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Linear Groups over $GF(2^k)$ Generated by a Conjugacy Class of a Fixed Point Free Element of Order 3

P. Fleischmann, W. Lempken, and A. E. Zalesskii

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0. INTRODUCTION

Motivated by applications to the problem of the classification of finite simple groups Wilson [23] has determined irreducible representations ϕ (over a field of characteristic 2) of quasi-simple Chevalley groups G over $GF(2^k)$ such that an element $g \in G$ of order 3 does not have eigenvalue 1. For $G \cong A_n$, the alternating group of degree n, a similar problem has been solved by Mullineux [20]. The classification of the finite simple groups allows one to approach many classical and new problems of recognition of finite linear groups with a given property by using representation theory. From this point of view Wilson and Mullineaux started to classify subgroups G of $GL(n, 2^k)$ which contain a fixed point free element of order 3. In this paper we provide a full classification of such groups G under the natural restriction that G is irreducible and is generated by the conjugacy class of this element. In the more general situation, where an arbitrary subgroup $G \leq GL(V)$ contains a noncentral fixed point free element g of order 3, our result can be used to identify the irreducible constituents of the restriction $V|_H$ where H is the normal subgroup of G generated by g.

Let $E(G)$ denote the layer of G, i.e., the maximal semisimple normal subgroup of G. Recall that if $E(G) \neq 1$ then $E(G)$ is generated by the components of G, i.e., the quasi-simple subnormal subgroups of G. Then the main result of this paper is the following.

THEOREM 0.1. *Let $G \subset GL(n, 2^k)$, $n > 1$, be an absolutely irreducible subgroup and $g \in G$ a fix point free element of order 3. Suppose that G is*

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generated by g^G, *the conjugacy class of* g *in* G. *Then exactly one of the following holds*:

(1) *The layer* $E(G)$ *is a finite group of Lie type in characteristic two. If* $G = E(G)$, *the complete list for* G *and the irreducible representations of* G *with the property in question are given in* [23]. *If* $E(G) < G$ *then* $E(G)$ *is of type* E_{A_l} *with* $E \in \{1, 2\}$ *and* $Eq \equiv 1(3)$ *and a similar list is given in Theorem* 8.9.

(2) $G \cong L_2(9)$, $U_3(3)$, $PSp_4(3)$, *or* $3_1U_4(3)$.

(3) $G \cong PSp_{2m}(3) \times \mathbb{Z}_3$ *with* $m \geq 2$.

(4) $G \cong Alt_m$ *with* $m \geq 5$.

(5) $G \cong J_2$, Co_1, *or* $3Suz$.

(6) $G = E : S$ *where* $E = F^*(G) = O_3(G)$ *is extraspecial of order* 3^{1+2m} *and exponent* 3, *where* $S \cong Sp_{2m}(3)$ *for some* $m \geq 1$.

Moreover, in most cases the conjugacy class of g *in* G *as well as the representation in question and hence the parameter* n *can be specified* (*for details consult the relevant sections*).

The paper is organized as follows: section 1 will give some necessary definitions and general observations about representations with the property in question. Section 2 collects some technical facts about automorphisms of finite groups of Lie type, which will be needed later. Section 3 contains the classification of those groups with the designed property and solvable generalized Fitting subgroup, whereas Sections 4 and 5 deal with those groups involving sporadic simple or alternating components. The remaining cases, involving components of Lie type defined in characteristic p, naturally split into three cases:

(i) If $p > 3$, the element g is semisimple and the representation is in "cross-prime" characteristic. This is analyzed in Section 6.

(ii) If $p = 3$, the representation still is in "cross-prime" characteristic, but g is unipotent. This is dealt with in Section 7.

(iii) If $p = 2$, then g is semisimple and the representation is in the "defining" characteristic, which allows the use of an ambient algebraic group and its rational representations. In particular we can use [23] and adapt Wilson's method to our slightly more general situation. This is done in Section 8.

1. PRELIMINARIES

We will use standard notation, in particular, $F(G)$ denotes the Fitting subgroup and $F^*(G)$ the generalized Fitting subgroup of G. Moreover, $G^\#$ denotes the set of nontrivial elements in G. We denote by $H : G$ the semi-direct product of H and G with normal subgroup H and $|G:X|$ the index of a subgroup X in G.

In the following all vector spaces, modules, and representations are finite dimensional. Groups under consideration are finite unless specified otherwise. Below we collect a number of preliminary results that provide information about subgroups of $GL(V)$, normalized by a fixed point free element $g \in GL(V)$. For the sake of convenience we first fix some additional notation.

DEFINITION 1.1. The triple (G, C, V) is said to satisfy the condition **FFG(p, r)** if the following hold:

(1) G is a finite subgroup of $GL(V)$ where V is a finite-dimensional vector space over the field $\mathbb{F}$ with characteristic p.

(2) G is generated by a conjugacy class $C = g^G$ such that g is a noncentral element of prime order r acting fixed point freely (i.e., without eigenvalue 1) on V.

If in addition V is a simple G-module, (G, C, V) is said to satisfy **FFG(p, r)***.

An involution of $GL(V)$ acting fixed point freely on the vector space V is in fact $-id|_V \in Z(GL(V))$; so we have

LEMMA 1.2. *If (G, C, V) satisfies $FFG(p, r)$ then r is an odd prime. Moreover, if (G, C, V) satisfies $FFG(p, r)^*$ then $O_p(G)$ acts trivially on V, and thus $O_p(G) = 1$.*

In situations with elements acting fixed point freely on modules the following results prove to be very useful.

LEMMA 1.3. *Let V be a finite dimensional vector space over a finite field of characteristic $p > 0$. Suppose that Q is a p'-subgroup of $GL(V)$ and that $R \leq GL(V)$ is cyclic of prime order r such that R normalizes Q and $C_R(Q) = 1$. Then the following hold:*

(1) (Higman Lemma) *If Q is abelian then $V|_R$ contains a regular R-submodule; in particular, $C_V(R) \neq 0$.*

(2) *Suppose that Q is a q-group for some prime $q \neq r$; if $q = 2$ and r is a Fermat prime assume in addition that Q is abelian. Then $C_V(R) \neq 0$.*

(3) *If p, r, and $|Q|$ are mutually coprime and $|Q|r$ odd then $C_V(R) \neq 0$.*

Proof. Consult [13, Theorem IX.1.10; 1, (36.2); and 4, Theorem 3.4] ∎

LEMMA 1.4. *Let $G \leq GL(V)$ where V is a vector space over a field of characteristic $p \neq 0$; moreover, let r be a prime different from p. Then the following hold*:

(1) *If $X < G$ is elementary abelian of order r^2 then $V = \langle C_V(x) \mid x \in X^{\#} \rangle$; in particular, X contains at most $r - 1$ different subgroups $\langle x \rangle$ of order r such that x acts fixed point freely on V.*

(2) *Suppose that $g \in G$ with $o(g) = r$ and $C_V(g) = 0$. If A is an abelian g-invariant r-subgroup of G, then $g \in C_G(A)$. In particular, if $R \in \mathrm{Syl}_r(G)$ then $\langle g^G \cap R \rangle$ is contained in every maximal abelian normal subgroup of R.*

(3) *Suppose that (G, C, V) satisfies $FFG(p, r)$; if $H \leq G$ such that $C_U(x) \neq 0$ for some simple H-module U involved in $V|_H$ and for any $x \in H$ with $o(x) = r$, then $H \cap C = \varnothing$.*

Proof. Part (1) is well known (e.g. see [7, Theorem 3.3.3] and (2) is clear by Lemma 1.3; (3) is obvious as well.

LEMMA 1.5. *Suppose that (G, C, V) satisfies $FFG(p, r)^*$ and let $g \in C$. Then the following statements hold*:

(1) *If $G = Z \times G_1$ with $Z \cong \mathbb{Z}_r$ then $g = z \cdot h$ with $z \in Z, h \in G_1$, and $o(z) = o(h) = r$; moreover, $\langle h \rangle^{\#}$ is not contained in h^G.*

(2) *If $G = Z * G_1$ with $Z \cong \mathbb{Z}_{r^2}$ and $Z_0 := Z \cap G_1 \cong \mathbb{Z}_r$ then $g = z \cdot h$ with $z \in Z, h \in G_1$, $o(z) = o(h) = r^2$, and $h^r = z^{-r} \in Z_0$; in particular, any $x \in Z_0$ has an rth root in G_1.*

Proof. Part (1) follows from Lemma 1.4(1) and Part (2) is obvious.

LEMMA 1.6. *Suppose (G, C, V) satisfies $FFG(p, r)$. Then the following hold*:

(1) *G is primitive on every simple submodule $W \leq V$.*

(2) *Let $Q \leq G$ be a q-group for a prime $q \notin \{p, r\}$; if $q = 2$ and r is a Fermat prime assume that Q is abelian. Then $N_G(Q) \cap C \subset G_G(Q)$.*

(3) *If Q is a $\{2, p, r\}'$-subgroup of G then $N_G(Q) \cap C \subset C_G(Q)$; in particular, $O_{\{2, p, r\}'}(G) \leq Z(G)$.*

(4) *Any abelian normal p'-subgroup of G is contained in $Z(G)$.*

Proof. (1) Suppose $W \leq V$ is not primitive; then, as $G = \langle g^G \rangle$, g permutes the imprimitivity spaces nontrivially. Let $v \neq 0$ be an element of such a space. Then the vectors $v, gv, \ldots, g^{r-1}v$ are linearly independent, and g permutes these vectors. It follows that g fixes $0 \neq v + gv + \cdots + g^{r-1}v$, in contradiction to $FFG(p, r)$.

(2) This follows from Lemma 1.3.

(3) This follows from Part (2) together with the Frattini argument.

(4) This follows from Part (2), Lemma 1.4(2), and the fact that $G = \langle C \rangle$.

COROLLARY 1.7. *Suppose that (G, C, V) satisfies $FFG(p, r)^*$. Then the following hold*:

(1) *Every abelian normal subgroup of G is central and cyclic.*

(2) *Let $q \in \{2, r\}$ and suppose that $Q := O_q(G) \nleq Z(G)$. Then Q is of symplectic type with $Q = Q_0 * Q_1$ where $Q_0 = Z(Q) \leq Z(G)$ and Q_1 is extraspecial.*

If $q = r$ then, in addition, $Q_1 = \Omega_1(Q) \triangleleft G$ and $\exp(Q_1) = r$.

If $q = 2$ then r is a Fermat prime.

(3) *If G has nontrivial layer $E(G)$ then $E(G)$ is a central product $E(G) = E_1 * \cdots * E_l$ with quasi-simple components E_i which are normal in G.*

Proof. Clearly, G acts irreducibly and faithfully on V, so $O_p(G) = 1$; hence, all abelian normal subgroups are p'-subgroups and central. By Schur's Lemma $Z(G)$ is scalar; hence $Z(G)$ is cyclic and (1) holds.

For (2) consult [12, Theorem III.13.10] and note that in the case $q = 2$ and Q_0 is noncyclic of order at least 16 the group $Z(C_Q(Q'))$ is cyclic of index 2 in Q_0.

For (3) we observe that any $g \in C$ permutes the components E_i of $E(G)$ via conjugation; so the claim easily follows from 1.3.

The next lemma shows that in order to analyze groups satisfying $FFG(p, r)^*$, we may assume that the module V is absolutely irreducible and $\mathbb{F}$ is a finite splitting field of G and all its subgroups. In particular we can assume that $\mathbb{F}$ contains all rth roots of unity.

LEMMA 1.8. *Suppose that (G, C, V) satisfies $FFG(p, r)^*$ with a module V over the finite field $\mathbb{F}$ and let $K \geq \mathbb{F}$ be some finite extension field. Then there is an irreducible faithful KG-module W satisfying $C_W(g) = 0$ for all $g \in C$, which is a direct summand of $V^K := V \otimes_{\mathbb{F}} K$.*

Proof. It is well known that $V^K = \sum^{\oplus} W^{\sigma}$, where W is some simple constituent of V^K and the σ's run through coset representatives of the stabilizer Γ_W in $\Gamma := Gal(K : \mathbb{F})$ (see [1, 26.2, p. 124]). Since $g \in C$ does not have eigenvalue 1 on V we get $C_W(g) = 0$. Since the actions of Γ and G on V^K commute, $C_G(W) = \bigcap_{\sigma} C_G(W^{\sigma}) = C_G(V^K) = 1$.

LEMMA 1.9. *Suppose (G, C, V) satisfies $FFG(p, r)^*$ and the field $\mathbb{F}$ contains all rth roots of unity. Let $N \trianglelefteq G$ and S be a simple constituent of the restriction $V|_N$. Then for any $g \in C$ there is a simple $\langle N, g \rangle$-module $U \leq V|_{\langle N, g \rangle}$ with $U|_N \cong S$.*

Proof. We apply a "correct version" of [1, 27.17, p. 134]: The statement there is: "Let V be an irreducible $\mathbb{F}G$-module. Assume G is finite and a semidirect product of $H \trianglelefteq G$ by X of prime order p. Assume $\dim(V) \neq p(\dim C_V(X))$. Then V is a homogeneous $\mathbb{F}H$-module and if $\mathbb{F}$ is finite of order prime to p then V is an irreducible $\mathbb{F}H$-module."

