Projectivity of Prime Quotients and Simple Lattices

F. A. Smith

Department of Mathematics, Kent State University, Kent, Ohio 44242

Communicated by R. H. Bruck
Received August 8, 1968

INTRODUCTION

It is clear that the only simple distributive lattices have at most two elements. Thus, for a nontrivial lattice to be simple it must be nondistributive. Most of the results about simplicity of lattices concern themselves with showing a certain type of lattice is simple if and only if prime quotients are projective. The purpose of this paper is to find a local condition which can be applied to discover when prime quotients are projective. We prove the following theorems.

THEOREM 1. A finite dimensional antidistributive semimodular lattice is simple if and only if it is locally complemented.

THEOREM 3. A finite dimensional antidistributive lattice is simple if each finite subset is contained in a locally complemented interval sublattice which satisfies the Jordan-Dedekind chain condition.

1. DEFINITION AND PRELIMINARIES

Let $L$ be a lattice. If $x, y \in L$ we write $x \succ y$ or $y \prec x$ if $x \geq y$ implies $a = y$. If $x \succ y$ let $[y, x] = \{a \in L: y \leq a \leq x\} = x \uparrow y$, and let $\theta_{x\uparrow y}$ be the smallest congruence relation on $L$ such that $x = y(\theta_{x\uparrow y})$. If the quotients $a \uparrow b$ and $c \uparrow d$ are translates we write $a \uparrow b \equiv c \uparrow d$ and if $a \uparrow b$ and $c \uparrow d$ are projective, we write $a \uparrow b \equiv P c \uparrow d$.

Definition. (i) A lattice is antidistributive if it is not distributive and when $a \succ b \succ c \succ d$, either $[c, a]$ or $[d, b]$ is a nondistributive sublattice.

(ii) An element $x \in L$ has a local complement if there are elements $y, z \in L$ such that $y \succ x \succ z$ and $x$ has a complement in $[z, y]$. If each $x \in L$, $0 \neq x \neq 1$ has a local complement, $L$ is locally complemented.

Copyright © 1974 by Academic Press, Inc.
All rights of reproduction in any form reserved.
(iii) A lattice is of finite length $n$ if there is a chain in $L$ of length $n + 1$ and every chain has at most length $n + 1$.

(iv) A lattice is finite dimensional if every interval sublattice has finite length.

(v) A lattice is semimodular if whenever $x, y > z$, we have $x \lor y > x, y$ and dually when $x, y < z$, we have $x \land y < x, y$.

**Proposition.** An element $x \in L$ has a local complement if and only if there is an $x'$ such that $x \lor x' > x > x \land x'$. Moreover, if $a > x > b$ and $L$ is semimodular, then $[b, a]$ is modular and if $[b, a]$ is nondistributive, then $a \mid x \lor T a \mid x' \lor b$ for all $x' \neq x$ such that $a > x' > b$.

**Theorem 1.** A finite dimensional antidistributive semimodular lattice is simple if and only if it is locally complemented.

**Proof.** If $x \in L$ does not have a local complement, then by semimodularity, $L = \{a \in L: a \geq x\} \cup \{b \in L: b \leq x\}$ and $L$ is not simple. For the converse, let $a > b > c$. We need only show $a \mid b \lor b \mid c$. Let $b'$ be a locally complement of $b$. Now $b \lor b' > b > b \land b'$ and by the preceding $[b \land b', b \lor b']$ is modular. If $b \lor b' \neq a$, let $x = b' \lor a$. Now $x > a$, $b \lor b' > b > c$ so $[c, a]$ or $[b, x]$ is nondistributive. If $[c, a]$ is nondistributive the preceding remark shows $a \mid b \lor b \mid c$. If $[b, x]$ is nondistributive, then $x > b \lor b' > b' > b \land b'$ so $[b \lor b', b \lor b']$ or $[b', x]$ is nondistributive. If $[b', x]$ is nondistributive then $a \mid b \lor b \mid b \lor b' \mid b \lor b' \mid b \lor b$ and if $[b \land b', b \lor b']$ is nondistributive then $a \mid b \lor b \mid b \lor b' \mid b \lor b \mid b \land b'$ or $a \mid b \lor b \mid b \land b'$. If $b \land b' = c$ we are finished if not $a > b > c$, $b \land b' > b' \land c$ so $[b' \land c, b]$ or

![Figure 1](image)
[c, a] is nondistributive, whence \( b \mid b \wedge b' P b \mid c \) and \( a \mid b P b \mid c \). Finally, if \( b \lor b' = a \) and \( b \wedge b' = c \), either \( a = 1 \), \( c = 0 \), there is an \( x \) such that \( x > a > b > c \) or there is a \( y \) such that \( a > b > c > y \). The first is clear. If \( x > a > b, b' > c \), then \([c, a] \) is nondistributive and \( a \mid b P b \mid c \) or \([b, x] \) and \([b', x] \) are nondistributive and \( a \mid b P x \mid a P a \mid b' P b \mid c \). The last case is clearly the dual of the second.

If we delete semimodularity from the above theorem neither direction is true in general. However, we do have the following.

**Remark.** If \( L \) is an antidiistributive locally complemented lattice of length \( \leq 3 \), \( L \) is simple, and Fig. 1 is the graph of an antidiistributive locally complemented lattice of length 4 which is not simple.

