2.6 ([6])**.** *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. For $s_1 \in E$, let the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ be defined by (1.2), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the following conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* (iii) $\lim_{n\to\infty} \beta_n = 0$.
Then $\{s_n\}_{n=1}^{\infty}$ is bounded if and only if $\{s_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

3. Rate of Convergence

In this section, we study the rate of convergence of the SP-iteration for continuous and nondecreasing functions; we also compare the rate of convergence of the SP-iteration with the Mann, Ishikawa and Noor iterations. We show that the SP-iteration converges faster than the others. For analysis of the rate of convergence, we use the concept introduced by Rhoades [4] as follows.

Definition 3.1. Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{y_n\}_{n=1}^{\infty}$ if

$$|x_n - p| \le |y_n - p| \quad \text{for all } n \ge 1.$$

The following lemmas are useful and crucial for our main results.

Lemma 3.2. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$. Let $\{u_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be defined by (1.1)–(1.4), respectively. Then the following hold:*

(i) *If $f(u_1) < u_1$, then $f(u_n) < u_n$ for all $n \ge 1$ and $\{u_n\}_{n=1}^{\infty}$ is nonincreasing.*
(ii) *If $f(u_1) > u_1$, then $f(u_n) > u_n$ for all $n \ge 1$ and $\{u_n\}_{n=1}^{\infty}$ is nondecreasing.*
(iii) *If $f(s_1) < s_1$, then $f(s_n) < s_n$ for all $n \ge 1$ and $\{s_n\}_{n=1}^{\infty}$ is nonincreasing.*
(iv) *If $f(s_1) > s_1$, then $f(s_n) > s_n$ for all $n \ge 1$ and $\{s_n\}_{n=1}^{\infty}$ is nondecreasing.*
(v) *If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \ge 1$ and $\{w_n\}_{n=1}^{\infty}$ is nonincreasing.*
(vi) *If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \ge 1$ and $\{w_n\}_{n=1}^{\infty}$ is nondecreasing.*
(vii) *If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \ge 1$ and $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.*
(viii) *If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \ge 1$ and $\{x_n\}_{n=1}^{\infty}$ is nondecreasing.*

Proof. (i) Let $f(u_1) < u_1$. Then $f(u_1) < u_2 \le u_1$. Since f is nondecreasing, we have $f(u_2) \le f(u_1)$. Thus $f(u_2) < u_2$. Assume that $f(u_k) < u_k$. Then $f(u_k) < u_{k+1} \le u_k$. Since $f(u_{k+1}) \le f(u_k)$, we have $f(u_{k+1}) < u_{k+1}$. By mathematical induction, we obtain $f(u_n) < u_n$ for all $n \ge 1$. It follows that $u_{n+1} \le u_n$ for all $n \ge 1$, that is $\{u_n\}_{n=1}^{\infty}$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(s_1) < s_1$. Then $f(s_1) < t_1 \le s_1$. Since f is nondecreasing, we have $f(t_1) \le f(s_1) < t_1 \le s_1$. This implies $f(t_1) < s_2 \le s_1$. Thus $f(s_2) \le f(s_1) < t_1 \le s_1$. If $f(t_1) < s_2 \le t_1$, then $f(s_2) \le f(t_1) < s_2$. Otherwise, if $t_1 < s_2 \le s_1$, then $f(s_2) \le f(s_1) < t_1 < s_2$. Hence, we have $f(s_2) < s_2$. By continuing in this way, we can show that $f(s_n) < s_n$ for all $n \ge 1$. This implies $t_n \le s_n$ for all $n \ge 1$. Since f is nondecreasing, we have $f(t_n) \le f(s_n) < s_n$ for all $n \ge 1$. It follows that $s_{n+1} \le s_n$ for all $n \ge 1$, that is $\{s_n\}_{n=1}^{\infty}$ is nonincreasing.

(iv) By using the same argument as in (iii), we obtain the desired result.

