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On the classification of the Lie algebras L_r^s

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Abstract

Linear algebraic methods are used to classify the Lie algebras L_r^s, presented by generators and relations. They were introduced as an algebraic model for quantized Hamiltonians.

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1. Introduction

In this paper we are concerned with a class of Lie algebras introduced in [2–4] as generalizations for the coupled quantized harmonic oscillators of Hamiltonian model $H = K_0 + \lambda(K_+ + K_-)$, where the coupling parameter $\lambda \in \mathbb{R}^*$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ [7]. These Lie algebras depend on two parameters $r, s \in \mathbb{R}$. For any such r, s the Lie algebra L_r^s is presented by generators K_0, K_+, K_- and relations:

$$[K_+, K_-] = sK_0, \quad [K_0, K_\pm] = \pm rK_\pm, \quad \text{where } s, r \in \mathbb{R}. \tag{1}$$

Note that L_2^1 is just $\mathfrak{sl}(2, \mathbb{R})$. Faithful matrix representations of least degree of L_r^s were given in [3,4]. The representations were subject to the physical requirements, namely, $K_- = K_+^\dagger$, and K_0 is a real diagonal operator representing energy. The Lie algebras L_1^2 and L_1^{-2} correspond to the models of the two-level optical atom and the light amplifier, respectively [1,5,6].

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The aim of this paper is to classify the Lie algebras L_r^s from an algebraic point of view. The classification is given by the following:

Theorem 1. *For every* $r, s \in \mathbb{R}^*$

1. $L_0^s \simeq L_0^1$.
2. $L_r^0 \simeq L_1^0$.
3. $L_r^s \simeq L_2^1$.
4. $L_2^1 \simeq \mathfrak{sl}(2, \mathbb{R})$, L_0^1, L_1^0 *and* L_0^0 *are non-isomorphic Lie algebras.*

Corollary 2. *A system of representatives for the isomorphism classes of the Lie algebras of the form* L_r^s *consists of* L_2^1, L_0^1, L_1^0 *and* L_0^0.

2. Isomorphism classes for $rs \neq 0$

Lemma 3. *The Lie algebras* L_r^s *and* L_{rs}^1 *are isomorphic, where* $s \in \mathbb{R}^*$ *and* $r \in \mathbb{R}$.

Proof. Let K_0', K_+', K_-' and K_0, K_+, K_- be generators of L_{rs}^1 and L_r^s, respectively and satisfy (1). The mapping $\phi : L_{rs}^1 \to L_r^s$, defined by, $\phi(K_\pm') = K_\pm$ and $\phi(K_0') = sK_0$, is a Lie algebra isomorphism. $s \neq 0$, is a necessary condition for ϕ to be onto. □

From Lemma 3, for the case $rs \neq 0$, it is enough to discuss the case when L_c^1 and L_d^1 are isomorphic, where $cd \neq 0$. Throughout this section the following notations are used, L_c^1 is generated by K_0, K_+, K_- satisfying (1), and L_d^1 is generated by K_0', K_+', K_-' satisfying (1) with $cd \neq 0$. $\operatorname{ad} u$ is the adjoint representation of L_c^1 assigned to u; $u \in L_c^1$ defined by $\operatorname{ad} u(v) = [u, v]$, for every $v \in L_c^1$.

Lemma 4. *If* $u = \alpha K_0 + \beta K_+ + \gamma K_- \in L_c^1$; $\alpha, \beta, \gamma \in \mathbb{R}$, *then the characteristic polynomial of* $\operatorname{ad} u$ *is*

$$f(\lambda) \equiv \lambda\left[\lambda^2 - c(c\alpha^2 + 2\beta\gamma)\right]. \tag{2}$$

Proof. Using (1), the matrix of $\operatorname{ad} u$ in the ordered basis K_0, K_+, K_- is

$$\begin{bmatrix} 0 & -\gamma & \beta \\ -c\beta & c\alpha & 0 \\ c\gamma & 0 & -c\alpha \end{bmatrix}. \qquad \Box \tag{3}$$