Let $\mathbb{F} := GF(q)$, p be a prime with $ord_p q = p - 1$, $V := \mathbb{F}(\alpha)^+$ with $\alpha \in \overline{\mathbb{F}}^*$, and $ord(\alpha) = p$, $H := 1$, and $G := \langle \alpha \rangle$. Then V is an irreducible $\mathbb{F}G$-module of dimension $p - 1$. Clearly $V|_H$ is homogeneous without being irreducible. Note that if in the original statement of 27.17 we replace "$\mathbb{F}$ is finite of order prime to p," with "$\mathbb{F}$ is finite with $|\mathbb{F}| \equiv 1 \bmod p$," then the statement will be true and it is easy to see how to fix the proof given on page 135 of [1].

In our situation we know already that $V|_N$ is homogeneous. Now fix $g \in C$, define $H := N\langle g \rangle$, and let U be a simple submodule of the H-closure $\sum_{h \in H} S^h$. Then $C_U(g) = 0$ and the correct version of the statement above yields $U|_N \cong S$.

LEMMA 1.10. *Suppose (G, C, V) satisfies $FFG(2,3)^*$ and let $g \in C$. If $N \trianglelefteq G$ is a central product $N_1 * N_2$ of g-invariant subgroups N_1 and N_2, then $N_i \leq C_G(g)$ for some i; in particular if N_i is normal in G then $N_i \leq Z(G)$.*

Proof. We can assume that $\mathbb{F}$ is a splitting field for all subgroups of G. Suppose first that $g \notin N$. Let $H_i := \langle N_i, g \rangle$ and $H := \langle H_1, H_2 \rangle = N\langle g \rangle$. Let W be an N-simple summand of $V|_N$. Note that W is faithful since $V|_N$ is homogeneous. Then $W \cong S_1 \otimes S_2$ where S_i is a faithful simple N_i-module. For $i \in \{1, 2\}$ let $\tilde{S_i} \cong S_i$ be an N_i-submodule of W and consider the H_i-closure $U_i := \sum_j \tilde{S_i} \cdot g^j$. Since g acts fixed point freely on U_i, this module is not induced from an N_i-module; therefore the restriction $U_i|_{N_i}$ is homogeneous. Hence by [4, Proposition 2.2] we get $U_i|_{N_i} \cong S_i$. Consider the $\mathbb{F}H$-module $M := U_1 \otimes_{\mathbb{F}} U_2$ with $(u_1 \otimes u_2)((n_1, n_2), g^j) = (u_1 n_1 g^j) \otimes (u_2 n_2 g^j)$ for all $u_i \in U_i$. By Lemma 1.9 we know that $V|_H$ contains some extension U of W to an H-module. Since $\langle g \rangle \cong H/N$, application of Lemma 1.12 below shows that there is some linear character χ of H/N such that $M \otimes \mathbb{F}_\chi$ appears in $V|_H$. Suppose now that neither N_1 nor N_2 are centralized by g, then the set of eigenvalues of g on S_1 and S_2 must be $\{\omega, \omega^{-1}\}$ with $1 \neq \omega \in \mathbb{F}$ of order three. (Otherwise g would act by scalar multiplication on S_1, say, and $[N_1, g] \leq \ker N_1|_{S_1} = 1$.) Hence the set of eigenvalues of g on M is $\{1, \omega, \omega^2\}$ and the same is true for the set of eigenvalues of g on $M \otimes \mathbb{F}_\chi$, which is a contradiction. The same argument immediately applies if $G = N$. So we conclude that, say $N_i \leq C_G(g)$. If N_i is normal in G then $C_G(N_i)$ contains C and hence G.

LEMMA 1.11. *Suppose (G,C,V) satisfies FFG$(2,3)^*$, then $F^*(G) = Z(G) * E(G)$ with $E(G)$ quasi-simple or $F^*(G) = F(G) = Z(G) * E$ with $E := \Omega_1(O_3(G))$ extraspecial of exponent 3.*

Proof. Since $F^*(G) = F(G) * E(G) \neq Z(G)$ and $F(G) = \prod_{r \in \pi(G)} O_r(G)$ as well as $O_2(G) = 1$ the claims follow immediately from Lemma 1.10 and Corollary 1.7.

Finally we record a result of a more general nature.

LEMMA 1.12. *Let G be a finite group, N a normal subgroup of G, and $\mathbb{F}$ a splitting field for G and all its subgroups. Suppose that W is an $\mathbb{F}G$-module such that the restriction $V := W|_N$ is a simple $\mathbb{F}N$-module. Then the following hold:*

(1) *Suppose that U is a simple $\mathbb{F}G$-module and that V is a constituent in $U|_N$. Then there exists a simple $\mathbb{F}(G/N)$-module X such that $U \cong W \otimes_{\mathbb{F}} X$.*

(2) *If W' is another extension of V to G then $W' \cong W \otimes_{\mathbb{F}} X$ for a one-dimensional $\mathbb{F}(G/N)$-module X.*

Proof. Use Nakayama reciprocity and the simplicity of U to see that U is an epimorphic image of the induced module $V^G \cong (W_N)^G \cong W \otimes_{\mathbb{F}} (G/N)$. Now use a composition series of the $\mathbb{F}G$-module $\mathbb{F}(G/N)$ together with the Jordan–Hoelder theorem and [13, Theorem VII.9.12] to verify the claim in (1).

Note that (2) is an immediate consequence of (1).

2. SOME FACTS ON GROUPS OF LIE TYPE

In this section we collect some results on finite groups of Lie type which will be used later in our investigation and which might be of some general interest.

LEMMA 2.1. *Let G be a finite Lie type group defined over a field of characteristic p and let $\alpha \in \mathrm{Aut}(G)$ be an automorphism of prime order satisfying the following condition:*

$$\text{any } \alpha\text{-invariant } p\text{-subgroup of } G \text{ is contained in } H := C_G(\alpha). \qquad (*)$$

Then exactly one of the following holds:

(1) *p does not divide $|H|$ and α is the product of an inner and a diagonal automorphism of G; in particular, α does not normalize any nontrivial p-subgroup of G.*

(2) *G is of type A_1, ${}^2A_2$, ${}^2B_2$, or ${}^2G_2$ and $U \le H \le N_G(U)$ for some $U \in \mathrm{Syl}_p(G)$; moreover, α is induced by an element of $Z(U)$ and so is inner.*

Proof. Without loss we may assume that $Z(G) = 1$ and hence $G \le A := \mathrm{Aut}(G)$. If H is a p'-group then, by Theorem 9.1 of [8], we easily verify that α is the product of an inner and a diagonal automorphism of G.

Henceforth we may assume that p divides $|H|$. Let V be a nontrivial p-subgroup of H and $V \le U \in \mathrm{Syl}_p(G)$. Clearly, α normalizes both $X := N_G(V)$ and $Y := O_p(X)$; in particular, $V \le Y \le H$ and $Z(U) \le X$. Also, by [8, (13.4)], X is p-constrained with $O_{p'}(X) = 1$; hence $\langle \alpha^X \rangle \cap X \le C_X(Y) \le Y$ and thus $[X, \alpha] \le Y$. Consequently α normalizes and thus centralizes any Sylow p-subgroup of X; in particular, $Z(U) \le O^{p'}(X) \le H$.

Replacing V with $Z(U)$ in the arguments above we get $U \le O^{p'}(N_G(Z(U))) \le H$. This in turn implies $H \ge G_0 := \langle O^{p'}(N) \mid N \in P \rangle$, where P denotes the set of maximal parabolic subgroups of G containing $B := N_G(U) = U : T$; in particular, $\alpha \in C_A(U) = Z(U)$ and thus $o(\alpha) = p$ as well as $H \le N_0$ for some $N_0 \in P$ (see [8, (13.1) and (13.2)]).

Note that $N = O^{p'}(N)T$ for any $N \in P$. If G has rank at least 2 then $G = \langle N \mid N \in P \rangle$ and thus $G = G_0 T = HT \le N_0$; but this is absurd. Therefore G is a Lie type group of rank 1, i.e., G is of type A_1, ${}^2A_2$, ${}^2B_2$, or ${}^2G_2$.

LEMMA 2.2. *Suppose the triple (G, C, V) satisfies FFG$(2,3)^*$ where $E(G)/Z(E(G))$ is a simple group of Lie type. Then $g \in C$ induces an automorphism on $E(G)/Z(E(G))$ which is the product of an inner and a diagonal one.*

Proof. Suppose the claim is false and put $E := E(G)$, $Z := Z(G)$ and $G_0 := E\langle g \rangle$ for some fixed $g \in C$ as well as $H := C_E(g)$. Note that now $G \ne E$.

Since $C_G(E) = Z$, $Z(G_0) = Z(E) = Z \cap E$; in particular, as E is quasisimple, $G_0 = \langle C_0 \rangle$ with $C_0 := g^E = g^{G_0}$. Note that (G_0, C_0, V_0) satisfies the hypothesis of the lemma for some suitable constituent V_0 of $V|_{G_0}$. Therefore we may assume without loss that $G = G_0$.

Note that if E is isomorphic to one of the exceptional covering groups $3L_2(9)$, $3G_2(3)$, $3U_4(3)$, or $3\Omega_7(3)$, then $\mathrm{Out}(E)$ is a 2-group. As this conflicts with $o(g) = 3$ and $G \ne E$, E is a quasi-simple Lie type group not isomorphic to an exceptional covering of a Lie type group.

Suppose next that $E \cong {}^2B_2(q)$ with $q = 2^{2m+1} \ge 8$; so g induces a field automorphism on E and $H \cong {}^2B_2(q_0)$ with $q_0^3 = q$ (see [8, (9.1)). Since E is a $3'$-group, the Frattini argument together with Lemma 1.6(2) implies that $|E : H|$ is a power of 2, which of course is impossible. Hence E is not of type ${}^2B_2$. In particular, there exists a g-invariant $S \in \mathrm{Syl}_3(E)$ such that $S_0 := S \cap H \in \mathrm{Syl}_3(H)$ and $S_0 \ne 1$.

Now suppose that g induces a field or a graph-field automorphism on E and fix some $t \in S_0$ with $o(t) = 3$. By [8, (7.2)] we know that $\langle t \rangle \cdot g \subset C$; but then $\langle t, g \rangle \cong E(3^2)$ violates the conclusion of 1.4(1).

The last contradiction together with [8, (9.1)] now shows that g induces a graph automorphism on E and that $E \cong D_4(q)$ or ${}^3D_4(q)$ for some suitable prime power q. Furthermore, in any case there are exactly two G-classes of elements of order 3 contained in $E \cdot g$, say, with representatives g_1 and g_2; moreover, $C_E(g_1) \cong G_2(q)$ and $C_E(g_2) \cong PGL_3^{\epsilon}(q)$ where $\epsilon \in \{0, 1, -1\}$ with $q \equiv \epsilon \bmod 3$ and $PGL_3^0(q) \cong [q^5]: SL_2(q)$. In particular, without loss we may assume that $g \in \{g_1, g_2\}$.

Suppose next that 3 divides q. An easy order check shows that $|S : S_0| = |S_0| = q^6$; in particular, $|C \cap \tilde{S}| \geq q^6$ for $\tilde{S} := S\langle g \rangle$. By 1.4(2) we know that $\langle C \cap \tilde{S} \rangle$ is an elementary abelian subgroup of $\tilde{S}$ and is thus contained in $S_0\langle g \rangle$. Comparing orders we get $\langle C \cap \tilde{S} \rangle = S_0\langle g \rangle$; this in turn implies that S_0 is elementary abelian. As the groups $C_E(g_i)$ $(i = 1, 2)$ do not have abelian Sylow subgroups for the defining characteristic, we have derived a contradiction. Therefore 3 does not divide q.

If q is odd, 1.6(2) and 2.1 show that $\gcd(q, |H|) = 1$; as this contradicts the information on the possible H given above, q is a power of 2.

Now let $\mathbf{G}$ denote an algebraic group of type D_4 such that E is the group of fixed points of a (possibly twisted) Frobenius endomorphism F. Then there is $\gamma \in \mathrm{Aut}(\mathbf{G})$, a diagram automorphism of order 3, commuting with F and normalizing an F-maximally split torus $\mathbf{T}$ contained in an F-stable Borel subgroup. Looking at the proof of [8, (9.1), p. 104], we see that the elements g_1 and g_2 can be chosen as γ and $t\gamma$ with $t \in \mathbf{T}$. In particular, g_1 and g_2 normalize $\mathbf{T}$ and act on this group in the same way as γ.

Now, according to [8] (9.1) pg. 105, we see $T := \mathbf{T}_F \cong (\mathbb{Z}_{q-E})^4$ if $\mathbb{Z}_E$ is of type $D_4(q)$ and $T := \mathbf{T}_F \cong \mathbb{Z}_{q^3-E} \times \mathbb{Z}_{q-E}$ if E is of type ${}^3D_4(q)$ with $E \equiv q \bmod 3$. In the first case g_1 and g_2 permute three cyclic factors of T transitively; in the second case we have $t^{t_i} = t^{q^{-1}}$ for a generator of the factor $\mathbb{Z}_{q^3-1}$. Hence the g_i's do not centralize T. Since q is even, T is an abelian group of odd order which is normalized but not centralized by the g_i's. As this conflicts with 1.6(2) and 1.4(2) we have derived a final contradiction.

