Categories of Finite Dimensional Weight Modules over Type I Classical Lie Superalgebras

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INTRODUCTION

Since the generalization of highest weight representation theory to Lie superalgebras in [K1]–[K3], a number of interesting results on the highest weight representation theory of classical Lie superalgebras have been obtained. In contrast with the case of semisimple Lie algebras, the highest weight representation theory of a classical Lie superalgebra which is not a Lie algebra is nontrivial even in the finite dimensional case. Two of the basic problems of the finite dimensional representation theory, namely the problem of classifying the finite dimensional weight representations and the problem of finding the character of a finite dimensional simple highest weight representation, remain unsolved in general. It was shown in [VHKT2] that there is no character formula of Kac–Weyl type that can cover all simple finite dimensional highest weight representations even for the algebra sl(m, n). Realizing the failure of the Kac–Weyl type formulas, several authors have developed new approaches to the finite dimensional representations of classical Lie superalgebras recently (see [VHKT1], [VHKT2], [HKV], [V], [PS1], [PS2], [Ser]). Our work is motivated by [VHKT2] and [Ser].

The approach in [VHKT2] is purely combinatorial. The conjectured character formula presented in [VHKT2] for the finite dimensional simple highest weight representations of sl(m, n) is given combinatorially in terms of the characters of the Kac modules.

More recently, a Kazhdan–Lusztig theory was developed for the Lie superalgebra gl(m, n) in [Ser]. The idea is to define some polynomials called Kazhdan–Lusztig polynomials, and show that the values of these
polynomials at $-1$ together with the characters of the Kac modules provide the characters of the finite dimensional simple highest weight representations of $\mathfrak{gl}(m, n)$; then based on a conjecture about the semisimplicity of certain reflection functor, one can proceed to compute the polynomials.

It is natural to ask whether one can combine these two approaches. Since in the case of a semisimple Lie algebra, the Kazhdan–Lusztig theory is developed within the Bernstein–Gelfand–Gelfand (BGG) category $\mathcal{O}$, one would like to see whether a theory similar to the category $\mathcal{O}$ theory can be developed for the category $\mathcal{F}$ of the finite dimensional weight representations of a classical Lie superalgebra, and if so, to explore how this theory can help in bridging the approach of [VHKT2] and the approach of [Ser]. The purpose of this study is to address these problems for type I classical Lie superalgebras, as well as $\mathfrak{gl}(m, n)$.

The work of [BGG] on the projective objects in category $\mathcal{O}$ is one of the most interesting parts of the category $\mathcal{O}$ theory. This work was generalized to graded Lie algebras (including Kac–Moody algebras) by [R-CW]. By using an approach similar to the approaches of [BGG] and [R-CW], we will show that results similar to the category $\mathcal{O}$ theory hold for the category $\mathcal{F}$ over $\mathfrak{gl}(m, n)$ or over a type I classical Lie superalgebra. To be more precise, we will show that there are enough projective objects in $\mathcal{F}$ and the indecomposable projective objects are in one-to-one correspondence with the simple objects of $\mathcal{F}$, and that under a certain condition, an analog of BGG’s duality theorem also holds in $\mathcal{F}$ if one replaces Verma modules by Kac modules. Then we will describe a classification scheme for the objects of $\mathcal{F}$ and define the Kazhdan–Lusztig polynomials for some of these Lie superalgebras by using the cohomology groups $\text{Ext}^n(K(\lambda), L(\mu))$ (unfortunately, in some cases, this definition produces power series instead of polynomials). We will show that in the case of $\mathfrak{gl}(m, n)$, the polynomials we define in this paper coincide with the polynomials defined in [Ser]. In the case of $\mathfrak{sl}(1, n)$ ($n \geq 2$) and $\mathbb{C}(n)$, these polynomials can be computed easily, and we will give the results. In the Lie algebra case, $\text{Ext}^1$ plays a crucial role for the Kazhdan–Lusztig polynomials. We will discuss $\text{Ext}^1$ briefly at the end. It would be interesting if one could develop an algorithm to use $\text{Ext}^1$ to define the polynomials recursively as in the Lie algebra case (cf. [Ser, Sect. 7]).

This paper is organized as follows. In Section 1, we introduce the necessary notation and terminology. In Section 2, we construct the projectives in $\mathcal{F}$ and prove the duality theorem (under a certain condition). In Section 3, we describe a classification scheme for the objects of $\mathcal{F}$. In Section 4, we treat $\mathfrak{sl}(2, 1)$ as an example. It is interesting to note that one can also describe the injective objects of $\mathcal{F}$ in this case (see Theorem 4.1). In Section 5, we define the Kazhdan–Lusztig polynomials. In Section 6, we
compute these polynomials for \( sl(1,n) \) and \( C(n) \); the results show that for these algebras, the polynomials are either 0 or \( q^k \). In Section 7, we show that for the algebra \( gl(m,n) \), the polynomials defined in Section 5 are the same as the polynomials defined in [Ser]. Finally, a brief discussion of \( \mathrm{Ext}^1(K(\lambda), L(\mu)) \) is given in Section 8.

1. DEFINITIONS AND NOTATION

A classical Lie superalgebra of type I over the complex number field \( \mathbb{C} \) is one of the Lie superalgebras

\[
\begin{align*}
A(m,n) &= \left. \frac{\mathfrak{sl}(m+1,n+1)}{\mathfrak{sl}(1,1)} \right|_{m \neq n}, \quad m,n \geq 0, \\
A(n, n) &= \frac{\mathfrak{sl}(n+1,n+1)}{\mathfrak{sl}(2,2n-2)}, \quad n > 0, \\
C(n) &= \mathfrak{osp}(2,2n-2), \quad n \geq 2.
\end{align*}
\]

Since our discussions in this paper hold for \( gl(m,n) \) also, we include \( gl(m,n) \) in our list of the Lie superalgebras.

Let \( G = G_{-1} + G_0 + G_{+1} \) be the natural consistent \( \mathbb{Z} \)-grading of \( G \). Then \( G_0 \) is the even part of \( G \) and \( G_{-1} + G_{+1} = G_1 \) is the odd part of \( G \). Let \( G = G^+ + H + N^+ \) be the usual triangular decomposition of \( G \) with \( H \) a Cartan subalgebra and \( B = H + N^+ \) the distinguished Borel subalgebra. Let \( R \) be the roots of \( G \) with respect to this triangular decomposition, let \( R_0, R_1 \) be the sets of even and odd roots, respectively. Let \( R^+, R^+_0, R^+_1 \) be the subsets of positive roots in \( R, R_0, R_1 \), respectively, and let \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \) be the set of simple roots.