(v) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \le w_1$. Since f is nondecreasing, we have $f(r_1) \le f(w_1) < r_1 \le w_1$. This implies $f(r_1) < q_1 \le w_1$. Thus $f(q_1) \le f(w_1) < r_1 \le w_1$. For q_1, we consider the following two cases:

Case 1: $f(r_1) < q_1 \le r_1$. Then $f(q_1) \le f(r_1) < q_1 \le r_1 \le w_1$. This implies $f(q_1) < w_2 \le w_1$. Thus $f(w_2) \le f(w_1) < r_1 \le w_1$. It follows that if $f(q_1) < w_2 \le q_1$, then $f(w_2) \le f(q_1) < w_2$, if $q_1 < w_2 \le r_1$, then $f(w_2) \le f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \le w_1$, then $f(w_2) \le f(w_1) < r_1 < w_2$. Thus, we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \le w_1$. Then $f(q_1) \le f(w_1) < r_1 \le w_1$. This implies $f(q_1) < w_2 \le w_1$. Thus $f(w_2) \le f(w_1) < r_1 < q_1 \le w_1$. It follows that if $f(q_1) < w_2 \le q_1$, then $f(w_2) \le f(q_1) < w_2$ and if $q_1 < w_2 \le w_1$, then $f(w_2) \le f(w_1) < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Cases 1 and 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \ge 1$. This implies $r_n \le w_n$ for all $n \ge 1$. Since f is nondecreasing, we have $f(r_n) \le f(w_n) < w_n$ for all $n \ge 1$. Thus $q_n \le w_n$ for all $n \ge 1$, then $f(q_n) \le f(w_n) < w_n$ for all $n \ge 1$. Hence, we have $w_{n+1} \le w_n$ for all $n \ge 1$, that is $\{w_n\}_{n=1}^{\infty}$ is nonincreasing.

(vi) By using the same argument as in (v), we obtain the desired result.

(vii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1$. By (1.4), we have $f(z_1) < y_1 \leq z_1$. This implies $f(y_1) \leq f(z_1) < y_1$. Since $f(y_1) < x_2 \leq y_1$, we have $f(x_2) \leq f(y_1)$. Thus $f(x_2) < x_2$. Assume that $f(x_k) < x_k$. Then $f(x_k) < z_k \leq x_k$. Since f is nondecreasing, we have $f(z_k) \leq f(x_k) < z_k$. By (1.4), we have $f(z_k) < y_k \leq z_k$. This implies $f(y_k) \leq f(z_k) < y_k$. Since $f(y_k) < x_{k+1} \leq y_k$, we have $f(x_{k+1}) \leq f(y_k)$. Thus $f(x_{k+1}) < x_{k+1}$. By induction, we can conclude that $f(x_n) < x_n$ for all $n \geq 1$. Thus together with (1.4), we have $z_n \leq x_n$ for all $n \geq 1$. It follows that $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. This implies that $y_n \leq x_n$ for all $n \geq 1$. Hence, we have $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. It follows that $x_{n+1} = (1-\alpha_n)y_n + \alpha_n f(y_n) \leq x_n$ for all $n \geq 1$. Thus $\{x_n\}_{n=1}^{\infty}$ is nonincreasing.

(viii) By using the same argument as in (vii), we obtain the desired result. □

Lemma 3.3. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined by (1.1)–(1.3) or (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0, 1)$. Then the following are satisfied:*

(i) *If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.*
(ii) *If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.*

Proof. We shall prove only the case that $\{x_n\}_{n=1}^{\infty}$ is defined by (1.4) because other cases can be proved similarly.

(i) Suppose that $p \in F(f)$ and $x_1 > p$. Since f is nondecreasing, we have $f(x_1) \geq f(p) = p$. By (1.4), we have $z_1 \geq p$. Thus $f(z_1) \geq p$. This implies by (1.4) that $y_1 \geq p$. Thus $f(y_1) \geq p$, it follows that $x_2 \geq p$. Assume that $x_k \geq p$. Thus $f(x_k) \geq f(p) = p$. By (1.4), we have $z_k \geq p$. Thus $f(z_k) \geq p$. This implies that $y_k \geq p$. Thus $f(y_k) \geq p$, it follows that $x_{k+1} \geq p$. By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$.