**Proof.** It is not simple since \( a \not\equiv b(\theta_{1|\omega}) \).

### 2. Lattices Satisfying the Jordan–Dedekind Chain Condition

**Definition.** A lattice \( L \) satisfies the Jordan–Dedekind chain conditions if all maximal chains between the same endpoints have the same finite length.

**Proposition.** If \( L \) satisfies the Jordan–Dedekind chain condition, \( L \) is finite dimensional and if \( a \succ x \succ b \), then \([b, a] \) is modular.

Throughout the remainder of this section we shall assume \( L \) is a locally complemented antidiistributive lattice which satisfies the Jordan–Dedekind chain condition.

**Lemma.** If \( L \) has finite length \( k \geq 3 \), \( 1 \succ x_{1} \succ x_{2} \succ \cdots \succ x_{k} = 0 \) and \( \theta_{x_{i}|x_{i-1}} = \theta_{x_{1}|x_{0}} \) for every local complement \( x' \) of \( x \).

**Proof.** It is clear \( \theta_{x_{i}|x_{i-1}} = \theta_{x_{1}|x_{0}} \) for \( i = 1, 2, \ldots, n \). If \( n = 1 \) there is a \( a \in L \) such that \( a \succ x \lor x' \succ x, x' \succ 0 \). Thus, by antidiistributivity and the two propositions, \( x \lor x' \mid x P x \mid 0 \). Assume \( n \geq 2 \), and let \( x' \) be a local complement of \( x \). Now there is a \( v \in L \) such that \( x \lor x' \succ x, x' \succ x \land x' \lor v \), so \( x \lor x' \mid x P x \mid x \land x' \). Now if \( x_{1} \land x' = 0 \) then \( x_{1} \mid 0 T x \mid x \land x' \) and \( \theta_{x_{1}|x_{0}} = \theta_{x_{0}|x_{0}} \). If \( x_{1} \land x' \neq 0 \) then \( x_{1} \leq x' \) since \( x_{1} > 0 \). Thus since the Jordan–Dedekind chain condition is satisfied, there is an \( i \) such that \( x_{i} \land x' = x_{i-1} \), so \( x \mid x \land x' T x_{i} \mid x_{i-1} \) and \( \theta_{x_{i}|x_{i-1}} = \theta_{x_{0}|x_{0}} \).

**Corollary.** If \( L \) is of finite length \( n \geq 3 \), \( 1 \succ a \succ x_{1} \succ x_{2} \succ \cdots \succ x_{n} = 0 \) and \( \theta_{x_{i}|x_{i-1}} = \theta_{x_{1}|x_{0}} \), then \( \theta_{x_{i}|x_{j}} = \theta_{x_{1}|x_{0}} \).
Proof. First notice we can find $x_i$, $i = n + 1, \ldots, k$, such that each $x_i$ has a local complement $x_i'$ such that

$$x_i \lor x_i' = x_{i+1}, \quad n \leq i < k$$

so $\theta_{x_i|x_{i-1}} = \theta_{1|0}$ for $1 \leq i \leq k$.

If $a \lor x_{k-1} = 1$, then $1 \mid x_{k-1} \mid a \mid x$ and $\theta_{a|x} = \theta_{1|x_{k-1}} = \theta_{1|0}$. If $a \lor x_{k-1} \neq 1$, then $a \leq x_{k+1}$ and by the dual of the argument in the preceding theorem $\theta_{a|x} = \theta_{1|x_{n-1}} = \theta_{1|0}$. We are now able to prove the following theorem.

**Theorem 2.** If $L$ is a locally complemented antidistributive lattice of finite length which satisfies the Jordan-Dedekind chain condition, then $L$ is simple.

Proof. We need only show $\theta_{a|b} = \theta_{1|0}$ for every prime quotient $a \mid b$. If $a > b$ there is a chain $0 = x_0 < x_1 < \cdots < x_k = 1$ such that $a \mid b = x_i \mid x_{i-1}$ for some $i$. But by the preceding corollary, $\theta_{x_i|x_0} = \theta_{x_i|x_1} = \cdots = \theta_{a|b} = \theta_{1|0}$, so $L$ is simple.

**Definition.** If $L$ is a lattice of finite length, let $L[i]$ be the subset of elements which belong to some maximal chain of length $i$.

Figure 2
**Corollary.** If $L$ is a lattice of finite length such that for each $i$, $L[i]$ is a locally complemented antidistributive lattice, which satisfies the Jordan–Dedekind chain condition, then $L$ is simple.

**Theorem 3.** A finite dimensional antidistributive lattice is simple if each finite subset is contained in a locally complemented interval sublattice which satisfies the Jordan–Dedekind chain condition.

**Proof.** Let $a > b$ and $x, y \in L$. Now there is an interval sublattice containing $a, b, x, y$ and satisfying the hypothesis of Theorem 2 so $x \equiv y(\theta_{a|b})$ and $L$ is simple.

Notice in the above theorem that assuming the Jordan–Dedekind condition globally is the same as in the interval sublattice. However we may not assume $L$ is locally complemented since Fig. 2 is the graph of a nonsimple antidistributive locally complemented lattice, which satisfies the Jordan–Dedekind chain condition since $\theta_{a|b} > \theta_{b|c}$.

**References**