Let \( \rho_0 \) (resp. \( \rho_1 \)) be the half-sum of all the positive even (resp. odd) roots, and let \( \rho = \rho_0 - \rho_1 \).

Let \( \lambda \in H^* \) be a linear function on \( H \). Let \( C\lambda \) be the one-dimensional \( B \)-module defined by

\[
h(\lambda) = \lambda(h) \lambda, \quad h \in H; \quad N^+(v_\lambda) = 0; \quad \deg(v_\lambda) = 0.
\]

Let \( M(\lambda) = \text{Ind}_H^G C\lambda \). Then \( M(\lambda) \) is the Verma \( G \)-module with highest weight \( \lambda \). The \( G \)-module \( M(\lambda) \) has a unique maximal submodule \( M' \), and the quotient \( L(\lambda) = M(\lambda)/M' \) is the simple highest weight \( G \)-module corresponding to \( \lambda \).

Let \( \{ e_i, f_i, h_i \} \) be the set of generators of \( G \) as described in Section 2.5 in [K1], and let \( \alpha_i \) be the odd simple root. For \( \lambda \in H^* \), let \( a_i = \lambda(h_i), \) \( 1 \leq i \leq r \). The following theorem is a part of [K1, Theorem 8].

**Theorem 1.1.** The \( G \)-module \( L(\lambda) \) is finite dimensional if and only if \( a_i \in \mathbb{Z}_+ \) for \( i \neq s \).
Let \( P = G_0 + G_{+1} \). Let \( \lambda \in H^* \) and let \( L_0(\lambda) \) be the simple highest weight \( G_0 \)-module with highest weight \( \lambda \). We extend \( L_0(\lambda) \) to a \( P \)-module by letting \( G_{+1} L_0(\lambda) = 0 \), and define the Kac module corresponding to \( \lambda \) by

\[
K(\lambda) = \text{Ind}_G^P L_0(\lambda).
\]

The \( G \)-module \( K(\lambda) \) has a unique maximal submodule \( K' \) such that the quotient \( K(\lambda)/K' \cong L(\lambda) \).

Fix a bilinear form \((\ ,\ )\) as described in [K1, Sect. 2.5]. Then an element \( \lambda \in H^* \) is called typical if \( (\lambda + \rho, \beta) \neq 0 \) for all \( \beta \in R^+_1 \); otherwise \( \lambda \) is called atypical.

**Definition 1.2.** An element \( \lambda \in H^* \) is called a dominant integral if the \( G \)-module \( L(\lambda) \) is finite dimensional.

We denote the set of all dominant integral elements of \( H^* \) by \( \Sigma^+ \).

Let \( \lambda \in \Sigma^+ \). Then by [K3, Prop. 2.9], the \( G \)-module \( K(\lambda) \) is simple if and only if \( \lambda \) is typical.

A \( G \)-module \( M \) is called a weight \( G \)-module if \( M \) is a direct sum of weight subspaces, i.e.,

\[
M = \bigoplus M_\mu, \quad \text{where } M_\mu = \{ v \in M : hv = \mu(h)v \text{ for all } h \in H \}.
\]

Let \( \mathcal{F} \) be the category of finite dimensional weight \( G \)-modules. For \( M \in \mathcal{F} \), we denote by \( [M : L(\lambda)] \) multiplicity of \( L(\lambda) \) in the composition series of \( M \). A \( G \)-module \( M \in \mathcal{F} \) is said to have a Kac Composition Series (KCS) if \( M \) admits a filtration of \( G \)-modules

\[
M = M^0 \supset M^1 \supset \cdots \supset M^k \supset M^{k+1} = 0
\]

such that \( M^i/M^{i+1} \cong K(\mu_i), 0 \leq i \leq k \). If \( M \) has a KCS, we denote by \( (M : K(\lambda)) \) the number of indices \( 1 \leq i \leq k \) such that \( \mu_i = \lambda \).

## 2. Projectives of \( \mathcal{F} \) and a Duality Theorem

In this section, we modify the approach of [BGG] and [R-CW] to prove that \( \mathcal{F} \) has enough projectives and that a duality theorem holds; we will also discuss briefly the indecomposable injective objects of \( \mathcal{F} \).

Let \( \mu \in \Sigma^+ \); we define a \( G \)-module \( A(\mu) \) by

\[
A(\mu) = \text{Ind}_G^P L_0(\mu) = U(G) \otimes_{U(G_0)} L_0(\mu).
\]\n
(2.1)

Note that since \( L_0(\mu) \) is finite dimensional, \( A(\mu) \) is finite dimensional by the PBW theorem of Lie superalgebras.
Lemma 2.1. Let $M \in \mathcal{F}$, and let $\mu \in \Sigma^+$. Then
\[
\text{Hom}_G(A(\mu), M) \cong \text{Hom}_{G_0}(L_0(\mu), M)
\]
as vector spaces over $\mathbb{C}$.

Proof. We define a map $f: \text{Hom}_G(A(\mu), M) \to \text{Hom}_{G_0}(L_0(\mu), M)$ by
\[
f(\phi)(v) = \phi(1 \otimes v),
\]
where $\phi \in \text{Hom}(A(\mu), M)$, $v \in L_0(\mu)$, and define a map $g: \text{Hom}_{G_0}(L_0(\mu), M) \to \text{Hom}_G(A(\mu), M)$ by
\[
g(\varphi)(x \otimes v) = x\varphi(v),
\]
where $\varphi \in \text{Hom}_{G_0}(L(\mu), M)$, $x \in U(G)$, $v \in L_0(\mu)$. Then the maps $f$ and $g$ are well-defined and they are inverses of each other. Thus the lemma follows.

Lemma 2.2. For $\mu \in \Sigma^+$, the $G$-module $A(\mu)$ defined by (2.1) is a projective object of $\mathcal{F}$.

Proof. This follows from Lemma 2.1 and the fact that $L_0(\mu)$ is a projective object in the category of finite dimensional weight $G_0$-modules.

Remark. Lemma 2.2 also follows from the general fact that the functor $\text{Ind}^G_S$ from a subalgebra $S$ to $G$ sends projective objects to projective objects, but Lemma 2.1 provides more detail in this special case.