(ii) Suppose that $p \in F(f)$ and $x_1 < p$. By using the same argument as in (i), we can show that $x_n \leq p$ for all $n \geq 1$. □

Lemma 3.4. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be the sequences defined by (1.1)–(1.4), respectively. Then the following are satisfied:*

(i) *If $f(u_1) < u_1$, then $x_n \leq w_n \leq s_n \leq u_n$ for all $n \geq 1$.*
(ii) *If $f(u_1) > u_1$, then $x_n \geq w_n \geq s_n \geq u_n$ for all $n \geq 1$.*

Proof. (i) Let $f(u_1) < u_1$. Since $u_1 = s_1 = w_1 = x_1$, we get $f(s_1) < s_1$, $f(w_1) < w_1$ and $f(x_1) < x_1$. First, we show that $x_n \leq w_n$ for all $n \geq 1$. By (1.4), we have $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1$. This implies $f(z_1) < y_1 \leq z_1 \leq x_1$. Using (1.3) and (1.4), we have $z_1 - r_1 = (1-\gamma_1)(x_1 - w_1) + \gamma_1(f(x_1) - f(w_1)) = 0$, that is $z_1 = r_1$, and we get $y_1 - q_1 = (1-\beta_1)(z_1 - w_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$. Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. This implies

$$
\begin{aligned}
x_2 - w_2 &= (1-\alpha_1)(y_1 - w_1) + \alpha_1(f(y_1) - f(q_1)) \\
&= (1-\alpha_1)(y_1 - x_1) + \alpha_1(f(y_1) - f(q_1)) \\
&\leq 0,
\end{aligned}
$$

that is $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. By Lemma 3.2(v) and (vii), we have $f(w_k) < w_k$ and $f(x_k) < x_k$. This implies $f(x_k) < z_k \leq x_k \leq w_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus $y_k - w_k = (z_k - w_k) + \beta_k(f(z_k) - z_k) \leq 0$ and $z_k - r_k = (1-\gamma_k)(x_k - w_k) + \gamma_k(f(x_k) - f(w_k)) \leq 0$. That is $y_k \leq w_k$ and $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have $y_k - q_k = (1-\beta_k)(z_k - w_k) + \beta_k(f(z_k) - f(r_k)) \leq 0$, so $y_k \leq q_k$, which implies $f(y_k) \leq f(q_k)$. It follows that

$$
\begin{aligned}
x_{k+1} - w_{k+1} &= (1-\alpha_k)(y_k - w_k) + \alpha_k(f(y_k) - f(q_k)) \\
&\leq 0,
\end{aligned}
$$

that is $x_{k+1} \leq w_{k+1}$. By mathematical induction, we obtain $x_n \leq w_n$ for all $n \geq 1$.

Next, we show that $w_n \leq s_n$ for all $n \geq 1$. Using (1.2) and (1.3), we have $r_1 - s_1 = (w_1 - s_1) + \gamma_1(f(w_1) - w_1) \leq 0$, that is $r_1 \leq s_1$. Since $f(r_1) \leq f(s_1)$, we obtain $q_1 - t_1 = (1-\beta_1)(w_1 - s_1) + \beta_1(f(r_1) - f(s_1)) \leq 0$, so $q_1 \leq t_1$, which implies $f(q_1) \leq f(t_1)$. It follows that

$$
\begin{aligned}
w_2 - s_2 &= (1-\alpha_1)(w_1 - s_1) + \alpha_1(f(q_1) - f(t_1)) \\
&\leq 0,
\end{aligned}
$$

that is $w_2 \leq s_2$. Assume that $w_k \leq s_k$. Thus $f(w_k) \leq f(s_k)$. By Lemma 3.2(iii) and (v), we have $f(s_k) < s_k$ and $f(w_k) < w_k$. This implies $r_k - s_k = (w_k - s_k) + \gamma_k(f(w_k) - w_k) \leq 0$, thus $r_k \leq s_k$. Since $f(r_k) \leq f(s_k)$, we have $q_k - t_k = (1-\beta_k)(w_k - s_k) + \beta_k(f(r_k) - f(s_k)) \leq 0$, so $q_k \leq t_k$, which implies $f(q_k) \leq f(t_k)$. It follows that

$$
\begin{aligned}
w_{k+1} - s_{k+1} &= (1-\alpha_k)(w_k - s_k) + \alpha_k(f(q_k) - f(t_k)) \\
&\leq 0,
\end{aligned}
$$

that is $w_{k+1} \leq s_{k+1}$. By mathematical induction, we obtain $w_n \leq s_n$ for all $n \geq 1$.