THEOREM 2.3. *Let G be a quasi-simple Lie type group over a field of characteristic p, F an algebraically closed field of characteristic $f \neq p$, and ϕ an irreducible representation of G over F. Then the following hold*:

(1) [25] *Suppose that there exists an element $g \in G$ of order p such that the minimal polynomial of $\phi(g)$ is of degree less than p. Then $p > 2$ and G is isomorphic to $SL_2(p)$, $SU_3(p)$, $SL_2(p^2)$, $Sp_4(p)$, or $Sp_{2n}(p)$ with g being a transvection except for $G = Sp_4(p)$ where g can also be a nontransvection.*

(2) [26] *Suppose that there exists an element $g \in G$ of order p such that $\phi(g)$ does not have the eigenvalue* 1. *Then G is isomorphic to $SL_2(p)$, $SU_3(p)$, $SL_2(p^2)$, or $Sp_4(p)$.*

Originally in [25] only groups G have been considered that can be obtained from algebraic groups. For our purpose we need to refine this for central extensions of these groups with $f = 2$, so we can assume that the center of the extension is of odd order. These possibilities occur only for groups $G \in \{A_1(9), B_3(3), G_2(3), SU_4(3)\}$. The Brauer character tables in [17] show that the result is true for the exceptional central extensions as well. Observe that [25] and [26] are published in Russian; an English exposition is available in [27].

3. SOLVABLE $F^*(G)$

The running hypothesis for this section will be as follows:

(SOL) G is a finite group such that the triple (G, C, V) satisfies $FFG(2,3)^*$ and $E := \Omega_1(O_3(G))$ is extraspecial of exponent 3 and order 3^{1+2m} $(m \geq 1)$.

Remark. Note that by (SOL) together with Lemma 1.11 and Lemma 1.2 we also have

$$Z := Z(G) = \langle z \rangle \text{ is cyclic of odd order,}$$

$$Z_0 := Z(E) = Z \cap E \cong \mathbb{Z}_3,$$

and

$$F := F^*(G) = F(G) = Z * E.$$

LEMMA 3.1. *$|E : C_E(g)| = 3$ for any $g \in C$; in particular, g acts as a transvection on E/Z_0.*

Proof. Let $g \in C$ and put $E_0 := C_E(g)$ as well as $\tilde{E} := E/Z_0$. Clearly, $Z_0 \leq E_0$ and $g \notin E$.

Suppose now that there exists $x \in G$ with $g^x = z_0 \cdot g$. As $Z_0 \leq Z$, $g^{x^2} = z_0^2 \cdot g$ and hence $\langle z_0, g \rangle$ is an elementary abelian group of order 9 violating Part (1) of 1.4. Thus we have

$$(1) \quad C_G(g \bmod Z_0) = C_G(g),\ E_0 = C_E(g \bmod Z_0), \text{ and } [E, g] \nleq Z_0,$$

where $C_G(g \bmod Z_0) = \{x \in G : [g, x] \in Z_0\}$. Now consider $\tilde{E}$ as a $\langle g \rangle$-module; so $\tilde{E} \cong \sum_i \mu_i M_i$ where M_i is an indecomposable $\mathbb{F}_3\langle g \rangle$-module of dimension i for $1 \leq i \leq 3$ and where μ_i is the multiplicity of M_i as a direct summand of $\tilde{E}_{\langle g \rangle}$. In view of (1) and the results in 1.10 we get $\mu_2 + \mu_3 = 1$.

Assume next that $\mu_2 = 0$ and $\mu_3 = 1$. So there exists a g-invariant subgroup H in E containing Z_0 such that $H/Z_0 \cong M_3$ as $\langle g \rangle$-modules.

Clearly, by (1), $H_0 := C_H(g)$ contains Z_0 with index 3, so $H_0 = \langle h_0, z_0 \rangle$. Since $h_0^H \subseteq H_0 \setminus Z_0$, $C_H(H_0) = C_H(h_0) > H_0$. Hence there is $v \in C_H(H_0) \setminus H_0$ with ${}^g v \equiv v \bmod(H_0)$ and $H_1 := \langle v, H_0 \rangle$ is an abelian g-invariant subgroup of H. Hence, by 1.4(2), g centralizes H_1, which is a contradiction.

We have shown that $\mu_3 = 0$ and $\mu_2 = 1$. In view of (1) the claimed results are immediate.

LEMMA 3.2. *We have $G = E:S$ with $S \cong Sp_{2m}(3)$. Moreover, if $m \geq 2$ and $g \in C$ with $E \cdot g = E \cdot s$ for some $s \in S$ then $C \cap E \cdot g = (es)^E = e \cdot s^E$ with $e \in Z_0^\#$. In particular, $G = G'$ unless $m = 1$ and $|G:G'| = 3$.*

Proof. Put $\overline{G} := G/F$ and note that $C_G(E) = C_G(F) = Z \leq F$. As G is generated by C and as $\mathrm{Out}(E) \cong Sp_{2m}(3):2$, $\overline{G}$ is isomorphic to a subgroup of $Sp_{2m}(3)$ generated by transvections. Since $O_3(\overline{G}) = 1$ and since E/Z_0 is an indecomposable $\overline{G}$-module (cf. 1.10), application of a result of McLaughlin [18] now yields:

(1) $\overline{G} \cong Sp_{2m}(3)$; in particular, $\overline{G}$ acts irreducibly on E/Z_0, and

$$\text{and } Z(\overline{G}) \cong \mathbb{Z}_2 \text{ acts invertingly on } E/Z_0.$$

Clearly, G/Z is a split extension of E/Z_0 by $\overline{G}$. Since $\overline{G}$ has a trivial Schur multiplier, we then even get:

(2) $$G = F:S \text{ with } S \cong Sp_{2m}(3),\ G' = E:S', \text{ and}$$
$$|G:G'| = |Z:Z_0| \cdot |S:S'| \in \{1,3\}.$$

If $m = 1$ then $S \cong SL_2(3)$ with $|S:S'| = 3$; so all the claims are obvious in this case. Henceforth we suppose that $m \geq 2$.

Assume now that $|G:G'| = 3$ and fix $g \in C$. Since $S = S'$, we have $Z \cong Z_9$, $G' = E:S$, and $G = Z * G'$. Without loss we may assume that $g = z \cdot y$ with $y \in G'$, $o(y) = 9$, and $y^3 = z^6 \in Z_0$; also put $y = e \cdot s$ with $e \in E$ and $s \in S$. Since $3 = o(g) = o(\bar{g}) = o(s)$, we get $y^3 = e \cdot e^{s^2} \cdot e^s$ and $[e, s] \neq 1$.

Recall that $C_S(s) \cong 2 \times (3^{1+2l}:Sp_{2l}(3))$ with $l = m-1$; considering the action of $C_S(s)$ on E we find

$$E_1 := C_E(s) \text{ has index 3 in } E,\ E_0 := Z_0[E,s] = Z(E_1) \cong E(3^2), \text{ and}$$
$$C_S(s) \text{ acts irreducibly on } E_1/E_0.$$

Clearly, as $o(y) = 9$, $e \in E \setminus E_1$. Now there exists $f \in E_1 \setminus C_{E_1}(e)$ and thus $y^f = a \cdot y$ with $a := [e,f] \in Z_0^\#$. Hence we get $g^f = a \cdot g$ and $g^{f^2} = a^2 \cdot g$. But now $\langle Z_0, g \rangle$ violates the conclusion of 1.4(1). Therefore we have

(3) $$Z = Z_0 \text{ and } G = E:S \text{ for any } m \geq 2.$$

Again fix $g \in C$ and put $g = e \cdot s$ with $e \in E$ and $s \in S$. Clearly, as $o(g) = 3$, we also have $o(s) = 3$. Now let $f \in E \setminus E_1$, and so $e_0 := [f, s] \in E_0 \setminus Z$. Since $f^{s^2} = f \cdot e_0^2$ and $C_E(e_0) = E_1$, we obtain $1 \neq (f \cdot s)^3 = [f, e_0] \in Z$. Therefore we have $e \in E_1$.

If $e \in E_1 \setminus E_0$, then there exists $f \in E_1 \setminus C_{E_1}(e)$ with $g^f = a \cdot g$ and $a := [e, f] \in Z^{\#}$; this is impossible for the same reasons as those just given above. Therefore we have $e \in E_0$. Since $E_0 = Z[E, s]$, we may then assume without loss that $e \in Z$.

In order to show that $e \in Z_0^{\#}$ assume that $e = 1$ and thus $s \in C$. If $m \geq 3$ then S contains a subgroup $S_0 = \langle s_0^S \rangle \cong Sp_6(3)$; hence the triple (S_0, s_0^S, V) satisfies $FFG(2,3)$, contrary to the information on $Sp_6(3)$ given in [17]. Therefore we have $m = 2$.

Now let W be an irreducible S-submodule of $V_0 := V|_S$. Again using [17] for $S \cong Sp_4(3)$ we observe that W is uniquely determined with $\dim(W) = 6$ and that any irreducible constituent of V_0 is isomorphic to W. Clearly, W extends to a module W_1 for $S_1 := Z \otimes S$ such that a generator z of Z acts by scalar multiplication with a primitive third root of unity. By Nakayama reciprocity we know that V is an epimorphic image of the induced module $V_1 := W_1^G$. Note that $GF(4)$ is a splitting field for S_1 and G, and so W_1 and V_1 can be realized over $GF(4)$.

Using the computer algebra system MAGMA [2] we easily construct both G (as a subgroup of $PSp_6(3)$) and the $GF(4)G$-module V_1; similarly, we can determine the socle series of V_1 and find that V_1 has Loewy length 3 with socle layers isomorphic to $36 \oplus 180$, 54, and $36 \oplus 180$, respectively. Moreover, the element s has nontrivial fixed points on each irreducible constituent of V_1. Thus we reach a contradiction, finally proving the claim.

We are left to show that for the groups $G = E : S$ with $E \cong 3^{1+2m}$ and $S \cong Sp_{2m}(3)$ as described above in the previous lemma there actually does exist a triple (G, C, V) satisfying $FFG(2,3)^*$.

Observe that $\mathbb{F} := GF(4)$ is a splitting field for E and let ω be a primitive element in $\mathbb{F}^*$. Recall that E has exactly two faithful complex irreducible characters each of which has degree $n := 3^m$; furthermore, these characters can be distinguished by and relate to the two nontrivial irreducible characters of $Z(E)$ (cf. [12, Satz V.16.14]). From this we deduce the existence of a simple faithful $\mathbb{F}G$-module V with $\dim(V) = n$. So without loss we consider E as a subgroup of $L := GL(V)$.

Clearly, $Z := Z(E) = C_L(E) = Z(L) \cong \mathbb{Z}_3$ and thus $N_L(E)/E$ is isomorphic to a subgroup of $\mathrm{Out}_Z(E) := \{\sigma \in \mathrm{Out}(E) | \sigma|_Z = id|_Z\} \cong Sp_{2m}(3)$. On the other hand, each $\sigma \in \mathrm{Out}_Z(E)$ gives rise to an equivalent $\mathbb{F}E$-module V^σ; thus, for each such σ we find a corresponding element $\lambda(\sigma) \in N_L(E) \setminus E$. Hence we have $N_L(E)/E \cong Sp_{2m}(3)$. Obviously, $N_L(E)$

splits over E and so we put $G = N_L(E) = E : S$, where $S \cong Sp_{2m}(3)$ is a suitable complement.

Next we want to find the appropriate class C of elements of order 3 in G such that $G = \langle C \rangle$ and elements of C act fixed point freely on V. For this purpose we use induction on m.

Put $L_1 := GL_3(4)$ and

$$E_1 := \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \leq L_1.$$

We easily verify that E_1 is extraspecial of order 27 and exponent 3; moreover, $G_1 := N_{L_1}(E_1) \cong 3^{1+2} : Sp_2(3)$ and $g_1 := \mathrm{diag}(\omega, \omega, \omega^2) \in G_1 \setminus E_1$ with $o(g_1) = 3$, $|E_1 : C_{E_1}(g_1)| = 3$ as well as $G_1 = \langle g_1^{G_1} \rangle$. Since E_1 acts irreducibly on the underlying three-dimensional vector space V_1, the triple $(G_1, g_1^{G_1}, V_1)$ satisfies $FFG(2,3)^*$. So we are done in the case $m = 1$.

Suppose next that $m = l + 1 \geq 2$ and let $G_2 := N_{L_2}(E_2) = E_2 : S_2$ with $E_2 \cong 3^{1+2l}$ acting irreducibly on $V_2 = \mathbb{F}^{3l}$, $S_2 \cong Sp_{2l}(3)$, and $L_2 := GL(V_2)$.

Now consider the Kronecker product $H := G_1 \otimes G_2$. Since $E_1 \otimes E_2 \cong 3^{1+2m}$ acts irreducibly on $V_1 \otimes V_2$, we may identify V with $V_1 \otimes V_2$ and E with $E_1 \otimes E_2$; hence $E \lhd H \leq G$. Moreover, $g_m := g_1 \otimes 1 \in H$ with $|E : C_E(g_m)| = 3$ and $C_V(g_m) = 0$. Put $C := (g_m)^G$ and observe that $\langle C \rangle$ is a normal subgroup of G not contained in E; so we get $G = \langle C \rangle$. In particular, (G, C, V) now satisfies $FFG(2,3)^*$.

Finally, suppose that W is a simple $\mathbb{F}G$-module such that (G, C, W) satisfies $FFG(2,3)^*$ where $\mathbb{F}$ is a splitting field of characteristic 2 for G and all its subgroups. Note that the construction given above also works over $\mathbb{F}$.

Denoting the set of eigenvalues of the action of an element $y \in G$ on a G-module X by $\sigma_X(y)$, we may assume that $\sigma_W(z) = \sigma_V(z) = \{\omega\}$ for a fixed generator z of Z and a fixed primitive third root ω of unity.