Lemma 2.3. For $\mu \in \Sigma^+$, the $G$-module $A(\mu)$ defined by (2.1) has a KCS.

Proof. Since $G_{-1}$ and $G_{-1}$ are subalgebras, $U(G_{-1})$ and $U(G_{-1})$ are defined. Note that by the PBW theorem of Lie superalgebras,
\[
A(\mu) = U(G) \otimes U(G_0) L_0(\mu) = U(G_{-1}) \otimes U(G_{+1}) \otimes L_0(\mu)
\]
as vector space. Let $v_0$ be a highest weight vector of $L_0(\mu)$. Let $u_1, \ldots, u_r$ be a basis of $U(G_{+1})$ given by the PBW theorem, and let this basis be ordered in such a way that
\[
G_+ u_i \subset \langle u_{i+1}, \ldots, u_r \rangle.
\]
Note that $U(G_{+1}) \otimes L_0(\mu)$ has a natural $P$-module structure. Let $V_i$ be the $P$-submodule of $U(G_{+1}) \otimes L_0(\mu)$ generated by $u_i \otimes v, \ldots, u_r \otimes v$, $1 \leq i \leq r$. Then $G_{+1} V_i \subset V_{i+1}$ and
\[
U(G_{+1}) \otimes L_0(\mu) = V_1 \supset V_2 \supset \cdots \supset V_{r+1} = 0.
\]

(2.2)
In fact, the inclusions are clear, and we only need to check that \( U(G_{+1}) \otimes L_0(\mu) = V_1 \). To verify this, note that \( G_{+1} \) is a simple \( G_0 \)-module. So if \( u \in G_{+1} \) is a lowest weight vector, then by using the action of \( G_0 \), we see that \( V_1 \triangleright Cu \otimes L_0(\mu) \), and therefore, as \( P \)-modules \( U(G_{+1}) \otimes L_0(\mu) = V_1 \).

Now since \( U(G_{+1}) \otimes L_0(\mu) \) is a finite dimensional \( G_0 \)-module, by refining (2.2) if necessary, we can find a filtration of \( P \)-modules
\[
U(G_{+1}) \otimes L_0(\mu) = U_1 \triangleright U_2 \triangleright \cdots \triangleright U_{i+1} = 0
\]
such that \( U_i/U_{i+1} \cong L_0(\lambda_i) \) as \( G_0 \)-modules for some \( \lambda_i \in \Sigma^+ \), \( 1 \leq i \leq t \), and \( G_{+1}U_i \triangleright U_{i+1} \). If we let \( A' = U(G) \otimes_p U_i \), we get a filtration of \( G \)-modules
\[
A(\mu) = A^1 \triangleright A^2 \triangleright \cdots \triangleright A^{t+1} = 0
\]
such that \( A^i/A^{i+1} \cong K(\lambda_i), 1 \leq i \leq t \). The lemma is now proved.

Let \( \mu \in \Sigma^+ \). Then by Lemma 2.3, we see that \( A(\mu) \) has a maximal submodule \( A' \) such that \( A(\mu)/A' \cong L(\mu) \). Consider the decomposition of \( A(\mu) \) into indecomposable components. Each of these indecomposable components is an indecomposable projective object in \( \mathcal{F} \). On the other hand, if \( I \) is an indecomposable projective object in \( \mathcal{F} \), then since \( I \) is an image of a direct sum of some \( A(\mu) \)'s, \( I \) must be an indecomposable component of one of these \( A(\mu) \)'s. Hence the direct summands of the \( A(\mu) \)'s, \( \mu \in \Sigma^+ \), exhaust all the indecomposable projective objects in \( \mathcal{F} \).

**Lemma 2.4.** Each indecomposable projective object in \( \mathcal{F} \) is singly generated and has a unique maximal proper submodule.

**Proof.** Since each indecomposable projective object of \( \mathcal{F} \) is a direct summand of some \( A(\mu) \), which is singly generated, the indecomposable projective objects in \( \mathcal{F} \) are singly generated. Being a finite dimensional \( G \)-module, each indecomposable projective object in \( \mathcal{F} \) has a maximal submodule. The proof of the uniqueness of the maximal submodule is similar to the proof of Lemma 11 in [R-CW].

The following proposition gives a description of the indecomposable projective objects of \( \mathcal{F} \).

**Proposition 2.5.** (i) There exists a one-to-one correspondence between the simple objects of \( \mathcal{F} \) and the indecomposable projective objects of \( \mathcal{F} \), hence the indecomposable projective objects of \( \mathcal{F} \) are indexed by the elements of \( \Sigma^+ \). We denote the indecomposable projective object that has \( L(\mu) \) as a quotient by \( I(\mu) \).

(ii) The projective object \( I(\mu), \mu \in \Sigma^+ \), has a KCS.
Proof. By Lemma 2.1,
\[ \dim \text{Hom}_G(A(\mu), L(\mu)) = \dim \text{Hom}_G(L_0(\mu), L(\mu)) \geq 1. \]

Therefore by Lemma 2.4, there exists an indecomposable component, say \( I \), in the decomposition of \( A(\mu) \), such that \( \dim \text{Hom}_G(I, L(\mu)) = 1 \). Let \( \pi : I \to L(\mu) \) be the projection. Now if \( I' \) is an indecomposable projective object in \( \mathcal{F} \) that has \( L(\mu) \) as a quotient, and \( \pi' : I' \to L(\mu) \) is the projection, then by projectivity, there exists \( \varphi : I \to I' \) such that \( \pi' \circ \varphi = \pi \).

If \( \varphi(I) \neq I' \), then \( \varphi(I) \) must be contained in a maximal submodule of \( I' \).

But by Lemma 2.4, \( I' \) has a unique maximal submodule, so \( \ker \pi' \) is the maximal submodule of \( I' \) and hence \( \pi = 0 \), which is impossible. So \( \varphi \) is onto, and by the projectivity of \( I' \), \( I' \) is a summand of \( I \). Hence \( I = I' \).

Thus we can denote \( I \) by \( I(\mu) \). Now if \( M \) is an indecomposable projective object of \( \mathcal{F} \), then \( M \) is a direct sum of two \( G \)-modules, then each component has a KCS. In fact, if we choose a maximal weight vector \( v \) of \( M \) in such a way that \( v \in M \) or \( \langle v \rangle \) generated by \( v \) is a Kac module and \( M/\langle v \rangle \equiv M_1/\langle v \rangle \oplus M_2 \), hence we can use induction on the length of the KCS of \( M \) to prove that each \( M \) has a KCS.