Lemma 5. *For* $cd \neq 0$, *there exists a nonzero element* $u \in L_c^1$, *for which* $\operatorname{ad} u$ *has eigenvalues* $0, d, -d$. *More precisely, an element* $u = \alpha K_0 + \beta K_+ + \gamma K_-$ *satisfies this property if and only if* $c(c\alpha^2 + 2\beta\gamma) = d^2$. *For such* u, *if* U *is the matrix whose columns are the eigenvectors of* $\operatorname{ad} u$ *corresponding to the eigenvalues* 0, d *and* $-d$ *respectively, then* U *is of the following forms*:

(i) *If* $\gamma = 0$ *and* $\alpha = \frac{d}{c}$, *then*

$$U = \begin{bmatrix} d & 0 & -2\beta d \\ c\beta & 1 & -c\beta^2 \\ 0 & 0 & 2d^2 \end{bmatrix}.$$

If $\gamma = 0$ *and* $\alpha = -\frac{d}{c}$, *then*

$$U = \begin{bmatrix} -d & 2d\beta & 0 \\ c\beta & -c\beta^2 & 1 \\ 0 & 2d^2 & 0 \end{bmatrix}.$$

(ii) *If* $\gamma \neq 0$, $\beta = 0$ *and* $\alpha = \frac{d}{c}$, *then*

$$U = \begin{bmatrix} d & 2d\gamma & 0 \\ 0 & -2d^2 & 0 \\ c\gamma & c\gamma^2 & 1 \end{bmatrix}.$$

If $\gamma \neq 0$, $\beta = 0$ *and* $\alpha = -\frac{d}{c}$, *then*

$$U = \begin{bmatrix} -d & 0 & -2d\gamma \\ 0 & 0 & -2d^2 \\ c\gamma & 1 & c\gamma^2 \end{bmatrix}.$$

(iii) *If* $\gamma \neq 0$, $\beta \neq 0$ *and* $\alpha = 0$, *then*

$$U = \begin{bmatrix} 0 & d & -d \\ \beta & -c\beta & -c\beta \\ \gamma & c\gamma & c\gamma \end{bmatrix}.$$

(iv) *If* $\gamma \neq 0$, $\beta \neq 0$ *and* $\alpha \neq 0$, *then*

$$U = \begin{bmatrix} \alpha & -2\beta(c\alpha - d) & 2\gamma(c\alpha - d) \\ \beta & -2c\beta^2 & -(c\alpha - d)^2 \\ \gamma & (c\alpha - d)^2 & 2c\gamma^2 \end{bmatrix}$$

where, $u = \alpha K_0 + \beta K_+ + \gamma K_-$; $\alpha, \beta, \gamma \in \mathbb{R}$.

Proof. From (2), $\operatorname{ad} u$ has eigenvalues $0, d, -d$, if and only if,

$$c(c\alpha^2 + 2\beta\gamma) = d^2. \tag{4}$$

The augmented matrix of the linear system $(\operatorname{ad} u - dI_3)X = 0$, is

$$\left[\operatorname{ad} u - dI_3 \mid 0\right] \equiv \left[\begin{array}{ccc|c} -d & -\gamma & \beta & 0 \\ -c\beta & c\alpha - d & 0 & 0 \\ c\gamma & 0 & -(c\alpha + d) & 0 \end{array}\right]$$

$$\equiv \left[\begin{array}{ccc|c} 1 & \frac{\gamma}{d} & -\frac{\beta}{d} & 0 \\ 0 & \alpha - \frac{d}{c} + \frac{\beta\gamma}{d} & -\frac{\beta^2}{d} & 0 \\ 0 & \frac{\gamma^2}{d} & \alpha + \frac{d}{c} - \frac{\beta\gamma}{d} & 0 \end{array}\right].$$

So, if $\gamma = 0$, then from (4), $\alpha = \pm\frac{d}{c}$. The eigenvectors corresponding to the eigenvalue d are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{for } \gamma = 0 \text{ and } \alpha = \frac{d}{c}$$

and

$$\begin{bmatrix} 2d\beta \\ -c\beta^2 \\ 2d^2 \end{bmatrix} \quad \text{for } \gamma = 0 \text{ and } \alpha = -\frac{d}{c},$$

respectively. Similarly, it can be proved that,

$$\begin{bmatrix} -2d\beta \\ -c\beta^2 \\ 2d^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are eigenvectors corresponding to the eigenvalue $-d$, for $\gamma = 0$, $\alpha = \frac{d}{c}$ and $\gamma = 0$, $\alpha = -\frac{d}{c}$, respectively. This proves (i), noticing that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

is an eigenvector for the eigenvalue 0.