Clearly, $W|_E$ is isomorphic to a direct sum of copies of $V_0 := V|_E$. By 1.12 we have $V \cong V \otimes X$ for some simple $\mathbb{F}G$-module X with $E \leq \mathrm{Ker}(X)$. Since $\sigma_W(g) = \{\omega, \omega^2\} = \sigma_V(g)$, we get $\sigma_X(g) = \{1\}$; as $G = \langle C \rangle$, X is the trivial G-module and thus $W \cong V$.

Note that the extension V of V_0 to G is unique for $m \geq 2$ and that for $m = 1$ there exist $|G : G'| = 3$ nonisomorphic extensions of V_0 to G which are related by tensoring with a suitable one-dimensional G-module (cf. 1.12(2)).

So we have shown:

THEOREM 3.3. *Let G be a finite group with $F^*(G) = F(G)$. Then there exists a triple (G, C, V) satisfying $FFG(2,3)^*$ if and only if the following hold*:

(1) *$G = E : S$ with $E \cong 3^{1+2m}$ and $S \cong Sp_{2m}(3)$ for some $m \geq 1$.*

(2) *$\dim(V) = 3^m$. Moreover, V is determined uniquely up to duality and isomorphism if $m \geq 2$; if $m = 1$, then up to duality and isomorphism there are exactly three possible choices for V.*

Remark. Note that the "if" part is also contained in [24].

As an immediate consequence of 3.3 and 3.2 we obtain

COROLLARY 3.4. *If $H \cong \mathbb{Z}_3 \times PSp_{2m}(3)$ for some $m \geq 2$ then there exists a triple (H, D, W) satisfying $FFG(2,3)^*$.*

4. SPORADIC COMPONENTS

By 1.11 we know that if (G, C, V) is a triple satisfying $FFG(2,3)^*$ with $F^*(G) \neq F(G)$ then $F^*(G) = Z(G) * E(G)$ where $E(G)$ is quasi-simple. In this section we shall deal with a special instance of this situation assuming throughout the following:

HYPOTHESIS (SPO). *G is a finite group such that $F^*(G) = Z(G) * E(G)$ and $E(G)$ is quasi-simple involving one of the 26 sporadic (nonabelian) simple groups; furthermore, (G, C, V) is a triple satisfying $FFG(2,3)^*$.*

Making use of some well-known properties of the sporadic simple groups first of all we deduce

LEMMA 4.1. *$G = E(G)$; moreover, if $Z(G) \neq 1$ then $Z(G) \cong \mathbb{Z}_3$ and $G/Z(G)$ is isomorphic to M_{22}, J_3, McL, Suz, $O'N$, Fi_{22}, or Fi'_{24}.*

Proof. Put $Z := Z(G)$ and $Z_0 := Z(E(G)) = Z \cap E(G)$; note that Z is cyclic and thus without loss $Z = \langle z \rangle$ as well as $Z_0 = \langle z_0 \rangle$.

Now recall that $C_G(F^*(G)) \leq F^*(G)$. Since $|\mathrm{Out}(S)| \leq 2$ for any sporadic simple group S and since $G = \langle g^G \rangle$ with $o(g) = 3$, obviously $G = F^*(G)$. Clearly, we also have $O_2(G) = 1$. Since the Schur multiplier of any sporadic simple group has order dividing 12 (cf. [3]), we get $Z_0 \leq Z_3$; moreover, if $Z_0 \cong \mathbb{Z}_3$ then $E(G)/Z_0$ is isomorphic to one of the groups listed above in the claim.

Assume now that $Z \nleq E(G)$. Since $|G : G'| \in \{1, 3\}$, we get $|G : G'| = |Z : Z_0| = 3$ and $G' = E(G)$.

Suppose that $Z_0 = 1$; so $G = Z \times G'$ where G' is simple and $Z \cong \mathbb{Z}_3$. According to the information given in [3] every element of order 3 in a sporadic simple group is conjugate to its inverse; therefore 1.5(1) now yields a contradiction.

We may assume now that $Z_0 \cong \mathbb{Z}_3$ and $Z \cong \mathbb{Z}_9$; in particular, $G = Z * G'$. Again by the information in [3], the triple covers of the seven sporadics listed above do not contain elements of order 9 which cube to elements in Z_0. As this contradicts 1.5(2), $Z = Z_0 \leq E(G)$ and thus $G = E(G)$. Now the remaining claims are immediate.

In the next few steps we are going to eliminate most of the possible choices for the group G in question.

LEMMA 4.2. *There is no triple (H, D, W) satisfying $FFG(2,3)$ with H isomorphic to one of the following groups*:

(1) M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_3, HS, McL;

(2) $3M_{22}$, $3J_3$, $3McL$;

(3) $L_2(19)$, $L_4(3)$, $U_4(3)$, $3_2U_4(3)$, ${}^2F_4(2)'$, $L_5(2)$;

(4) $U_6(2)$.

Proof. The 2-modular character tables of the groups listed in (1)–(3) are well known and can be found in [17]. So a straightforward inspection yields the claim of these groups.

As for $U_6(2)$, we observe that it contains a subgroup $X \cong U_4(3)$ with $|X|_3 = |U_6(2)|_3$; so Part (3) together with 1.4(3) yields the claim in (4).

LEMMA 4.3. *There is no triple (H, D, W) satisfying $FFG(2,3)$ whenever H is isomorphic to He, Ru, Suz, $O'N$, Co_3, Co_2, Fi_{22}, HN, Ly, Th, Fi_{23}, J_4, Fi'_{24}, BM, or M.*

Proof. Let H be any of the sporadic groups listed in the claim above and let $R \in \mathrm{Syl}_3(H)$. Now we inspect the various possibilities for H.

(1) $H = He$. Now H contains a subgroup $L \cong 2^2L_3(4)$ and so without loss $R_0 := R \cap L \cong E(3^2)$. As R is extraspecial of order 27 and as $N_L(R_0)$ acts transitively on the nontrivial elements of R_0, application of 1.4 yields the claim in this case.

(2) $H = Ru, Co2$, or J_4. Observe that H contains a subgroup X with $R < X$ and $X \cong {}^2F_4(2)'$, McL, or M_{24}, respectively; so 1.4(3) and 4.2 apply.

(3) $H = Suz$. Note that $X = C_H(3A) \cong 3_2U_4(3)$ and without loss $R < X$. Since $H > Y \cong 3^5 : M_{11}$ with $O_3(Y)$ containing 22 elements of type 3A in H, $\langle (3A)^H \cap X \rangle = X$. Again 1.4(3) and 4.2(3) apply.

(4) $H = O'N$. As H contains only one class of elements of order 3 and as H contains a subgroup isomorphic to M_{11}, we apply 1.4(3) and 4.2(1).

(5) $H = Co_3$. Note that $X := C_H(2B) \cong 2 \times M_{12}$ and that H contains a subgroup $Y \cong McL$; moreover, X contains elements of type 3B and 3C and Y contains elements of type 3A in H. As H has only three classes of elements of order 3, 1.4(3) and 4.2(1) apply.

(6) $H = Fi_{22}$. Observe that $X := C_H(2A) \cong 2U_6(2)$ contains elements of types 3A, 3B, and 3C in H. Restricting complex irreducible characters of degrees 78 and 429 to a subgroup $Y \cong {}^2F_4(2)'$ of H we easily verify that Y contains elements of type 3D in H. As H contains exactly four classes of elements of order 3, 1.4(3) and 4.2 yield the claim.

(7) $H = HN$. First observe that $X := C_H(2A) \cong 2HS:2$ contains elements of type 3A of H. Restricting an irreducible complex character of degree 133 of H to a subgroup $Y \cong \mathrm{Alt}_{12}$ we easily verify that elements of types 3A, 3B, and 3C in Y are all contained in the 3A-class of H and that elements of type 3D in Y are 3B-elements in H. Next, by information in [17], the 3D-elements of Y have nontrivial fixed points on every simple $\overline{\mathbb{F}}_2Y$-module. So 1.4(3) and 4.2(1) yield the claim.

(8) $H = Ly$. Recall that H has exactly two classes of elements of order 3 and that without loss $R \leq X := C_H(3A) \cong 3McL$; moreover, the elements of $O_3(X)^{\#}$ are H-conjugate to elements of $X \setminus O_3(X)$. Now 1.4(3) and 4.2(2) apply.

(9) $H = Th$. Note that H has three classes of elements of order 3 and that H contains subgroups $X \cong L_2(19)$ and $Y \cong 2^5L_5(2)$ such that X contains elements of type 3B and Y contains elements of types 3A and 3C of H. So 1.4(3) and 4.2(3) apply again.

(10) $H = Fi_{23}$. Since $C_H(2A) \cong 2Fi_{22}$ contains elements of each H-class of elements of order 3, part (6) and 1.4(3) apply.

(11) $H = Fi_{24}'$. First of all we note that H contains exactly five classes of elements of order 3. Moreover, $X := C_H(2A) \cong 2Fi_{22}:2$ contains elements of types 3A, 3B, 3C, and 3D in H. Next, observe that H contains a maximal subgroup $Y \cong 3^7O_7(3)$ with $R < Y$. Clearly, $Q := O_3(Y) \lhd R$ and $C_H(Q) = Q$. Since Q contains no elements of type 3E of H (cf. [3]), we appeal to 1.4 and part (6).

(12) $H = BM$. Observe that H contains only two classes of elements of order 3. Moreover, $C_H(2C) = (2^2 \times X):2$ with $X \cong F_4(2)$ containing elements of types 3A and 3B of H. Since $X > Y \cong L_4(3)$ and $|X|_3 = |Y|_3$, we apply 1.4(3) and 4.2(3).

(13) $H = M$. Note that H contains exactly three classes of elements of order 3. Moreover, $C_H(2A) \cong 2BM$ contains elements of types 3A and 3B of H. Next we observe that H contains a maximal 3-local subgroup $X \cong 3^8O_8^-(3)2$ with $R < X$ and $Q := O_3(X) = C_H(Q) \lhd R$ containing only elements of type 3A and 3B in H. So we apply part (12) together with 1.4.

LEMMA 4.4. *There is no triple (H, D, W) satisfying $FFG(2,3)$ with H isomorphic to $3O'N$, $3Fi_{22}$, or $3Fi_{24}'$.*

Proof. Suppose without loss that (H, D, W) satisfies $FFG(2,3)^*$ with H isomorphic to $3O'N$, $3Fi_{22}$, or $3Fi_{24}'$. Fix $d \in D$ and put $Z := \langle z \rangle :=$

$Z(H)$, $\tilde{H} := H/Z$ as well as $n := \dim(W)$. Since W is a simple H-module and since $d \notin Z$, we may assume without loss that

(1) $z|_W = \omega \cdot I_n$ and $d|_W = \operatorname{diag}\big(\omega \cdot I_k, \omega^2 \cdot I_l\big)$ where $k + l = n$ with $k \neq 0 \neq 1$ and where ω is a primitive third root of unity.

From 1.4(1) we readily deduce

(2) $$d \cdot Z \cap D = \{d\}.$$

If $H \cong 3O'N$, then there exists only one class of elements of order 3 in $H \setminus Z$, but this contradicts (2).

Assume next that $H \cong 3Fi_{22}$. From (2) and the information in [3] we deduce that $\bar{d}$ is of type 3A in $\overline{H}$. Since $\bar{d}$ is $\overline{H}$-conjugate to its inverse, we see that $d \cdot Z = \{x, xz, xz^2\}$ where x is H-conjugate to its inverse and where xz^2 is H-conjugate to $(xz)^{-1} = x^2z^2$.

If $d = xz$, then d^2 is H-conjugate to $xz^2 = zd$; but then $\langle z, d\rangle \cong E(3^2)$ violates the conclusion of 1.4(1). We derive a similar contradiction if $d = xz^2$. Hence we have $d = x$; in particular, $\{d, d^2\} \subset D$. Moreover, if $h \in H$ with $\bar{h}$ of type 3A in $\overline{H}$ and $h^2 \in h^H$ then $h \in D$.

Next we observe that $\overline{H}$ contains a subgroup $\overline{X} \cong O_8^+(2)$; moreover, $\overline{X}$ has Schur multiplier of order 4. Therefore, H contains a subgroup $X \cong O_8^+(2)$. From [3] we also find that X contains exactly five classes of elements of order 3, say with representatives $d_1, \dots, d_5$. Moreover, restricting an irreducible complex character of degree 78 of $\overline{H}$ to X we see that $\{d_1, d_2, d_3\} \subset D$, that $\bar{d}_4$ is of type 3B or 3D in $\overline{H}$, and that $\bar{d}_5$ is of type 3C in $\overline{H}$.

Using the information of [17] on X we find that X has no irreducible module over $\overline{\mathbb{F}}_2$ on which d_1, d_2, and d_3 act fixed point freely at the same time. This contradiction now proves:

(3) $$H \cong 3Fi'_{24}.$$

Using (2) together with the information in [3] we now find that without loss

(4) $d \in x \cdot Z$ or $d \in y \cdot Z$ where $\bar{x}$ is of type 3B and $\bar{y}$ is of type 3C in $\overline{H}$; moreover x^2 is H-conjugate to x, xz^2 is H-conjugate to $(xz)^{-1}$, y^2 is H-conjugate to y, and yz^2 is H-conjugate to $(yz)^{-1}$.