To prove (ii), we note that if \( \mu \in \mathcal{F} \) has a KCS, and \( M = \mathcal{F} \oplus M_2 \) is a direct sum of two \( G \)-modules, then each component has a KCS. In fact, if we choose a maximal weight vector \( v \) of \( M \) in such a way that \( v \in M_1 \) or \( \langle v \rangle \), then the \( G \)-submodule \( \langle v \rangle \) generated by \( v \) is a Kac module and \( M/\langle v \rangle \equiv M_1/\langle v \rangle \oplus M_2 \), hence we can use induction on the length of the KCS of \( M \) to prove that each \( M \) has a KCS. Since \( I(\mu) \) is an indecomposable component of \( A(\mu) \), it follows from Lemma 2.3 that \( I(\mu) \) has a KCS. The proof of the proposition is now complete. \( \blacksquare \)

Lemma 2.6. Let \( \lambda, \mu \in \Sigma^+ \); then
\[ (A(\mu) : K(\lambda)) = \dim \text{Hom}_G(A(\mu), K(\lambda)). \]

Proof. From the proof of Lemma 2.3, we see that
\[ (A(\mu) : K(\lambda)) = \dim \text{Hom}_G(L_0(\mu), U(G_{+1}) \otimes L_0(\mu)) \]
\[ = \dim \text{Hom}_G(U(G_{-1}) \otimes L_0(\mu), L_0(\mu)). \]

The last step holds because \( U(G_{+1}) \otimes L_0(\mu) \) is a direct sum of simple \( G_{0} \)-modules. Hence by the fact that \( G_{-1} \) and \( G_{+1} \) are contragredient \( G_{0} \)-modules, we have
\[ (A(\mu) : K(\lambda)) = \dim \text{Hom}_G(L_0(\mu), U(G_{+1})^* \otimes L_0(\lambda)) \]
\[ = \dim \text{Hom}_G(L_0(\mu), U(G_{-1}) \otimes L_0(\lambda)). \]

Since \( U(G_{-1}) \otimes L_0(\lambda) \equiv K(\lambda) \) as a \( G_{0} \)-module, the lemma follows from Lemma 2.1. \( \blacksquare \)
Theorem 2.7 (The duality theorem). Assume that for any $\lambda \in \Sigma^+$, $[K(\lambda); L(\lambda)] = 1$. Then for $\lambda, \mu \in \Sigma^+$,

$$ (I(\mu); K(\lambda)) = [K(\lambda); L(\mu)]. \quad (2.3) $$

Proof. First we note that the theorem is true for $\lambda = \mu$, since both sides of (2.3) are equal to 1 in this case. Then we note that the theorem is also true if $\lambda \neq \mu$ and $\lambda \notin \mu + \Lambda$, where $\Lambda$ is the set of weights of $U(G_{+1})$, because both sides of (2.3) are 0 in this case. So we can assume that $\lambda \in \mu + \Lambda$. Since $L(\mu)$ is a quotient of $I(\mu)$ and $I(\mu)$ is projective, we have $[K(\lambda); L(\mu)] = \dim_c \text{Hom}_G(I(\mu), K(\lambda))$. We will show that

$$ (I(\mu); K(\lambda)) = \dim_c \text{Hom}_G(I(\mu), K(\lambda)). \quad (2.4) $$

Let us write

$$ A(\mu) = I(\mu) \oplus \bigoplus_{v \in \mu + \Lambda} m_v I(\nu). $$

Then by Lemma 2.6,

$$ (A(\mu); K(\lambda)) = (I(\mu); K(\lambda)) + \sum_{v \in \mu + \Lambda} m_v (I(\nu); K(\lambda)) $$

$$ = \dim_c \text{Hom}_G(I(\mu), K(\lambda)) $$

$$ + \sum_{v \in \mu + \Lambda} m_v \dim_c \text{Hom}_G(I(\nu), K(\lambda)). \quad (2.5) $$

Let $\beta = 2\rho_1$ (the sum of all positive odd roots). Then $\mu + \beta$ is the maximal weight of $\mu + \Lambda$, and $\dim_c \text{Hom}_G(I(\mu + \beta), K(\lambda)) = (I(\mu + \beta); K(\lambda))$ because they are either both equal to 1 or both equal to 0. Now by using (2.5) we can apply induction to prove (2.4) by going downward on the weights as follows. Suppose that (2.4) has been proved for all $\mu'$ such that $\mu' < \mu'' \leq \mu + \beta$. By applying (2.5) to $\mu = \mu'$, and noting that by the construction of $I(\nu)$, if $\nu > \lambda$, then $(I(\nu); K(\lambda)) = 0 = \dim_c \text{Hom}_G(I(\nu), K(\lambda))$, we have

$$ (I(\mu'); K(\lambda)) = \dim_c \text{Hom}_G(I(\mu'), K(\lambda)). $$

This completes the induction and proves (2.4). Hence the theorem follows as desired.

Remarks. (1) In proving the duality theorem, we need the condition that $[K(\lambda); L(\lambda)] = 1$ for all $\lambda \in \Sigma^+$, because the following example of [HKV, p. 486] shows a rather awkward situation in the highest weight representation theory of Lie superalgebras. Let $G = sl(2,2)$, let $\lambda = \frac{1}{2}$, $\mu = 0$. Then we have

$$ [K(\lambda); L(\lambda)] = 1, \quad [K(\mu); L(\mu)] = 0. $$

This shows that the duality theorem does not hold in this case.
[1; 0; 1]. Then \( \lambda \) is doubly atypical with atypical roots \( \beta_{12} \) and \( \beta_{21} \) \( \beta_{ij} = \varepsilon_i - \delta_j \), and the Kac module \( K(\lambda) \) has five composition factors with maximal weights given by

\[
\lambda, \lambda - \beta_{12}, \lambda - \beta_{21}, \lambda - \beta_{12} - \beta_{21}, \lambda - \beta_{12} - \beta_{21} - \beta_{22}.
\]

Note that \( \lambda - \beta_{12} - \beta_{21} = \lambda \) because \( \varepsilon_1 + \varepsilon_2 - \delta_1 - \delta_2 = 0 \). We see that \([K(\lambda):L(\lambda)] = 2\).