Finally, we show that $s_n \le u_n$ for all $n \ge 1$. Using (1.1) and (1.2), we have $t_1 - u_1 = (s_1 - u_1) + \beta_1(f(s_1) - s_1) \le 0$, thus $t_1 \le u_1$. Since f is nondecreasing, we have $f(t_1) \le f(u_1)$. This implies

$$\begin{aligned} s_2 - u_2 &= (1-\alpha_1)(s_1 - u_1) + \alpha_1(f(t_1) - f(u_1)) \\ &\le 0, \end{aligned}$$

that is $s_2 \le u_2$. Assume that $s_k \le u_k$. Thus $f(s_k) \le f(u_k)$. By Lemma 3.2(i) and (iii), we have $f(u_k) < u_k$ and $f(s_k) < s_k$. This implies $t_k - u_k = (s_k - u_k) + \beta_k(f(s_k) - s_k) \le 0$, so $t_k \le u_k$, which implies $f(t_k) \le f(u_k)$. It follows that

$$\begin{aligned} s_{k+1} - u_{k+1} &= (1-\alpha_k)(s_k - u_k) + \alpha_k(f(t_k) - f(u_k)) \\ &\le 0, \end{aligned}$$

that is $s_{k+1} \le u_{k+1}$. By mathematical induction, we obtain $s_n \le u_n$ for all $n \ge 1$.
(ii) By using Lemma 3.2(ii, iv, vi, viii) and the same argument as in (i), we can show that $x_n \ge w_n \ge s_n \ge u_n$ for all $n \ge 1$. □

Proposition 3.5. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by one of the following iteration methods: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f.*

Proof. By Lemma 3.2(ii, iv, vi, viii), we have that $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^{\infty}$ does not converge to a fixed point of f.

Proposition 3.6. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by one of the following iteration method: $M(x_1, \alpha_n, f)$, $I(x_1, \alpha_n, \beta_n, f)$, $N(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ does not converge to a fixed point of f.*

Proof. By Lemma 3.2(i, iii, v, vii), we have that $\{x_n\}_{n=1}^{\infty}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in E : p = f(p)\}$, it follows that $\{x_n\}_{n=1}^{\infty}$ does not converge to a fixed point of f. □

Theorem 3.7. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be the sequences defined by (1.1)–(1.4), respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1)$. Then the following are satisfied:*

- (i) *The Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if the Mann iteration $\{u_n\}_{n=1}^{\infty}$ converges to p. Moreover, the Ishikawa iteration converges faster than the Mann iteration.*
- (ii) *The Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to p. Moreover, the Noor iteration converges faster than the Ishikawa iteration.*
- (iii) *The SP-iteration $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if the Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. Moreover, the SP-iteration converges faster than the Noor iteration.*

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$.
(i) ($\Rightarrow$) If the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then set $\beta_n = 0$ for all $n \ge 1$ in (1.2), we can get the convergence of the Mann iteration.
($\Leftarrow$) Suppose that the Mann iteration $\{u_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. We divide our proof into the following three cases: Case 1: $u_1 = s_1 > U$, Case 2: $u_1 = s_1 < L$, Case 3: $L \le u_1 = s_1 \le U$.
Case 1: $u_1 = s_1 > U$. By Proposition 3.5, we get $f(u_1) < u_1$ and $f(s_1) < s_1$. This implies by Lemma 3.4(i) that $s_n \le u_n$ for all $n \ge 1$. We note that $U < s_1$ and by using (1.2) and mathematical induction, we can show that $U \le s_n$ for all $n \ge 1$. Then, we have $0 \le s_n - p \le u_n - p$, so