Similarly, the proof can be completed, with the use of (4). □

Theorem 6. *L_c^1 and L_d^1 are isomorphic, whenever $cd \neq 0$.*

Proof. In order to show that L_c^1 and L_d^1 are isomorphic, it is enough to find elements $u, u_+, u_- \in L_c^1$ satisfying

$$[u, u_\pm] = \pm d u_\pm \quad \text{and} \quad [u_+, u_-] = u. \tag{5}$$

In particular this requires $\operatorname{ad} u$ to have eigenvalues $0, d, -d$. Lemma 5 provides all such elements u. As in case (i) of Lemma 5, with $\beta = 0$, choose $u = \frac{d}{c}K_0$, $u_+ = \lambda K_+$ and $u_- = 2d^2\mu K_-$. Using (1),

$$\begin{aligned}
[u, u_+] &= \left[\tfrac{d}{c}K_0, \lambda K_+\right] = d\lambda K_+ = du_+, \\
[u, u_-] &= \left[\tfrac{d}{c}K_0, 2d^2\mu K_-\right] = -d(2\mu d^2)K_- = -du_-, \\
[u_+, u_-] &= \left[\lambda K_+, 2d^2\mu K_-\right] = 2\lambda\mu d^2 K_0.
\end{aligned}$$

Taking λ and μ such that $\lambda\mu = \frac{1}{2cd}$, we have $[u_+, u_-] = \frac{d}{c}K_0 = u$. Hence the proof of the theorem follows. □

$L_2^1 \simeq \mathfrak{sl}(2, \mathbb{R})$ can be chosen as a representative for the isomorphism class of L_c^1, for $c \neq 0$.

3. Isomorphism classes for $rs = 0$

Lemma 7. *For any $s \in \mathbb{R}$, L_r^s has a trivial center if and only if $r \neq 0$.*

Proof. Let $Z = \alpha K_0 + \beta K_+ + \gamma K_- \in L_r^s$ be a central element. Using (1), $0 = [Z, K_0] = -r\beta K_+ + r\gamma K_-$, implies that $r\beta = r\gamma = 0$. If $r \neq 0$, then $\beta = \gamma = 0$, and hence, $Z = \alpha K_0$ which is central if and only if $\alpha = 0$. Conversely, if $r = 0$, then K_0 generates the center of L_r^s. □

Lemma 8. *L_0^s and L_0^1 are isomorphic, for $s \neq 0$.*

Proof. It follows directly from Lemma 3. □

Lemma 9. *L_r^0 and L_1^0 are isomorphic, for $r \neq 0$.*

Proof. It can be shown that $\phi : L_r^0 \to L_1^0$, defined by $\phi(K_0) = rK_0'$ and $\phi(K_\pm) = K_\pm'$ is an isomorphism, where K_0, K_+, K_-, satisfying (1) respectively, are generators of L_r^0 and K_0', K_+', K_-', satisfying (1) respectively, are generators of L_1^0. □

Theorem 10. *The Lie algebras L_2^1, L_0^1, L_1^0 and L_0^0 are nonisomorphic Lie algebras.*

Proof. L_0^0 is an abelian Lie algebra. From Lemma 7, L_0^1 has a nontrivial centre, while L_2^1 and L_1^0 have trivial centre. Let K_0', K_+', K_-', satisfying (1) respectively, be generators of L_1^0. We have $[L_2^1, L_2^1] = L_2^1$, but $[L_1^0, L_1^0] = \mathbb{R}K_+' + \mathbb{R}K_-' \neq L_1^0$. If $\phi : L_2^1 \to L_1^0$ were an isomorphism, then $[\phi(L_2^1), \phi(L_2^1)] = \phi(L_2^1)$ yields to the contradiction that $[L_1^0, L_1^0] = L_1^0$. □

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