(2) It is easy to see that \([K(\lambda):L(\lambda)] > 1\) can only happen for \( \text{sl}(m,m) \).

We now discuss the injective objects of \( \mathcal{F} \) briefly. Let

\[
0 \leftarrow C \leftarrow I(0) \leftarrow P_1 \leftarrow \cdots
\]

be the projective resolution of the trivial \( G \)-module \( C \). Then for any \( M \in \mathcal{F} \), we have the injective resolution for \( M \):

\[
0 \longrightarrow M \longrightarrow \text{Hom}_C(I(0), M) \longrightarrow \text{Hom}_C(P_1, M) \longrightarrow \cdots. \tag{2.6}
\]

Hence there are enough injective objects in \( \mathcal{F} \). By putting \( M = L(\lambda) \) in (2.6), we see that the indecomposable injective object that contains \( L(\lambda) \) (i.e., the injective hall of \( L(\lambda) \)) is a direct summand of \( \text{Hom}_C(I(0), L(\lambda)) \).

Since any indecomposable injective object contains a unique simple submodule, we have proved the following proposition.

**Proposition 2.8.** The indecomposable injective objects of \( \mathcal{F} \) are indexed by the elements of \( \Sigma^+ \), and each indecomposable injective object of \( \mathcal{F} \) is a direct summand of \( \text{Hom}_C(I(0), L(\lambda)) \) for some \( \lambda \in \Sigma^+ \).

3. A CLASSIFICATION OF THE OBJECTS OF \( \mathcal{F} \)

By applying the results of Section 2, we describe a classification scheme for the objects of \( \mathcal{F} \). Note that by Proposition 2.5, every object of \( \mathcal{F} \) is a quotient of some projective \( G \)-module which is a direct sum of certain \( I(\mu) \)'s. We extend the partial order on \( \Sigma^+ \) (inherited from \( H^* \)) to the set \( F(\Sigma^+) \) of all the finite subsets of \( \Sigma^+ \) as follows.

Let \( A, B \in F(\Sigma^+) \). If \( |A| < |B| \) (or \( |A| = |B| \)) then \( A < B \) (or \( A \geq B \)); if \( |A| = |B| \), then \( A \leq B \) (or \( A \geq B \)) if and only if after a suitable arrangement of the elements of \( A \) and \( B \), say \( A = \{ \mu_1, \ldots, \mu_k \} \) and \( B = \{ \lambda_1, \ldots, \lambda_k \} \), the relations \( \mu_i \leq \lambda_i \) (or \( \mu_i \geq \lambda_i \)) \( 1 \leq i \leq k \), hold. This partial order of \( F(\Sigma^+) \) induces a partial order on the set \( \mathcal{P} \) of all the projective objects of \( \mathcal{F} \).

**Lemma 3.1.** Let \( M \in \mathcal{F} \). Then there exists a unique minimal object in \( \mathcal{P} \) that has \( M \) as a quotient.
Proof. Suppose that \( Q_1 \) and \( Q_2 \) are minimal among the objects of \( \mathcal{P} \) that have \( M \) as a quotient. Let \( \pi_1 : Q_1 \to M \) and \( \pi_2 : Q_2 \to M \) be the natural projections. By projectivity, there exists a \( G \)-module homomorphism \( \phi : Q_1 \to Q_2 \) such that \( \pi_1 = \pi_2 \circ \phi \). Suppose that \( Q_2 = \bigoplus_{1 \leq i \leq k} m_i I(\mu_i) \). Then by Proposition 2.5, each \( I(\mu_i) \) is singly generated, and by the minimality, a generator of \( I(\mu_i) \) cannot be in \( \ker \pi_2 \). Hence by considering \( Q_1, Q_2, \) and \( M \) as \( G \)-modules, we see that some set of generators of \( Q_2 \) is contained in \( \phi(Q_1) \). Therefore \( \phi \) is onto. Similarly, there is an onto homomorphism from \( Q_2 \) to \( Q_1 \). But \( Q_1 \) and \( Q_2 \) are finite dimensional \( G \)-modules, so \( \phi \) must also be one-to-one and hence is an isomorphism. Thus the lemma follows, as desired.

**Definition 3.2.** Let \( M \in \mathcal{F} \). The unique minimal object in \( \mathcal{P} \) that has \( M \) as a quotient is called the projective cover of \( M \) in \( \mathcal{F} \).

**Definition 3.3.** An object \( M \in \mathcal{F} \) is called a local cover if it has a unique maximal submodule \( M' \) such that \( M/M' \equiv L(\mu) \) for some \( \mu \in \Sigma^+ \). If this is the case, we call \( M \) a local cover of \( L(\mu) \).

It is clear that any highest weight \( G \)-module in \( \mathcal{F} \) is a local cover, and in particular, Kac modules are local covers. The indecomposable projective objects \( I(\mu) \) are also local covers.

**Lemma 3.4.** Let \( \mu \in \Sigma^+ \) and let \( M \) be a local cover of \( L(\mu) \). Then \( M \) is a quotient of \( I(\mu) \).

**Proof.** Let \( M' \) be the unique maximal submodule of \( M \). Then any nonzero vector \( v \in M_{\mu} - M' \) is a generator of \( M \), since otherwise the submodule generated by \( v \) is a proper submodule and must be contained in \( M' \). Let \( \varphi : M \to L(\mu) \) and \( \phi : I(\mu) \to L(\mu) \) be the canonical projections; then there is a \( \phi' : I(\mu) \to M \) such that \( \phi = \varphi \circ \phi' \). Let \( v \) be a generator of \( I(\mu) \) of weight \( \mu; \) then \( \phi(v) \) generates \( L(\mu) \) and hence \( \phi'(v) \) generates \( M \), thus \( \phi' \) is onto, and the lemma follows.

**Proposition 3.5.** Let \( M \in \mathcal{F} \). Let \( S = \{v_1, \ldots, v_k\} \) be a set of generators of \( M \) that consists of primitive vector \( v_i \) of weight \( \mu_i, 1 \leq i \leq k \), such that the submodule generated by each \( v_i \) is a local cover. Suppose further that \( |S| = k \) is minimal among such generating sets of \( M \). Then \( \bigoplus_{1 \leq i \leq k} I(\mu_i) \) is the projective cover of \( M \) in \( \mathcal{F} \).