$$|s_n - p| \le |u_n - p| \quad \text{for all } n \ge 1. \tag{3.1}$$

It follows that $s_n \to p$. That is the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to the same fixed point p. Moreover, by (3.1), we see that the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges faster than the Mann iteration $\{u_n\}_{n=1}^{\infty}$.
Case 2: $u_1 = s_1 < L$. By Proposition 3.6, we get $f(u_1) > u_1$ and $f(s_1) > s_1$. This implies by Lemma 3.4(ii) that $s_n \ge u_n$ for all $n \ge 1$. We note that $s_1 < L$ and by using (1.2) and mathematical induction, we can show that $s_n \le L$ for all $n \ge 1$. Then, we have $|s_n - p| \le |u_n - p|$ for all $n \ge 1$. It follows that $s_n \to p$ and the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges faster than the Mann iteration $\{u_n\}_{n=1}^{\infty}$.
Case 3: $L \le u_1 = s_1 \le U$. Suppose that $f(u_1) \ne u_1$. If $f(u_1) < u_1$, we have by Lemma 3.2(i) that $\{u_n\}_{n=1}^{\infty}$ is nonincreasing with limit p. By Lemmas 3.3(i) and 3.4(i), we have $p \le s_n \le u_n$ for all $n \ge 1$. It follows that $|s_n - p| \le |u_n - p|$ for all $n \ge 1$. Hence, we have that $s_n \to p$ and the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges faster than the Mann iteration $\{u_n\}_{n=1}^{\infty}$. If $f(u_1) > u_1$, we have by Lemma 3.2(ii) that $\{u_n\}_{n=1}^{\infty}$ is nondecreasing with limit p. By Lemmas 3.3(ii) and 3.4(ii), we have $p \ge s_n \ge u_n$ for all $n \ge 1$. It follows that $|s_n - p| \le |u_n - p|$ for all $n \ge 1$. Hence, we have that $s_n \to p$ and the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges faster than the Mann iteration $\{u_n\}_{n=1}^{\infty}$.

(ii) ($\Rightarrow$) If the Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then set $\gamma_n = 0$ for all $n \geq 1$ in (1.3), we can get the convergence of the Ishikawa iteration.
($\Leftarrow$) Suppose that the Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. By using Lemmas 3.2(iii, iv), 3.3 and 3.4, Propositions 3.5, 3.6 and the same proof as in (i), we obtain the desired result.
(iii) ($\Rightarrow$) If the SP-iteration $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then set $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 1$ in (1.4), we can get the convergence of the Mann iteration. Then, the result is obtained directly by (i) and (ii).
($\Leftarrow$) Suppose that the Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. By using Lemma 3.2(v, vi), Lemmas 3.3 and 3.4, Propositions 3.5 and 3.6 and the same proof as in (i), we obtain the desired result. □

The speed of convergence for the Mann, Ishikawa, Noor and SP-iterations also depends on the choice of $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ in the interval $[0, 1)$. We present such result only for the SP-iteration. The others are very similar.

Theorem 3.8. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\alpha_n^*\}_{n=1}^{\infty}$, $\{\beta_n^*\}_{n=1}^{\infty}$, $\{\gamma_n^*\}_{n=1}^{\infty}$ be sequences in $[0, 1)$ such that $\alpha_n \leq \alpha_n^*$, $\beta_n \leq \beta_n^*$ and $\gamma_n \leq \gamma_n^*$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be defined by $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$, respectively. If $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then $\{x_n^*\}_{n=1}^{\infty}$ converges to p. Moreover, $\{x_n^*\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$, provided that $x_1^* = x_1 \in E$.*

Proof. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. We divide our proof into the following three cases:
Case 1: $x_1 = x_1^* > U$. By Proposition 3.5, we have $f(x_1) < x_1$. This implies by Lemma 3.2(vii) that $f(x_n) < x_n$ for all $n \geq 1$. It follows by (1.4), we can show that $f(z_n) < z_n$ and $f(y_n) < y_n$ for all $n \geq 1$. Using (1.4), we have

$$\begin{aligned} z_1^* - z_1 &= (x_1^* - x_1) + \gamma_1^*(f(x_1^*) - x_1^*) + \gamma_1(x_1 - f(x_1)) \\ &= (\gamma_1^* - \gamma_1)(f(x_1) - x_1) \\ &\leq 0, \end{aligned}$$

that is $z_1^* \leq z_1$. Since f is nondecreasing, we have $f(z_1^*) \leq f(z_1)$. By $z_1 - f(z_1) > 0$, it follows that