Similarly to the above we verify now that $\{d, d^2\} \subset D$; i.e., $d \in \{x, y\}$. As $\overline{H}$ contains a subgroup isomorphic to Fi_{23} and this group has trivial Schur multiplier, H contains a subgroup $F \cong Fi_{23}$. Restricting an irreducible complex character of H of degree 783 to F we easily verify now that F contains H-conjugates of both x and y. Now 4.3 together with 1.4(3) yields a contradiction; this finally proves the claim.

We are now ready for the final attack.

THEOREM 4.5. *Suppose that (G, C, V) satisfies Hypothesis (SPO). Then exactly one of the following three cases occurs*:

(1) $G \cong J_2$, *C is the $3A$-class in G*, $\dim(V) = 6$, *and V is determined, uniquely up to duality.*

(2) $G \cong Co_1$, *C is the $3A$-class in G, and V is isomorphic to the mod-2 reduction of the Leech lattice of dimension* 24.

(3) $G \cong 3Suz$ and $\dim(V) = 12$; *moreover, elements in C are G-conjugate to their inverses and project onto $3A$-elements in $G/Z(G)$. The module V is determined uniquely up to duality.*

Proof. By 4.1, 4.2, 4.3, and 4.4 we are left to consider triples (G, C, V) with $G \cong J_2$, Co_1, or $3Suz$. Moreover, the claims in (1) follow from [17] and those in (3) follow from the 2-modular character-table of $3Suz$ as can be found in GAP.

Assume now that $G \cong Co_1$ and fix $g \in C$. Restricting an irreducible complex character of G of degree 276 to a subgroup $X \cong Co_3$ we easily verify that X contains elements of type 3B, 3C, and 3D of G. As G has exactly four classes of elements of order 3, 4.3 and 1.4(3) now imply that C can only be the 3A-class in G.

Using the information given in [3] we easily verify that G contains a subgroup $H = \langle H \cap C \rangle \cong \mathrm{Alt}_4$. Moreover, $O_2(H)$ acts quadratically on V; i.e., $[V, O_2(H), O_2(H)] = 1$ (e.g., see [10, Theorem 8.1]). Application of Theorem 1 and Theorem 3 in [19] now shows that V has to be the mod-2 reduction of the Leech lattice of dimension 24. Using [3] again we finally verify that elements of type 3A do indeed act fixed point freely on V.

5. ALTERNATING COMPONENTS

In this short section we deal with triples (G, C, V) satisfying $FFG(2,3)$ such that the group $E(G)/Z(E(G))$ is isomorphic to an alternating group. More precisely, we prove

THEOREM 5.1. *Suppose that (G, C, V) satisfies $FFG(2,3)^*$ and that the only nonabelian simple composition factor in $E(G)$ is isomorphic to an alternating group of degree $n \geq 5$. Then $G \cong \mathrm{Alt}_n$. Moreover, if $g \in C$ is not a 3-cycle, then $n = 6$, g is a product of two disjoined 3-cycles, and* $\dim(V) = 4$; *if $g \in C$ is a 3-cycle, then the list of admissible modules V is given in* [20].

Proof. Observe that G has a cyclic center; so we put $Z := Z(G) = \langle z \rangle$. Also note that $O_2(G) = 1$. Since automorphism groups and Schur multipliers of alternating groups are well known and since $G = \langle C \rangle$, we easily deduce

$$(1)\qquad G = Z * E(G),\ Z_0 := Z \cap E(G) = Z(E(G)) \leq \mathbb{Z}_3, \text{ and}$$
$$E(G)/Z_0 \cong \mathrm{Alt}_n; \text{ moreover, if } Z_0 \cong Z_3, \text{ then } n \in \{6, 7\}.$$

Now fix $g \in C$ and suppose that $Z_0 \cong \mathbb{Z}_3$ and thus $n \in \{6, 7\}$. From [17] we readily see that $G \neq E(G)$ and hence $|G : G'| = |Z : Z_0| = 3$ as well as $Z \cong \mathbb{Z}_9$. So without loss $g = z \cdot h$ with $h \in E(G)$ and $o(h) = 9$. On the other hand, neither $3\,\mathrm{Alt}_6$ nor $3\,\mathrm{Alt}_7$ contains elements of order nine (e.g., see [3]). This contradiction implies $Z_0 = 1$ and $G' = E(G) \cong \mathrm{Alt}_n$.

Assume next that $Z \neq 1$ and thus $G = Z \times E(G)$ with $Z \cong \mathbb{Z}_3$. Since elements of order 3 in Alt_n are real for $n \geq 5$ (see [16, 1.2.10]), application of 1.5(1) yields a contradiction. Thus we have $G = E(G) \cong \mathrm{Alt}_n$.

Suppose now that $g \in C$ is not a 3-cycle. If $n \leq 8$, then the remaining claims follow directly from [17]. Henceforth we may assume that $n \geq 9$. But now we easily verify that there exist subgroups $X \cong E(3^2)$ in G with $|X \cap C| \geq 6$. As this contradicts 1.4(1), we are done.

6. COMPONENTS OF CHARACTERISTIC p-TYPE, $p > 3$

In this section we consider the case where $E(G)$ is a quasi-simple group of Lie type defined over a field of characteristic $p > 3$.

LEMMA 6.1. *Let X be a quasi-simple group of Lie type over a field of characteristic $p \neq 3$. Suppose that $\tau \in \mathrm{Aut}(X)$ is an element of order 3 which does not stabilize any nontrivial p-subgroup of X. Then τ is the product of an inner and a diagonal automorphism of X. Moreover one of the following is true*:

(1) *X is of type $A_1(q)$ and $q \equiv -1 \bmod 3$.*

(2) *X is of type ${}^2A_2(q)$ with $q \equiv -1 \bmod 3$ and τ stabilizes a subgroup isomorphic to $SL_2(q)$ in X.*

(3) *X is of type $A_2(q)$ with $q \equiv 1 \bmod 3$ or of type ${}^2A_2(q)$ with $q \equiv -1 \bmod 3$ and τ stabilizes a nontrivial abelian 3-group of X without centralizing it.*

Proof. Let D denote the subgroup of $\mathrm{Aut}(X)$ generated by inner and diagonal automorphisms. By Theorem 9.1 of [8] we easily verify that any element of $\mathrm{Aut}(X) \setminus D$ centralizes a nontrivial p-subgroup of G; therefore $\tau \in D$.

Without loss of generality we can assume that $X = G'/Z$ where $Z \leq Z(G')$ and G is the group of F-fixed points $G = \mathbf{G}_F$, where $\mathbf{G}$ is a connected reductive algebraic group with Frobenius endomorphism F. Now τ is induced by an element $g \in \mathbf{G}$ with $g^3 \in Z(\mathbf{G})$ and $g^{-1}F(g) \in Z(\mathbf{G})$ (see [21]). Hence $C_{\mathbf{G}}(g)$ is F-stable and, as $p \neq 3$, g is semisimple. Since g does not normalize a nontrivial p-subgroup of G it is regular and contained in the unique maximal torus $\mathbf{T} := C_G(g)^o$ of $\mathbf{G}$. Let $\Phi := \Phi(\mathbf{G}, \mathbf{T})$ denote the root system of $\mathbf{G}$ with respect to $\mathbf{T}$. Regularity of g implies that no root in Φ annihilates g. Since $g^3 \in Z(\mathbf{G})$, we have $\alpha(g) \in \{\omega, \omega^{-1}\}$ for each root $\alpha \in \Phi$, where ω denotes a primitive third root of unity in the algebraic closure of $\mathbb{F}_q$.

Suppose that $\alpha, \beta, \gamma \in \Phi$ with $\alpha + \beta$, $\alpha + \beta + \gamma \in \Phi$, and $\beta + \gamma \in \Phi$ if $\beta \neq \gamma$. Then $(\alpha + \beta)(g) = \alpha(g) \cdot \beta(g)$ and we conclude that $\alpha(g) = \beta(g) = \gamma(g) = \omega$, say. But then $(\alpha + \beta + \gamma)(g) = \omega^3 = 1$, a contradiction. Note that root systems of types A_3, B_2, C_2, and G_2 always have a triple of roots as above: In the case of A_3 take the three fundamental roots; in the case B_2, C_2, or G_2 take $\gamma = \beta$ with $\alpha + \beta$ and $\alpha + 2\beta \in \Phi$. So we conclude that Φ does not contain any of these root systems as a closed subsystem and therefore Φ must be contained in a system of type A_2. Hence $\mathbf{G} \in \{SL_2, SL_3\}$ and g is not contained in any proper F-stable parabolic subgroup.

Consequently we have $G = G' \leq \tilde{G} \in \{GL_2(q), GL_3(q), GU_3(q)\}$; moreover, τ is induced by an element $g \in \tilde{G}$ which is not contained in any proper parabolic subgroup of $\tilde{G}$.

If $\tilde{G} \in \{GL_2(q), GL_3(q)\}$ then g must be contained in a Coxeter torus which is cyclic of order $q^2 - 1$ or $q^3 - 1$, respectively. In both cases $Z(\tilde{G})$ coincides with the subgroup of order $q - 1$; hence 3 divides $q + 1$ if $\tilde{G} \cong GL_2(q)$ and 3 divides $q - 1$ if $\tilde{G} \cong GL_3(q)$. In particular we are done in case $\tilde{G} \cong GL_2(q)$.

If $\tilde{G} \cong GU_3(q)$, then $\tilde{G}$ has three conjugacy classes of maximal tori of orders $(q+1)^2(q-1), (q+1)^3$, and $(q+1)(q^2 - q + 1)$, respectively, where the third class consists of Coxeter tori. Since the tori in the first class are quasi-split and therefore contained in a Borel group of $\tilde{G}$, g must belong to a torus of the second or third kind; in particular, 3 must divide $q + 1$.

Suppose that g belongs to a torus T_2 of the second kind and consider $\tilde{G}$ as the subgroup of σ-fixed points $\{x \in GL_n(q^2) | x^\sigma = x\}$ where $x^\sigma := (x^{(q)})^{-tr}$. Here (q) denotes the operation which raises each entry of x to the qth power and *tr* means a transposition of matrices. Now T_2 can be described as $T_2 := \{\operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) | \alpha_i \in \mathbb{F}_{q^2}^*, \alpha_i^{q+1} = 1\}$. In particular g normalizes the subgroups $S := \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) | a \in SU_2(q)\}$, which is isomorphic to $SL_2(q)$.

Henceforth we can assume that $\tilde{G} \cong GL_3(q)$ or $\tilde{G} \cong GU_3(q)$ and g is an element of a Coxeter torus. Let $\epsilon = 1$ in the case of $GL_3(q)$, $\epsilon = -1$ in the case of $GU_3(q)$, and let $q - \epsilon = 3^s a$ and $\gcd(a, 3) = 1$. Then the Coxeter tori are cyclic of order $q^3 - \epsilon = 3^{s+1}a'$ and $\gcd(a', 3) = 1$. Since g is not in $Z(\tilde{G})$, its eigenvalues are ν, $\nu^{q\epsilon}$, ν^{q^2} with $\nu \in \overline{\mathbb{F}}_q^*$ of order 3^{s+1}. In particular, $\det(g) \neq 1$ and so $g \in \tilde{G} \setminus G$ and g induces a diagonal automorphism on X.

Let

$$\tilde{T} := \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_2 \end{pmatrix} \middle| \overline{\mathbb{F}}_q \ni t_i^{q-\epsilon} = 1 \right\} \leq \tilde{G},$$

define $T := \tilde{T} \cap G \cong \mathbb{Z}_{q-\epsilon}^2$ and $N := N_G(T)$. Then we can find $S \in \mathrm{Syl}_3(G)$ such that $S = O_3(T) : \langle w \rangle$ and

$$w = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and without loss $g \in O_3(\tilde{T})S = O_3(\tilde{T})\langle w \rangle$. Since $g \notin \tilde{T}$ we have $g = tw$ with a 3-element $t \in \tilde{T} \setminus T$. Let $c \in \mathbb{F}_q^*$ or $\mathbb{F}_{q^2}^*$ be of order 3^s and

$$x = \begin{pmatrix} c & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in T,$$

then g normalizes the abelian group $A := \langle xZ, x^g Z, x^{g^2} Z \rangle \leq G$ but does not centralize A unless $s = 1$ and $|Z| = 3$. So we assume this latter case now. Then, as g has order 3^2 and $O_3(\tilde{T}) = O_3(T)\langle y \rangle$ with

$$y := \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that $g = tw$ with $t \in O_3(\tilde{T}) \setminus O_3(T)$ and $twZ \neq wtZ$ and g normalizes the abelian group $\langle wZ, xZ \rangle$ without centralizing it.

THEOREM 6.2. *Let G be a finite group such that $E(G)$ is a quasisimple group of Lie type over $\mathbb{F}_q$ with $q = p^f$ and $p > 3$. Then there exists a triple (G, C, V) satisfying $FFG(2,3)^*$ if and only if $G \cong L_2(5)$ and* $\dim(V) = 2$.

Proof. Suppose that (G, C, V) satisfies $FFG(2,3)^*$; let $H := E(G)$ and $g \in C$. Since $C_G(H) = C_G(F^*(G)) = Z(G)$, g induces an automorphism α of order 3 on H which, by 1.6(2), satisfies condition (*) of 2.1; so α does not normalize any nontrivial p-subgroup of H. So, by 6.1, H is of type $A_1(q)$ or ${}^2A_2(q)$ with $q \equiv -1 \bmod 3$; moreover, α is the product of an inner and a diagonal automorphism of H.