**Proof.** For \( v_i \in S \), let \( V_i = (v_i) \) be the submodule of \( M \) generated by \( v_i \). Then \( M = \sum_{1 \leq i \leq k} V_i \). By Lemma 3.4, for each \( i \), there is an onto \( G \)-module homomorphism \( \varphi_i : I(\mu_i) \to V_i \); hence \( \varphi = \sum_{1 \leq i \leq k} \varphi_i \) is an onto \( G \)-module homomorphism from \( \bigoplus_{1 \leq i \leq k} I(\mu_i) \) to \( M \). Hence by the minimality assumption of the proposition, the projective module \( \bigoplus_{1 \leq i \leq k} I(\mu_i) \) must be the projective cover of \( M \).
PROPOSITION 3.6. Let \( M_1 \) and \( M_2 \in \mathcal{F} \), let \( P_1 \) and \( P_2 \) be the projective covers of \( M_1 \) and \( M_2 \), respectively, and let \( N_1, N_2 \) be the corresponding submodules of \( P_1 \) and \( P_2 \), respectively, such that \( P_1/N_1 \cong M_1, P_2/N_2 \cong M_2 \). Then \( M_1 \cong M_2 \) if and only if \( P_1 = P_2 \) and there is an automorphism \( \sigma \) of \( P_1 \) such that \( \sigma(N_1) = N_2 \).

Proof. This is clear.

We need to describe the automorphism groups of the objects of \( \mathcal{F} \).

PROPOSITION 3.7. Let \( Q \in \mathcal{F} \), and let \( Q = \bigoplus_{1 \leq i \leq r} m_i I(\mu_i) \) for some \( \mu_i \in \Sigma^+, m_i \in \mathbb{N}, 1 \leq i \leq r \). Let \( d_i = m_i (\dim I(\mu_i) - 1) \) and let \( d = \sum_{i=1}^r d_i \). Then \( \text{Aut}(Q) \cong \text{GL}(\mathbb{C}^{m_1}) \times \cdots \times \text{GL}(\mathbb{C}^{m_r}) \times \mathbb{C}^d \).

Proof. Let \( Q_i = m_i I(\mu_i) \) (a direct sum of \( m_i \) copies of \( I(\mu_i) \)), \( 1 \leq i \leq r \). Then it is clear that \( \text{Aut}(Q) \cong \text{Aut}(Q_1) \times \cdots \times \text{Aut}(Q_r) \). Thus we may assume that \( Q = \bigoplus_{1 \leq i \leq r} I(\mu_i) \) is a direct sum of \( r \) copies of \( I(\mu) \) (we keep the indices for our convenience). Let \( \dim I(\mu) = m \) and let \( e = r(m - 1) \). Let \( I' \) be the maximal submodule of \( I(\mu) \), let \( Q' = \bigoplus_{1 \leq i \leq r} I_i' \), and let \( Q = Q/Q' \). Then \( Q \cong \bigoplus_{1 \leq i \leq r} L(\mu_i) \). Let \( v_i \in I(\mu_i) \) be a generator of \( I(\mu) \). If \( \sigma \in \text{Aut}(Q) \), then the set \( \{ \sigma(v_i) : 1 \leq i \leq r \} \) generates \( Q \), and hence the images \( \overline{\sigma(v_i)} \) of \( \sigma(v_i) \) in \( Q \) form a basis of \( Q \). Thus there is a \( \varphi \in \text{GL}(\mathbb{C}^r) \) such that \( \varphi(v_i) = \overline{\sigma(v_i)} \), and if we denote by \( \varphi' \) the linear map on \( \sum_{1 \leq i \leq r} \mathbb{C} v_i \) induced by \( \varphi \), then \( \varphi'(v_i) + x_i = \sigma(v_i) \) for some \( x_i \in Q \). Now suppose that \( \phi \in \text{GL}(\mathbb{C}^r) \) and \( y_i \in Q \) (\( 1 \leq i \leq r \)); then \( w_i = \phi(v_i) + y_i \) form a set of generators of \( Q \). If we define \( \sigma : Q \to Q \) by

\[
\sigma: \sum_{i=1}^r u_i v_i \mapsto \sum_{i=1}^r u_i w_i, \quad u_i \in U(G), \quad 1 \leq i \leq r,
\]

then \( \sigma \in \text{Aut}(Q) \). Therefore \( \text{Aut}(Q) \cong \text{GL}(\mathbb{C}^r) \times \mathbb{C}^e \). Now the proposition follows.

4. PROJECTIVES AND INJECTIVES OF \( \mathcal{F} \) FOR \( \text{sl}(2,1) \)

We fix \( G = \text{sl}(2,1) \) in this section. Let

\[
e_1 = E_{12}, \quad e_2 = E_{22}, \quad e_3 = E_{13}, \quad f_1 = E_{21}, \quad f_2 = E_{32}, \quad f_3 = E_{31},
\]

\[
h_1 = E_{11} - E_{22}, \quad h_2 = E_{22} + E_{33},
\]

Then \( \{ e_1, e_2, e_3, f_1, f_2, f_3, h_1, h_2 \} \) is a basis of \( G \), and \( \langle h_1, h_2 \rangle = H \) is a Cartan subalgebra of \( G \) with basis \( \{ h_1, h_2 \} \). We choose \( \alpha = e_1 - e_2, \beta = e_2 - e_1 \) as a simple root system for \( G \). Corresponding to the chosen
simple root system, the set of positive even root and the set of positive odd roots are \( R_0^+ = \{ \alpha \} \) and \( R_1^+ = \{ \beta, \alpha + \beta \} \), respectively. Thus \( \rho = \rho_0 - \rho_1 = -\beta \). For a weight \( \lambda \in H^* \), let \( a = \lambda(h_1) \) and \( b = \lambda(h_2) \), and write \( \lambda = (a, b) \). Then we have \( \alpha = (2, -1), \beta = (-1, 0), \) and \( \rho = (1, 0) \).

According to [K3], a weight \( \lambda \in H^* \) is typical if and only if \((\lambda + \rho)(h_1 + h_2) \neq 0 \) and \((\lambda + \rho)(h_2) \neq 0 \). That is, if \( \lambda = (a, b) \), then \( \lambda \) is typical if and only if \( a + b + 1 \neq 0 \) and \( b \neq 0 \). Also, \( L(\lambda) \) is finite dimensional if and only if \( a \in \mathbb{Z}_+ \). Hence a weight \( \lambda = (a, b) \in \Sigma^+ \) is atypical if and only if

(i) \( b = 0 \); or

(ii) \( b = -a - 1 \) is a negative integer.