$$\begin{aligned} y_1^* - y_1 &= (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1(z_1 - f(z_1)) \\ &\leq (z_1^* - z_1) + \beta_1^*(f(z_1^*) - z_1^*) + \beta_1^*(z_1 - f(z_1)) \\ &= (1 - \beta_1^*)(z_1^* - z_1) + \beta_1^*(f(z_1^*) - f(z_1)) \\ &\leq 0, \end{aligned}$$

that is $y_1^* \leq y_1$. Since $f(y_1^*) \leq f(y_1)$ and $y_1 - f(y_1) > 0$, we have

$$\begin{aligned} x_2^* - x_2 &= (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1(y_1 - f(y_1)) \\ &\leq (y_1^* - y_1) + \alpha_1^*(f(y_1^*) - y_1^*) + \alpha_1^*(y_1 - f(y_1)) \\ &= (1 - \alpha_1^*)(y_1^* - y_1) + \alpha_1^*(f(y_1^*) - f(y_1)) \\ &\leq 0, \end{aligned}$$

that is $x_2^* \leq x_2$. Assume that $x_k^* \leq x_k$. Since $f(x_k^*) \leq f(x_k) < x_k$, we have $z_k^* - z_k \leq (1 - \gamma_k^*)(x_k^* - x_k) + \gamma_k^*(f(x_k^*) - f(x_k)) \leq 0$, that is $z_k^* \leq z_k$. Since $f(z_k^*) \leq f(z_k) < z_k$, we have $y_k^* - y_k \leq (1 - \beta_k^*)(z_k^* - z_k) + \beta_k^*(f(z_k^*) - f(z_k)) \leq 0$, so $y_k^* \leq y_k$, then $f(y_k^*) \leq f(y_k) < y_k$. It follows that

$$\begin{aligned} x_{k+1}^* - x_{k+1} &\leq (1 - \alpha_k^*)(y_k^* - y_k) + \alpha_k^*(f(y_k^*) - f(y_k)) \\ &\leq 0, \end{aligned}$$

that is $x_{k+1}^* \leq x_{k+1}$. By mathematical induction, we obtain $x_n^* \leq x_n$ for all $n \geq 1$. We note that $U < x_1^*$ and by using (1.4) and mathematical induction, we can show that $U \leq x_n^*$ for all $n \geq 1$. Hence, we have $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \to p$ and $\{x_n^*\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.
Case 2: $x_1 = x_1^* < L$. By Proposition 3.6, we get $f(x_1) > x_1$. In the same way as Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$. We note that $x_1^* < L$ and by using (1.4) and mathematical induction, we can show that $x_n^* \leq L$ for all $n \geq 1$. This implies $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. It follows that $x_n^* \to p$ and $\{x_n^*\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$.
Case 3: $L \leq x_1 = x_1^* \leq U$. Suppose that $f(x_1) \neq x_1$. If $f(x_1) < x_1$, we have by Lemma 3.2(vii) that $\{x_n\}_{n=1}^{\infty}$ is nonincreasing with limit p. By Lemma 3.3(i), we have $p \leq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \leq x_n$ for all $n \geq 1$, so $p \leq x_n^* \leq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we have that $x_n^* \to p$ and $\{x_n^*\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$. If $f(x_1) > x_1$, we have by Lemma 3.2(viii) that $\{x_n\}_{n=1}^{\infty}$ nondecreasing with limit p. By Lemma 3.3(ii), we have $p \geq x_n^*$ for all $n \geq 1$. By using the same argument as in Case 1, we can show that $x_n^* \geq x_n$ for all $n \geq 1$, so $p \geq x_n^* \geq x_n$ for all $n \geq 1$. It follows that $|x_n^* - p| \leq |x_n - p|$ for all $n \geq 1$. Hence, we obtain that $x_n^* \to p$ and $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{w_n\}_{n=1}^{\infty}$.

Table 1
Comparison of rate of convergence of the Mann, Ishikawa, Noor and SP-iterations for the given function in Example 3.10.