Assume first that H is of type $A_1(q)$.

Recall that $q \notin 9$ and thus $L_2(q)$ has Schur multiplier of order 2; since $O_2(G) = 1$, we get $Z(H) = 1$ and hence $H \cong L_2(q)$. Clearly, g induces an inner automorphism on H and thus $G = F^*(G) = Z \times H$ where $Z := Z(G)$ is cyclic.

Assume now that $Z \neq 1$ and thus $Z \cong \mathbb{Z}_3$. Since elements of order 3 are conjugate to their inverses in H, application of 1.5(1) yields a contradiction. So we have shown that $G = H \cong L_2(q)$ with $q \equiv 2 \bmod 3$.

Now we apply the results in [5] to determine the irreducible 2-modular Brauer-characters of G and their values on elements of order three. A straightforward check then reveals that $FFG(2,3)^*$ is satisfied if and only if $q = 5$ and $\dim(V) = 2$.

Henceforth we may assume that H is of type ${}^2A_2(q)$. Again by 6.1, $g \in C$ induces an automorphism on H which stabilizes a subgroup $X \cong SL_2(q)$ of H without centralizing it. Hence, by the previous argument applied to $\langle X, g \rangle$, we conclude that $q = 5$ and $g \in X$; in particular, $g \in H$ and thus $G = H \cong (P)SU_3(5)$. A look at [17] now reveals that we have derived a contradiction, finally completing the proof.

7. COMPONENTS OF CHARACTERISTIC 3-TYPE

In this chapter we shall investigate finite groups G satisfying the following hypothesis:

(CH3) *There exists a triple* (G, C, V) *satisfying* $FFG(2,3)^*$ *such that* $E/Z(E)$ *is a simple Lie type group over a field of characteristic* 3 *where* $E := E(G)$.

Recall from 1.11 and 1.2 that $F^*(G) = Z * E$ where $Z := Z(G)$ is cyclic of odd order and $Z \cap E = Z(E)$.

THEOREM 7.1. *The finite group G satisfies* $(CH3)$ *if and only if one of the following five cases occurs*:

(1) $G \cong L_2(9) \cong \mathrm{Alt}_6$ *and* $\dim(V) = 4$.

(2) $G \cong U_3(3)$ *and* $\dim(V) = 6$.

(3) $G \cong PSp_4(3)$ *and* $\dim(V) \in \{4, 6\}$.

(4) $G \cong 3_1U_4(3)$ *and* $\dim(V) = 6$.

(5) $G \cong \mathbb{Z}_3 \times PSp_{2n}(3)$ *with* $n \geq 2$ *and* $\dim(V) = (3^n - 1)/2$; *moreover*, *V is the mod* 2 *reduction of a complex Weil representation of the same dimension.*

Proof. Suppose that G satisfies the hypothesis (CH3) and put $E := E(G)$, $Z := Z(G)$, and $Z_0 := Z \cap E = Z(E)$ as well as $\overline{G} := G/Z$; moreover, fix $g \in C$.

By 2.2 we know that elements of $G/F^*(G)$ can induce only diagonal automorphisms on $\overline{E}$. Since E is defined in characteristic 3, the group of outer diagonal automorphisms of $\overline{E}$ is a $3'$-group; therefore $G = F^*(G)$. In particular, $G = Z * E$.

Suppose first that 3 does not divide $|Z_0|$ and thus $G = Z_1 \times E$ with $|Z_1| = |G : G'| \in \{1, 3\}$; in particular, E is an epimorphic image of a quasi-simple Chevalley group.

If $Z_1 = 1$ then 2.3 yields that G is isomorphic to $L_2(9)$, $U_3(3)$, or $PSp_4(3)$; using information in [17] we now establish the claims in (1)–(3).

Assume next that $Z_1 = \langle z \rangle \cong \mathbb{Z}_3$. Application of 2.3 now yields $E \cong L_2(9)$, $U_3(3)$, or $PSp_{2n}(3)$ with $n \geq 2$. As elements of order 3 are conjugate to their inverses in both $L_2(9)$ and $U_3(3)$, we have $E \cong PSp_{2n}(3)$ with $n \geq 2$ by application of 1.5(1). By 2.4 we know already that this case actually does occur. In addition to that, [9] also gives a description of the module V under consideration; with this the claims in (5) follow.

Henceforth we may assume that 3 divides $|Z_0|$. An inspection of the Schur multipliers of the possible candidates for $\overline{E}$ (e.g., see [3]) then yields $\overline{E} \cong L_2(9)$, $U_4(3)$, $G_2(3)$, or $\Omega_7(3)$ and $Z_0 \cong \mathbb{Z}_3$. Assume next that $Z \neq Z_0$; so $Z \cong \mathbb{Z}_9$ and $G = Z * E$. Again we use [3] and verify that E does not contain elements of order 9 which cube to elements in $Z(E)$, contrary to 1.5(2). Hence we get Z_0 and $G = E$.

Suppose now that $G \cong 3\Omega_7(3)$. By 1.4(1) and the information in [3] we see that $\bar{g}$ must be of type $3B$ in $\overline{G}$; next we observe that g must be contained in a subgroup $Z \times H$ with $H \cong L_4(3)$ (note that H has Schur multiplier of order 2). But the previous arguments already have shown that neither H nor $Z \times H$ satisfies $FFG(2, 3)$. Therefore $G \not\cong 3\Omega_7(3)$.

Now we are in a position to apply the results in [17]; this eventually yields $G \cong 3_1U_4(3)$ and $\dim(V) = 6$. So (4) holds and we are done.

8. COMPONENTS OF CHARACTERISTIC 2-TYPE

In this final section we investigate finite groups G satisfying the following hypothesis:

(CH2) *There exists a triple* (G, C, V) *satisfying* $FFG(2, 3)^*$ *such that* $E/Z(E)$ *is isomorphic to a simple group of Lie type defined over a finite field* $\mathbb{F}$ *of characteristic* 2 *where* $E := E(G)$.

Recall again that $F^*(G) = Z * E$ by 1.11 where $Z := Z(G)$ is cyclic of odd order and $Z \cap E = Z(E)$. Moreover, by 2.2, $G/F^*(G) \hookrightarrow$

Outdiag $E(G)$, the subgroup of Out $E(G)$ generated by diagonal automorphisms. Since the group $E(G)$ is perfect it is easy to see that

$$\text{Outdiag } E(G) \leq \text{Outdiag } E(G)/Z(G),$$

which is solvable. Moreover, it is known that diagonal automorphisms of $E(G)$ are induced by semisimple elements of a finite (untwisted) group of Lie type $H \geq E(G)$ having the same root system as $E(G)$. For example, if $E(G) \cong SL_n(q)$ one can take H to be $GL_n(q)$ or one can pick elements from a suitable overgroup $SL_n(\tilde{q})$. In particular Outdiag $E(G)$ is abelian and we conclude that $G/F^*(G) \cong \langle \bar{g} \rangle$ with $g \in C$; hence

$$G = F^*(G) \cdot \langle g \rangle = \langle g^{F^*(G)} \rangle = \langle g^{E(G)} \rangle = E(G) : \langle g \rangle$$

with $\text{Spec}(g|_V) = \{\omega, \omega^{-1}\}$, ω being a primitive third root of unity in $\overline{\mathbb{F}}$. If $g \in E(G)$ then $G = E(G)$ and the triple (G, C, V) is determined by the results of Wilson [23]. So for the rest of this section we assume that $g \notin E(G)$.

For any $g \in C$ there is $h \in H$ such that h acts on $E(G)$ in the same way as g does and by Schur's lemma we get $g|_V = h|_V \cdot \lambda Id_V$ for some scalar $\lambda \in \overline{\mathbb{F}}$. Hence we get $h^3 \in Z(H)$ and $\#\text{Spec}(h|_V) = 2$. Obviously we can choose h and λ to have 3-power orders.

On the other hand, let $h \in N_H(E(G))$ with $h^3 \in Z(H)$ and $\text{Spec}(h|_V) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$. Then $\lambda_2 = \lambda_1 \cdot \omega$ and $g := \omega \cdot \lambda_1^{-1} Id_V \cdot h|_V \in GL(V)$ satisfy $g^3 = Id_V$, $\text{Spec}(g|_V) = \{\omega, \omega^{-1}\}$, and $(G, C, V) \in FFG(2,3)^*$ with $G = \langle E(G)_V, g|_V \rangle$ and $C = g^G$. Hence we have to find all triples (H, h, V) with $h^3 \in Z(H)$, V absolutely irreducible, and $\#\text{Spec}(h|_V) = 2$. We abbreviate the set of these triples as $IS(2,3)^*$ (incomplete spectra).

We need the following observation:

LEMMA 8.1. *Let H be a finite group of Lie type, defined over a field of characteristic p, such that the Weyl group W of H contains a central element w of order 2. Then every semisimple element (i.e., p'-element) of H is conjugate to its inverse. In particular, $Z(H) \setminus \{1\}$ consists of involutions only.*

Proof. Let $\mathbf{H}$ denote the corresponding algebraic group such that $H = \mathbf{H}^F$ with Frobenius endomorphisms F and suppose that $h \in H$ is semisimple. Then h is contained in an F-stable maximal torus $\mathbf{T} \leq \mathbf{H}$ with $W \cong N_{\mathbf{H}}(\mathbf{T})/\mathbf{T}$. It is known that $w = -Id$ in the reflection representation of W, which implies that ${}^w t = t^{-1}$ for all $t \in \mathbf{T}$. Now let $\tilde{n}$ be a preimage of w in $N_{\mathbf{H}}(\mathbf{T})$. By Lang's theorem, the F-stable coset $\tilde{n}\mathbf{T}$ contains an F-stable element $n \in H = \mathbf{H}^F$ and we get ${}^n h = h^{-1}$.

LEMMA 8.2. *If G satisfies Hypothesis* (CH2) *with $C \subseteq G \setminus E(G)$, then the root system of G is of type A_l, D_{2l+1}, or E_6.*

Proof. If the Weyl group of H (and of $E(G)$) contains $-Id$, then

$$|O_3(Z(H))| = 1 = |O_3(\text{Outdiag}(E(G)))|.$$

Since $g \notin E(G)$ we have $g|_V = h|_V \cdot \omega Id_V$ with $h \in E(G)$, $h^3 = 1$, and $\text{Spec}(h|_V) = \{1, \omega\}$. But this contradicts the fact that h and h^{-1} are conjugate by Lemma 8.1. The only indecomposable root systems such that the corresponding Weyl group does not contain a central element of order two are those of types A_l, D_{2l+1}, and E_6.

So we can assume that the root system Φ of $E(G)$ is of type as prescribed in Lemma 8.2. Note that all these root systems are "simply laced," i.e., all roots have the same length.

Let $\Pi := \{\alpha_1, \dots, \alpha_l\}$ be a base of Φ, $Q := \langle \Phi \rangle_Z$ the root lattice, and $P := \langle \omega_1, \dots, \omega_l \rangle_Z$ the weight lattice with $\{\omega_1, \dots, \omega_l\}$ the set of fundamental dominant weights, satisfying $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$ in the usual notation of Lie theory.

Note that Q and P are dual to each other; i.e., there are pairings $P \times Q \to \mathbb{Z}$, $(p, q) \mapsto \langle p, q \rangle$, $Q \times P \to \mathbb{Z}$, $(q, p) \mapsto \langle q, p \rangle$ such that

$$P = Q^{\perp} := \text{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}), Q = P^{\perp} := \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}).$$

For any $Q \le S \le P$ we define

$$S^{\perp} := \{y \in P | \langle x, y \rangle \in \mathbb{Z}, \forall x \in S\}.$$

Let $\mathbf{T} \le \mathbf{H}$ be a maximally split maximal F-stable torus of $\mathbf{H}$ with Weyl group $W = N_{\mathbf{H}}(\mathbf{T})/\mathbf{T}$ and $T := \mathbf{T}^F \cong (\mathbb{F}_q^*)^l$ with $q = 2^n$. Define $X := \text{Hom}(\mathbf{T}, \overline{\mathbb{F}}_2^*)$, then, if $\mathbf{H}$ is semisimple, we have $Q \le X \le P$ with $X = P$ if $\mathbf{H}$ is simply connected or $X = Q$ if $\mathbf{H}$ is of adjoint type. There is a well-known W-equivariant isomorphism

$$h(\ ) : \text{Hom}\left(X, \overline{\mathbb{F}}_q^*\right) \to \mathbf{T}$$

which restricts to an isomorphism

$$h(\ ) : \text{Hom}(X, \mathbb{F}_q^*) \to T$$

of W-modules. If $\kappa \in \mathbb{F}_q^*$ is a generator of $\mathbb{F}_q^*$, then the map

$$\chi : X^{\perp} \to \text{Hom}(X, \mathbb{F}_q^*), \tau \mapsto \chi(\tau) = (c \mapsto \kappa^{\langle \tau, c \rangle})$$

gives rise to the exact sequence of W-modules

$$1 \to (q-1) X^{\perp} \to X^{\perp} \to \text{Hom}(X, \mathbb{F}_q^*) \to 1.$$

for $\lambda \in X^{\perp}$, the element $h(\chi\lambda) \in T$ is the center of H if and only if $\lambda \in (q-1)P$.