Thus an atypical weight is singly atypical with atypical root \( \beta \) or \( \alpha + \beta \).

A straightforward computation shows that if \( \lambda \in \Sigma^+ \) is atypical, then the character of \( K(\lambda) \) is \( \text{ch} \ L(\lambda) + \text{ch} \ L(\lambda - \beta) \) or \( \text{ch} \ L(\lambda) + \text{ch} \ L(\lambda - \alpha - \beta) \) according to whether the atypical root is \( \beta \) or \( \alpha + \beta \). If \( v_\lambda \) denotes a highest weight vector of \( K(\lambda) \), then \( f_2v_\lambda \) or \((af_3 + f_2f_1)v_\lambda \) gives a primitive (maximal, to be more precise) vector of \( K(\lambda) \) which is not a highest weight vector.

Consider the indecomposable projective objects of \( \mathcal{F} \). Let \( \lambda = (a, b) \in \Sigma^+ \). If \( \lambda \) is typical, then \( L(\lambda) = K(\lambda) = I(\lambda) \). Suppose that \( \lambda \) is atypical. Let \( v_\lambda \in I(\lambda)_f \) be a generator of \( I(\lambda) \). We have two cases to consider.

**Case (i).** \( b = 0 \). The structure of \( I(\lambda) \) is given by the diagram

\[
\begin{array}{ccc}
v_\lambda & \longrightarrow & e_2v_\lambda \\
\downarrow & & \downarrow \\
f_2v_\lambda & \longrightarrow & e_2f_2v_\lambda
\end{array}
\]

where the vectors are primitive vectors.

**Case (ii).** \( a + b + 1 = 0 \). The structure of \( I(\lambda) \) is given by the diagram

\[
\begin{array}{ccc}
v_\lambda & \longrightarrow & e_3v_\lambda \\
\downarrow & & \downarrow \\
wv_\lambda & \longrightarrow & u_3v_\lambda
\end{array}
\]

where \( u = af_3 + f_2f_1 \in U(G) \).

**Theorem 4.1.** The set \( \mathcal{F} = \{ I(\lambda): \lambda \in \Sigma^+ \} \) is a complete set of indecomposable projective objects as well as a complete set of indecomposable injective objects of \( \mathcal{F} \).

**Proof.** We only need to prove that \( \mathcal{F} \) is a complete set of indecomposable injective objects of \( \mathcal{F} \). By the results on the structures of the \( I(\lambda) \)'s,
we see that each $I(\lambda)$ contains a unique simple $G$-module; this simple module is a copy of $L(\lambda)$. Hence we only need to prove that each $I(\lambda)$ is injective. We may assume that $\lambda$ is atypical, since if $\lambda$ is typical, $I(\lambda) = L(\lambda)$ is clearly injective. Now by Theorem 4.2 in [Su], any $M \in \mathcal{F}$ that contains $I(\lambda)$ as a submodule must have $I(\lambda)$ as a direct summand. Therefore $I(\lambda)$ is injective, and the theorem follows.

5. KAZHDAN–LUSZTIG POLYNOMIALS

Since there are enough projective objects in $\mathcal{F}$, for $M$ and $N \in \mathcal{F}$, we can define $\text{Ext}^n(M, N)$ as usual, i.e., let

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

be the projective resolution of $M$ in $\mathcal{F}$, and then by applying the functor $\text{Hom}_G(\cdot, N)$ to the resolution, we obtain

$$0 \rightarrow \text{Hom}_G(M, N) \rightarrow \text{Hom}_G(P_0, N) \rightarrow \text{Hom}_G(P_1, N) \rightarrow \cdots,$$

and we define $\text{Ext}^n(M, N) = \text{Ker} d_{n+1}/\text{Im} d_n$. We have the following theorem on $\text{Ext}^n$.

**Theorem 5.1** (cf. [R–CW, Sect. 7, Theorem 2]). Let $M \in \mathcal{F}$, and let $\mu \in \Sigma^+$. Then $\text{Ext}^n(K(\mu), M)$ is naturally isomorphic with $\text{Hom}_{G_0}(L_0(\mu), H^n(G_{+1}, M))$.

**Proof.** The proof is a modification of the proof given in [R–CW]. For $j = 0, 1, \ldots$, and $\mu \in \Sigma^+$, let

$$X_{j, \mu} = U(G_{+}) \otimes \Lambda^j(G_{+1}) \otimes L_0(\mu),$$

where $\Lambda$ satisfies

$$x_1 \wedge x_2 = x_2 \wedge x_1, \quad \text{for } x_1, x_2 \in G_{+1}.$$

Note that $[G_{+1}, G_{+1}] = (0)$, so we can define $\sigma_j \colon X_{j, \mu} \rightarrow X_{j-1, \mu}$ ($j \geq 1$), by