	Mann	Ishikawa	Noor	SP-iteration		
n	u_n	s_n	w_n	x_n	$\lvert f(x_n) - x_n\rvert$	$\lvert\frac{x_n-p}{x_{n-1}-p}\rvert$
10	1.13516510	1.10003535	1.09340527	1.02246989	1.7925E−02	6.7448E−01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
33	1.00006355	1.00004644	1.00004331	1.00000919	7.3542E−06	7.3219E−01
34	1.00004667	1.00003410	1.00003181	1.00000674	5.3932E−06	7.3335E−01
35	1.00003433	1.00002508	1.00002339	1.00000495	3.9611E−06	7.3447E−01

Table 2
Comparison of rate of convergence of the SP-iteration for the given function in Example 3.11.

	$\alpha_n = \frac{1}{n^{0.2}+1}$			$\alpha_n^* = 0.7$		
n	x_n	$\lvert f(x_n) - x_n\rvert$	$\lvert\frac{x_n-x_{n-1}}{x_n}\rvert$	x_n^*	$\lvert f(x_n^*) - x_n^*\rvert$	$\lvert\frac{x_n^*-x_{n-1}^*}{x_n^*}\rvert$
10	3.01562105	1.3019E−02	2.6712E−03	3.00031983	2.6652E−04	1.5452E−04
⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	3.00073037	6.0865E−04	1.0716E−04	3.00000027	2.2357E−07	1.2644E−07
19	3.00050902	4.2418E−04	7.3772E−05	3.00000011	9.2676E−08	5.2357E−08
20	3.00035599	2.9666E−04	5.1004E−05	3.00000005	3.8437E−08	2.1695E−08

Remark 3.9. From Theorem 3.8, the control sequences $\{\alpha_n^*\}_{n=1}^{\infty}$, $\{\beta_n^*\}_{n=1}^{\infty}$ and $\{\gamma_n^*\}_{n=1}^{\infty}$ in $[0, 1)$ can be chosen such that $SP(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$ converges to a fixed point of f but those sequences may not satisfy the control condition in Theorem 2.1.

Next, we will present two numerical examples. The first example shows that our iteration converges faster than the Mann, Ishikawa and Noor iterations for continuous and nondecreasing functions on an arbitrary interval and the second example compares the speed of convergence of the SP-iteration with respect to various control conditions. Throughout this section, we use $\alpha_n = \frac{1}{n^{0.2}+1}$, $\beta_n = \frac{1}{n^2+1}$ and $\gamma_n = \frac{1}{n^2+1}$.

Example 3.10. Let $f : [0, 8] \to [0, 8]$ be defined by $f(x) = \frac{x^2+9}{10}$. Then f is a continuous and nondecreasing function. The comparison of the convergences of the Mann, Ishikawa, Noor and SP-iterations to the exact fixed point $p = 1$ are given in Table 1, with the initial point $u_1 = s_1 = w_1 = x_1 = 4$.

From Table 1, we see that the SP-iteration converges faster than the Mann, Ishikawa and Noor iterations. The value $\lvert\frac{x_n-p}{x_{n-1}-p}\rvert$ is called the *order of convergence*. If $\lvert\frac{x_n-p}{x_{n-1}-p}\rvert \to K$ with $0 < K < 1$, the sequence $\{x_n\}_{n=1}^{\infty}$ is said to be *linear convergent*. From Table 1, the sequence $\{x_n\}_{n=1}^{\infty}$ is seem to be linear convergent.

Example 3.11. Let $f : [-6, \infty) \to [-6, \infty)$ be defined by $f(x) = \sqrt{x+6}$. Then f is a continuous and nondecreasing function. The comparison of the convergence for SP-iterations with new control conditions to the exact fixed point $p = 3$ is given in Table 2, with initial point $x_1 = x_1^* = 9$.

From Table 2, we see that $SP(x_1^*, \alpha_n^*, \beta_n, \gamma_n, f)$ converges faster than $SP(x_1, \alpha_n, \beta_n, \gamma_n, f)$. It clear that $\{\alpha_n^*\}_{n=1}^{\infty}$ does not satisfy the control condition in Theorem 2.1.

Acknowledgements

The authors would like to thank the referees for valuable suggestions on the paper and thank the Center of Excellence in Mathematics, the Commission on Higher Education, the Thailand Research Fund, and the Graduate School of Chiang Mai University for financial support.

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