In the following we will adapt some results of [23] to our situation. In that paper Wilson considers Lie type groups G in characteristic two, acting irreducibly on the $\overline{\mathbb{F}}_2[G]$-module V such that $1 \notin \mathrm{Spec}(g|_V)$ for some noncentral $g \in G$ with $g^3 = 1$. In our situation we can restrict to simply laced root systems, in fact to the types A_l, D_{2l+1}, and E_6, but we have to consider the slightly more general noncentral elements $h \in H$ with h^3 central and $\#\mathrm{Spec}(h|_V) = 2$. All arguments in [23] which refer to $h^3 = 1$ or $1 \notin \mathrm{Spec}(h|_V)$ have to be modified to work for $h^3 \in Z(H)$ and $\#\mathrm{Spec}(h|_V) = 2$.

We need the following version of Proposition 4.1 in [23].

THEOREM 8.3. *Let $(H, h, V) \in IS(2,3)^*$ with H a quasi-simple group of Lie type, such that the root system Φ of $\mathbf{H}$ is simply laced.*

Then Φ is of type A_l or D_l with V of highest weight $\lambda = 2^f\omega_j$ for some $f \geq 0$ and $j \in \{1, 2, \dots, l\}$. Moreover, the element h is $\mathbf{H}$-conjugate to $h(\chi\tau) \in T$ with $\tau \in X^{\perp} \cap (r\omega_i + (q-1))P$ for some $i \in \{1, \dots, l\}$.

Furthermore, the following restrictions hold:

(1) *If Φ is of type A_l, then $\{i, j\} \cap \{1, l\} \neq \emptyset$.*

(2) *If Φ has type D_4, then $\{i, j\} \subseteq \{1, l-1, l\}$ and $i \neq j$.*

(3) *If Φ is of type D_l with $l > 4$, then $\{1\} \subseteq \{i, j\} \subseteq \{1, l-1, l\}$ and $i \neq j$.*

Proof. The proof follows through the steps in [23]. In the same way as in Lemmas 3.2 and 3.3 of [23], one can use Steinberg's tensor product theorem to see that $(H, h, V) \in IS(2,3)^*$ implies that V has highest weight $2^f\lambda \in P$, $f \geq 0$, and $0 \leq \langle \lambda, \alpha_i \rangle \leq 2$ for all $i \in \{1, 2, \dots, l\}$. Here the arguments in [23] only use the fact that h is noncentral and has less than three eigenvalues.

Extending the field of definition for H if necessary, we can assume that 3 divides $q-1$. Since h^3 is central, h is semisimple and the arguments in [23, (4.2)] show that h is $\mathbf{H}$-conjugate to some $h(\chi\tau) \in T$ with $\tau \in X^{\perp}$. Now set $r := \frac{q-1}{3}$; then $h^3 \in Z(H) \not\ni h$ implies that $3 \cdot \tau \in (q-1)P$ and

$$\tau = r \cdot \sum_{b=1}^{l} c_b \omega_b$$

with $c_b \in \mathbb{Z}$ and $3 \nmid c_b$ for some b. Hence we are in the same situation as in the beginning of the proof of [23, (4.4)], which is in fact a statement about noncentral elements $h(\chi\tau) \in T$ with $h^3 \in Z(H)$ and which proves that for these elements one has

$$\tau \in X^{\perp} \cap \left(r\delta_S + (q-1)P\right)$$

with $S \subseteq \{1, \dots, l\}$, $|S| \leq 3$, and $\delta_S := \sum_{s \in S} \omega_s \in P$. (This holds even in the nonsimply-laced case).

Let $\lambda \in P$ be the highest weight of V and $P(\lambda)$ the set of all weights of V. Lemma 4.8 of [23] states that if Φ is simply laced there is a sequence $\beta_1, \dots, \beta_e$ of elements of Π such that $\lambda - \sum_{a=1}^{b} \beta_a \in P(\lambda)$ for all $1 \leq b \leq e$ and that $\lambda - \omega_0 \lambda = \sum_{a=1}^{e} \beta_a = \sum_{c=1}^{l} d_c \alpha_c$, where the d_c are strictly positive integers. Here ω_0 denotes the longest element in W.

Now we modify the argument in [23, (4.9)] to conclude $S = \{i\}$ and $d_i = 1$: In fact,

$$\begin{aligned} 2 &= \#\mathrm{Spec}(h(\chi\tau)|_V) = \#\{\kappa^{\langle\tau,\mu\rangle} \mid \mu \in P(\lambda)\} \\ &\geq \#\{\kappa^{\langle\tau,\lambda\rangle} \cdot \kappa^{-r(\#\{i=1,\dots,b \mid \beta_i \in S\})} \mid 0 \leq b < e\} \end{aligned}$$

implies that δ_S only involves one ω_i such that the corresponding α_i can be subtracted from λ only once, which means $d_i = 1$. Now we can repeat the original arguments in the proof of [23, (4.10)] and get exactly the same conclusion, which is given in the theorem above.

Remark 8.4. 1. In the case A_l we have $Q = \langle \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_l - \epsilon_{l+1} \rangle \leq \bigoplus_{i=1}^{l+1} \mathbb{Z}\epsilon_i$ and $\omega_j = \delta_j - \frac{j}{l+1}\delta_{l+1}$ with $\delta_j := \epsilon_1 + \cdots + \epsilon_j$; $w_0 = (1, l+1)(2, l) \cdots (i, l+2-i) \cdots \in W \cong \Sigma_{l+1}$. For $1 \leq j \leq \frac{l+1}{2}$ we have $\omega_j - w_0\omega_j = \alpha_1 + 2\alpha_2 + \cdots + \cdots + j(\alpha_j + \cdots + \alpha_{l+1-j}) + (j-1)\alpha_{l+1-j+1} + \cdots + 2\alpha_{l-1} + \alpha_l = \sum_{k=1}^{l} d_k^{(j)} \alpha_k$. Hence for $1 < j < l$ we get $d_k^{(j)} = 1 \Leftrightarrow k = 1$ or $k = l$, and for $j \in \{1, l\}$ we see $d_k^{(j)} = 1$ for all $1 \leq k \leq l$.

2. In case D_l we have

$$Q = \langle \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} + \epsilon_l \rangle \leq \bigoplus_{i=1}^{l} \mathbb{Z}\epsilon_i.$$

and $\omega_j = \delta_j$ for $1 \leq j \leq l-2$, $\omega_{l-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{l-1} - \epsilon_l)$, $\omega_l = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{l-1} + \epsilon_l)$.

For $j \leq l-2$ we have $\omega_j - w_0\omega_j = 2w_j = 2\alpha_1 + 4\alpha_2 + \cdots + 2(j-1)\alpha_{j-1} + 2j(\alpha_j + \cdots + \alpha_{l-2}) + j(\alpha_{l-1} + \alpha_l)$.

For l even we get $\omega_{l-1} - w_0(\omega_{l-1}) = 2\omega_{l-1} = \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + l\alpha_{l-1} + (l-2)\alpha_l$; $\omega_l - w_0(\omega_l) = 2\omega_l = \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + (l-2)\alpha_{l-1} + l_{\alpha l}$.

For l odd we have $w_0\omega_l = \omega_{l-1}$ and therefore $\omega_l - w_0(\omega_l) = \omega_{l-1} - w_0(\omega_{l-1}) = \omega_l + \omega_{l-1} = \alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2} + (l-1)\alpha_{l-1} + (l-1)\alpha_l$.

This explains the restrictions on the i, j in the above theorem.

The next results state that no "new cases" occur for groups with $E(G)$ having a root system of type D_l, l odd.

THEOREM 8.5. *Let $(G, C, V) \in FFG(2,3)^*$ with $G = E(G):\langle g\rangle$, $g \in C \setminus E(G)$. Then $E(G)$ is not of type $D_l(q)$ or ${}^2D_l(q)$ with l odd.*

Proof. Assume $E(G)$ is of type $D_l(q)$ or ${}^2D_l(q)$ with l odd. Then

$$|Z(E(G))| = \gcd(q \pm 1, 4) = 1 = |\mathrm{Outdiag}(E(G))|.$$

Hence $g|_V = h|_V \cdot \lambda Id_V$ with $h \in E(G)$, $h^3 = 1$, and $\mathrm{Spec}(h|_V) = \{1, \omega\}$. If $E(G) = D_l(q)$ define $H := E(G)$; if $E(G) = {}^2D_l(q)$ define $H := D_l(q^2) \geq E(G)$. Then H acts on the module V and $(H, h, V) \in IS(2,3)^*$. Now the following lemma implies that $1 \notin \mathrm{Spec}(h|_V)$l, which is a contradiction.

LEMMA 8.6. *Let $(H, h, V) \in IS(2,3)^*$ with H a quasi-simple group $D_l(q)$, l odd. Then $1 \notin \mathrm{Spec}(h|_V)$.*

Remark 8.7. If 3 divides $q - 1$, then the triple (H, h, V) appears in Table 2 of [23].

Proof. On extending q if necessary, we can assume that 3 divides $q - 1$. Moreover, $h^3 = 1$ and we can take $X = P$ and $X^\perp = Q$, because in characteristic two the adjoint and simply connected groups of type D are isomorphic as abstract groups. In particular we can take $X = \langle \omega_l, Q\rangle$ (see 8.4). Consider $h = h(\chi\tau) \in T$ with $\tau = (r\omega_i + (q-1)\pi)$ with $\pi \in P$. Note that $h(\chi\tau)$ and $h(\chi\sigma)$ are conjugate in H if and only if σ and $\tau \in P$ lie in the same orbit under the affine Weyl group $(q-1)Q : W$ [15]. Hence we can change τ to $\tau + (q-1)x$ with $x \in Q$. Note also that $X/Q = \langle \overline{\omega}_l\rangle \cong \mathbb{Z}_4$, because l is odd. Hence we can assume that $\tau = r\omega_i + 3rv\omega_l \in Q$ with $0 \leq v < 4$. We also can restrict to V_{ω_j} with $j \in \{1, l-1. l\}$. Let $j = 1$; then we have $i = l$ or $i = l - 1$. First assume that $i = l$; then $\tau = r(1 + 3v)\omega_l \in Q$. Since $r = (q-1)/3$ is odd, 4 must divide $1 + 3v$, which implies $v = 1$ and $\tau = 4r\omega_l$. But this is one of the elements appearing in Wilson's list [23, Table 2] and therefore has no eigenvalue 1. The argument for $i = l - 1$ is similar, or the result can be obtained by applying $w_0 \in W$.

Now let $j = l$. Then $\tau = r(\omega_1 + 3v\omega_l) \in Q$. Since r is odd, we conclude that $\overline{\omega}_1 \equiv v\overline{\omega}_l \bmod Q$, hence $0 \neq v$ is even and therefore $v = 2$. Since the dominant weight ω_l is minimal (i.e., $\lambda \prec \omega_l$ and λ dominant implies $\lambda = \omega_l$; see [11, p. 72, example 13), one has $P(\omega_l) = \omega_l^W = \{1/2(\Sigma_{i=1}^{l} \pm$

$\epsilon_i)\}$. For each $w \in W$ and $x \in P$ we have $(w-1)(x) \in Q$, hence

$$\begin{aligned}\langle \tau, w(\omega_l)\rangle &= \langle \tau, \omega_l + (w-1)(\omega_l)\rangle \\ &\equiv \langle \tau, \omega_l\rangle + r\langle w_1, w(\omega_l) - \omega_l\rangle \bmod (q-1)Z \\ &\equiv \langle \tau - r\omega_1, \omega_l\rangle + r\langle \omega_1, w(\omega_l)\rangle \\ &\equiv \frac{3rvl}{4} \pm \frac{r}{2} \equiv r\left(\frac{6l \pm 2}{4}\right) \bmod (q-1)Z.\end{aligned}$$

Hence $\mathrm{Spec}(h(\chi\tau|_V)) = \{\kappa^{r(\frac{6l\pm 2}{4})}\} = \{\omega^{\frac{a6l\pm 2}{4}}\} = \{\omega, \omega^{-1}\}$.

Again for $j = l-1$ one can argue similarly or apply w_0.

Now we are left with the case where $E(G)$ has a root system of type A_l. Let $n := l+1$, $\mathbf{H} := GL_n(\overline{\mathbb{F}}_q)$, and $\mathbf{S} := SL_n(\overline{\mathbb{F}}_q)$, both with Frobenius endomorphisms F_1: $(g_{ij}) \mapsto (g_{ij}^q)$ and F_{-1}: $(g_{ij}) \mapsto (g_{ij}^q)^{-tr}$. Then $H_\epsilon := H_\epsilon(q) := \mathbf{H}^{F_\epsilon} = GL_n(q)$, $S_\epsilon := S_\epsilon(q) := SL_n(q)$ for $\epsilon = 1$, and $:= U_n(q)$, $SU_n(q)$ for $\epsilon - 1$), respectively. Moreover, $E(G)$ is isomorphic to a quotient $E_s := E_s(\epsilon, q) := S_\epsilon/\langle a \cdot Id_n | a^s = 1\rangle$, where Id_n denotes the identity $(n \times n)$-matrix and s is a divisor of $z := \gcd(q-\epsilon, n)$. Similarly, H is isomorphic to a quotient $H_s := H_s(\epsilon, q) := H_\epsilon/\langle a \cdot Id_n | a^s = 1\rangle$. If $\mathcal{N}$ denotes the natural module then, up to duality, the module V_{ω_i} can be described as the exterior powers $\Lambda^i(\mathcal{N})$ with $1 \le i \le l$ and $M_i^* \cong M_{n-i}$. Clearly M_i is an G-module if and only if s divides i and, by the arguments above, all candidates V with $(G, C, V) \in FFG(2,3)^*$ or $(H, h, V) \in IS(2,3)^*$ arise in that way with G a suitable subsection of H_ϵ.

Due to 8.3 we need only consider the modules V_ω. Now we consider $V := M_i \cong \Lambda^i(\mathcal{N})$ with $s|i > 1$. By duality we can restrict to $i \le n/2$ and $n > 3$. Again let $\tilde{h} \in H_\epsilon(q)$ denote a preimage of $h \in H$ with $h^3 \in Z(H)$ and $\#\mathrm{Spec}(h|_V) = 2$. Then $\tilde{h}|_{\mathcal{N}}^3 = a \cdot Id_{\mathcal{N}}$ with $a \in \overline{\mathbb{F}}_{q\epsilon}$ and $\mathrm{Spec}(\tilde{h}|_{\mathcal{N}}) \subseteq \{\delta \cdot \omega^j | j = 0, 1, -1\}$. Let n_0, n_1, and n_{-1} with $n_0 + n_1 + n_{-1} = n$ denote the multiplicities of $\delta, \delta\cdot\omega$, and $\delta \cdot \omega^{-1}$ in $\mathrm{Spec}(\tilde{h}|_{\mathcal{N}})$, respectively. Then

$$\mathrm{Spec}\left(\tilde{h}|_{M_i}\right) = \delta^i \cdot \left\{\omega^{d_1 - d_{-1}} \mid 0 \le d_j \le n_j, 0 \le i - d_1 - d_{-1} \le n_0\right\}.$$

LEMMA 8.8. *Let* $1 < i \le n/2$ *and* $\#\mathrm{Spec}(\tilde{h}|_{M_i}) = 2$, *then* $\#\mathrm{Spec}(\tilde{h}|_{\mathcal{N}}) = 2$ *and* δ *can be chosen such that for* $\tilde{h}$ *or* $\tilde{h}^{-1}$ *we have* $n_0 = n-1$, $n_1 = 1$, *and* $n_{-1} = 0$; *i.e.*, $\tilde{h}$ *or* $\tilde{h}^{-1}$ *is* $\mathbf{H}$-*conjugate to*

$$\delta \cdot \mathrm{diag}(\omega, 1, 1, \dots, 1)$$

with $\mathrm{Spec}(\tilde{h}|_{M_i}) = \delta^i\{1, \omega\}$.

Proof. We will show that one of the following cases must occur:

(i) $n_0 = 0,\ n_1 = 1,\ n_{-1} = n - 1;$

(ii) $n_0 = 0,\ n_1 = n - 1,\ n_{-1} = 1;$

(iii) $n_0 = 1,\ n_1 = 0,\ n_{-1} = n - 1;$

(iv) $n_0 = 1,\ n_1 = n - 1,\ n_{-1} = 0;$

(v) $n_0 = n - 1,\ n_1 = 0,\ n_{-1} = 1;$

(iv) $n_0 = n - 1,\ n_1 = 1,\ n_{-1} = 0.$

From this the result follows after changing δ and replacing $\tilde{h}$ with $\tilde{h}^{-1}$ if necessary. We call $(d_1, d_{-1}) \in \mathbb{N}_0 \times \mathbb{N}_0$ admissible if the inequalities

$$0 \le d_j \le n_j, \qquad 0 \le i - d_1 - d_{-1} \le n_0$$

are satisfied. Suppose that (d_1, d_{-1}) is admissible with $0 < d_1$ and $0 < d_{-1}$. If $(d_1 - 1, d_{-1})$ or $(d_1, d_{-1} - 1)$ is also admissible, then so is $(d_1, d_{-1} - 1)$ or $(d_1 - 1, d_{-1})$ respectively, and we get the contradiction

$$\operatorname{Spec}\left(\tilde{h}|_{M_i}\right) \supseteq \delta^i \cdot \omega^{d_1 - d_{-1}} \cdot \{\omega, 1, \omega^{-1}\}.$$

Hence $i = n_0 + d_1 + d_{-1}$. A similar contradiction arises if $(d_1 + 1, d_{-1} - 1)$ and $(d_1 - 1, d_{-1} + 1)$ are admissible; so we conclude that $d_1 = n_1$ or $d_{-1} = n_{-1}$. By symmetry in the role of ω and ω^{-1}, we can assume $d_1 = n_1$. Then $i \le n/2$ and $n > 2$ imply that $d_{-1} \le n_{-1} - 2$ but then $d_1 = n_1 = 1$ (otherwise $(d_1, d_{-1}), (d_1 - 1, d_{-1} + 1)$, and $(d_1 - 2, d_{-1} + 2)$ were admissible). Since $(1, d_{-1}), (0, d_{-1} + 1), (1, d_{-1} + 1)$ cannot all be admissible, we conclude that $i - d_1 - d_{-1} - 1 < 0$, hence $n_0 = 0$. So we arrive at (i) and (ii) (by symmetry). Now assume that (i) and (ii) do not hold. Then we have

$$(d_1, d_{-1}) \text{ admissible } \Rightarrow d_1 = 0 \text{ or } d_{-1} = 0.$$

By symmetry we can assume that $(0, d_{-1})$ is admissible with $d_{-1} > 0$. If $n_1 > 0$, then $(1, d_{-1} - 1)$ is admissible, hence $d_{-1} = 1$; on the other hand, $(1, 1)$ cannot be admissible, so $i - 2 > n_0$, because $i > 1$ by hypothesis. This leads to the contradiction

$$i - 1 = i - d_{-1} \le n_0 < i - 2.$$

Thus we get $n_1 = 0$. Hence $n_{-1}, n_0 > 0$, because $h \notin Z(H)$. We claim that $n_0 = 1$ or $n_{-1} = 1$. Suppose $n_0 > 1 < n_{-1}$. Note that $(0, d_{-1} + 1), (0, d_{-1})$, and $(0, d_{-1} - 1)$ cannot all be admissible. Hence $d_{-1} = i$ or $d_{-1} = n_{-1}$ or $i - d_{-1} = n_0$. Suppose that $d_{-1} = i$. Then $(0, d_{-1} - 2)$ is not admissible, but $d_{-1} = i \ge 2$ implies the contradiction $i - d_{-1} + 2 = 2$

$> n_0$. Suppose that $d_{-1} = n_{-1}$. Again $(0, d_{-1} - 2)$ is not admissible and $d_{-1} = n_{-1} > 1$ implies that $i - d_{-1} + 2 > n_0 \Rightarrow i - n_{-1} \geq n_0 - 1$ which leads to the contradiction $n \leq 2$. Finally suppose that $i - d_{-1} = n_0 > 1$ and $d_{-1} < n_{-1}$. Then $i - d_{-1} - 2 \geq 0$ but $(0, d_{-1} + 2)$ is not admissible. Hence $d_{-1} + 1 = n_{-1}$ and we get $n = n_0 + n_{-1} = i - n_{-1} + 1 + n_{-1} = i + 1 \Rightarrow n/2 \geq i = n - 1$ which leads to the contradiction $n \leq 2$. The rest of the statement follows from symmetry.

As a consequence we obtain the following final result:

THEOREM 8.9. *Let $G = E(G):\langle g \rangle \leq GL(V)$ with $C := g^G$ and $g \notin E(G) = E_s(\epsilon, q)$. Then $(G, C, V) \in FFG(2,3)^*$ if and only if $q\epsilon \equiv 1 \bmod 3$ and one of the following is true*:

(1) *$s = 1 = i$, $V \cong \mathscr{N}$ up to duality, $E(G) = S_\epsilon$, $G = E(G):\langle g_m \rangle$,*

$$g_m = \mathrm{diag}(\omega, \dots, \omega, \omega^{-1}, \dots, \omega^{-1}) \in C$$

with m entries ω and $n - m$ entries ω^{-1} and $n - 2m \not\equiv 0 \bmod 3$.

(2) *Up to duality $V \cong \Lambda^i(\mathscr{N})$, with $s|i > 1$, $i \leq n/2$, and there is $g \in C$ such that g or g^{-1} is of the form $\omega\delta^{-i} \cdot Id_V \cdot t_V$ with $\delta \in \overline{\mathbb{F}}_q$, satisfying $\delta^{q\epsilon} = \delta$ and*

$$t := \mathrm{diag}(\omega, 1, \dots, 1) \cdot \delta \cdot Id_{\mathscr{N}} \in H_\epsilon.$$

The condition for $E(G) < G$ is: $\forall c \in \overline{\mathbb{F}}_q$ with $c^{q\epsilon - 1} = 1$ and $c^n = \omega$, we have $c^{-i} \neq \omega$.

Moreover, in all cases we have

$$G = Z(G) * E(G) \Leftrightarrow \frac{q - \epsilon}{\gcd(q - \epsilon, n)} \equiv 0 \mod 3.$$

Proof. Let $(G, C, V) \in FFG(2,3)^*$ with $g \in C$ and $g \notin E(G)$. Then $g|_V = \lambda \cdot Id_V \cdot h|_V$ with $h \in H_\epsilon$ such that $(H_\epsilon, h, V) \in IS(2,3)^*$.

Suppose that $q\epsilon \not\equiv 1 \bmod 3$. In this case all elements of order three in H_ϵ are conjugate to their inverses: indeed, let x be such an element; then it is $\mathbf{H}$ conjugate to an element $t \in \mathbf{T}$ and we get

$$x^{\mathbf{H}} = F_\epsilon(x^{\mathbf{H}}) = F_\epsilon(t^{\mathbf{H}}) = F_\epsilon(t)^{\mathbf{H}} = (t^{q\epsilon})^{\mathbf{H}} = (t^{-1})^{\mathbf{H}} = (x^{-1})^{\mathbf{H}}.$$

Since $C_{\mathbf{H}}(x)$ is connected, Lang's theorem implies $x^{H_\epsilon} = (x^{-1})^{H_\epsilon}$. Moreover, 3 does not divide $|Z(H_\epsilon)|$ and $|\mathrm{Outdiag}(E(G))|$ and we get $g|_V = \omega \cdot Id_V \cdot h'|_V$ with $h' \in E(G)$ and $h'^3 = 1$. Let $h \in H_\epsilon$ be an inverse image of h'. We can choose h to be of 3-power order and, as $h^3 \in Z(H_\epsilon)$, in fact to be of order 3. But then h is H_ϵ-conjugate to its inverse and we conclude that $\mathrm{Spec}(h'|_V) = \mathrm{Spec}(h|_V) = \{\omega, \omega^{-1}\}$, which is a contradiction. Hence $q\epsilon \equiv 1 \bmod 3$.

First we deal with the case $i = 1$ or, in other words, $V \cong \mathcal{N}$; note that $s = 1$ in this case and g is automatically contained in H_ϵ and is $\mathbf{H}$-conjugate to $g_m := \mathrm{diag}(\omega, \ldots, \omega, \omega^{-1}, \ldots, \omega^{-1}) \in T$ with ω appearing m times and ω^{-1} appearing $n - m$ times. Hence g and g_m are also conjugate under $S_\epsilon = E(G)$. On the other hand, if $q\epsilon \equiv 1 \bmod 3$ we can start with g_m, whose class $g_m^{\mathbf{H}}$ is F_ϵ-stable and therefore contains an element $g \in H_\epsilon$, by Lang's theorem. We then conclude that $(G, C, V) \in FFG(2,3)^*$ for $G := \langle S_\epsilon, g\rangle$, $C = g^G$, and $V := M_1$. Note that $g \notin S_\epsilon(q) = E(G)$ if and only if $\det(g) \neq 1$, which is equivalent to $\omega^{2m-n} \neq 1$.

Now we suppose that $1 < i \leq n/2$. From 8.8 we see that h or h^{-1} is $\mathbf{S}$-conjugate to

$$t := \mathrm{diag}(\omega, 1, \ldots, 1) \cdot \delta \cdot Id_{\mathcal{N}}.$$

Since $h = F_\epsilon(h)$ and $\omega^{q\epsilon} = \omega$, a look at $\mathrm{Spec}(h|_{\mathcal{N}})$ with its multiplicities shows that $\delta^{q\epsilon} = \delta$ and therefore $t \in H_\epsilon$. Hence h is conjugate to t under S_ϵ. Thus after a suitable conjugation in the image $E(G)$, we can assume that $g|_V = \lambda \cdot Id_V t|_V$. Looking at $\mathrm{Spec}(t|_V) = \delta^i\{1, \omega\}$, we get $\lambda = \delta^{-i}\omega$.

Note that $\det(t) = \delta^n\omega$ and that conjugation by t is an inner automorphism on $S_\epsilon(q)$ if and only if $yt^{-1} = \mu \cdot Id \in Z(H_\epsilon)$ for some $y \in T$ with $\det(y) = 1$. This in turn is equivalent to $\omega = c^n$ for some $c = 1/\mu\delta \in \overline{\mathbb{F}}$ with $c^{q\epsilon - 1} = 1$ or to $\frac{q-\epsilon}{\gcd(q-\epsilon, n)} \equiv 0 \bmod 3$. This criterion also works in the case $i = 1$.

Moreover, $E(G) = G$ if and only if $t|_V \cdot \delta^{-i}\omega \cdot Id_V = y|_V$ with some y as above. This is equivalent to $\delta^{-i}\omega \cdot Id_V = \mu^i \cdot Id_V$ or to $\omega = c^{-i}$ for some c with $c^n = \omega$ and $c^{q\epsilon} = c$.

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