$$\sigma_j(u \otimes x_1 \wedge \cdots \wedge x_j \otimes v) = \sum_{k=1}^j u x_k \otimes x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_j \otimes v.$$
Let $\sigma_i: X_{i,\mu} \to L_0(\mu)$ be the natural module action. Then the sequence
\[
\cdots \longrightarrow X_{j,\mu} \xrightarrow{\sigma_j} \cdots \xrightarrow{\sigma_1} X_{0,\mu} \xrightarrow{\epsilon} L_0(\mu) \longrightarrow 0
\]
is an exact sequence of $U(P)$-modules.
For $M \in \mathcal{F}$, let $M_j = \text{Hom}_{U(P)}(X_{j,\mu}, M)$, $j \geq 0$, let $\text{Hom}_{U(P)}(L_0(\mu), M) = M_{-1}$, and let
\[
M' : 0 \to M_{-1} \to M_0 \to \cdots
\]
be the resulting complex. We show that
\[
\text{Hom}_{G_\alpha}(L_0(\lambda), H^*(G_{+1}, M)) \cong H^*(M'). \quad (5.1)
\]
Put $X_n = U(G_{+1}) \otimes \Lambda^*(G_{+1})$. For $f \in \text{Hom}_{G_\alpha}(L_0(\mu), \text{Hom}_{G_{+1}}(X_n, M))$, we define $f' \in \text{Hom}_c(X_{n,\mu}, M)$ by
\[
f'(u \otimes v) = f(v)(u), \quad u \in X_n, v \in L_0(\mu).
\]
Then one can check that $f' \in M_n$. For example, if $x \in G_0$, $u \in X_n$, $v \in L_0(\mu)$, then
\[
f'(x(u \otimes v)) = f'(xu \otimes v) + f'(u \otimes xv)
= f(v)(xu) + f(xv)(u)
= f(v)(xu) + x(f(v)(u)) - f(v)(xu)
= x(f'(u \otimes v)).
\]
Now for $g \in \text{Hom}_{U(P)}(X_{n,\mu}, M)$, we define
\[
g^* \in \text{Hom}_c(L_0(\mu), \text{Hom}_{G_{+1}}(X_n, M))
\]
by
\[
g^*(v)(u) = g(u \otimes v), \quad v \in L_0(\mu), u \in X_n.
\]
Then for $x \in G_0$, $v \in L_0(\mu)$, and $u \in X_n$, we have
\[
g^*(xv)(u) = g(u \otimes xv) = g(x(u \otimes v) - xu \otimes v)
= x(g(u \otimes v)) - g(xu \otimes v)
= x(g^*(v)(u)) - (g^*(v) \circ x)(u)
= (xg^*(v))(u).
\]
Hence $g^* \in \text{Hom}_{G_{+1}}(L_0(\mu), \text{Hom}_{G_{+1}}(X_n, M))$. It is easy to see that $(f')^* = f$ and $(g^*)' = g$, so we have an isomorphism
\[
\text{Hom}_{G_{+1}}(L_0(\mu), \text{Hom}_{G_{+1}}(X_n, M)) \cong \text{Hom}_{U(P)}(X_{n,\mu}, M).
\]
Now (5.1) follows from the definitions of the complexes involved.
By (5.1), in order to prove the theorem, we only need to prove that 
\[ H^\ast(M) = \text{Ext}^\ast(K(\mu), M) \]. Let \( \tau \in \text{Hom}_{U(P)}(X_n, \mu, M) \) be such that \( \tau \circ \sigma_{n+1} = 0 \). Then we can construct the following diagram with exact rows,

\[
\begin{array}{ccccccccc}
\cdots & X_{n+1, \mu} & \xrightarrow{\sigma_{n+1}} & X_{n, \mu} & \xrightarrow{\sigma_n} & X_{n-1, \mu} & \cdots & \xrightarrow{\sigma_1} & X_{0, \mu} & \xrightarrow{\epsilon} & L_0(\mu) & \rightarrow 0 \\
& & & \tau & & \tau_{n-1} & & \tau_0 & & \tau_0 & & \\
0 & \rightarrow & M & \xrightarrow{\beta_n} & Y_{n-1, \mu} & \cdots & \xrightarrow{\beta_1} & Y_{0, \mu} & \xrightarrow{\beta_0} & L_0(\mu) & \rightarrow 0 \\
\end{array}
\]

where \( (\tau_{n-1}, \beta_n) \) is the push-out of \( (\sigma_i, \tau) \), \( Y_{i, \mu} = X_{i, \mu} \), and \( \tau_i = \text{identity} \) for \( i = 0, \ldots, n - 2 \). Tensoring the bottom row with \( U(G) \) over \( U(P) \), we get an exact sequence of \( G \)-modules

\[
0 \rightarrow U(G) \otimes_{U(P)} M \xrightarrow{1 \otimes \beta_n} U(G) \otimes_{U(P)} Y_{n-1, \mu} \xrightarrow{1 \otimes \beta_1} \cdots \xrightarrow{1 \otimes \beta_1} U(G) \otimes_{U(P)} Y_{0, \mu} \xrightarrow{1 \otimes \beta_0} K(\mu) \rightarrow 0.
\]

Let \( \pi: U(G) \otimes_{U(P)} M \rightarrow M \) be the onto \( G \)-module homomorphism defined by \( \pi(u \otimes m) = um, u \in U(G), m \in M \). Let \( K = \ker \pi \), let

\[
V_{n-1, \mu} = U(G) \otimes_{U(P)} Y_{n-1, \mu} / (1 \otimes \beta_n)(K),
\]

\[
V_{i, \mu} = U(G) \otimes_{U(P)} Y_{i, \mu}, \quad 0 \leq i < n - 1,
\]

and let \( d_i = 1 \otimes \beta_i \) (\( 0 \leq i < n - 1 \)). Then \( V_{i, \mu} \in \mathcal{F} \) for \( 0 \leq i \leq n - 1 \), and

\[
0 \rightarrow M \rightarrow V_{n-1, \mu} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} V_{0, \mu} \xrightarrow{d_0} K(\mu) \rightarrow 0
\]

is an exact sequence of \( G \)-modules depending only on the cohomology class of \( \tau \). We denote this exact sequence by \([\tau]\).

On the other hand, given a representation

\[
E_n: 0 \rightarrow M \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} V_0 \xrightarrow{d_0} K(\mu) \rightarrow 0
\]

of an element in \( \text{Ext}^\ast(K(\mu), M) \), set \( V_n = M \). Then we can define \( G_0 \)-module homomorphisms \( \varphi_i: \text{Im} \ d_i \rightarrow V_i, \ 0 \leq i \leq n \), such that \( d_i \circ \varphi_i(z) = z \) for all \( z \in \text{Im} \ d_i \). Let \( 1 \otimes v_\mu \) be a generator of \( K(\mu) \) and let \( u_0 = \varphi_0(1 \otimes v_\mu) \). Then \( u_0 \) induces an element \( \phi_0 \) in \( \text{Hom}_{U(P)}(X_0, \mu, V_0) \) via \( x \otimes v_\mu \mapsto xyu_0 \), where \( x \in U(G_{+1}), y \in U(G_0) \). Fix elements \( x \in U(G_{+1}) \) and \( y \in G_{+1} \). Then

\[
d_0 \circ \phi_0 \circ \sigma_i(x \otimes y \otimes v_\mu) = d_0 \circ \phi_0(x y \otimes v_\mu) = d_0(x y u_0) = xy(1 \otimes v_\mu) = 0.
\]
Hence \( \phi_0 \circ \sigma_1(x \otimes y \otimes v_\mu) \in \text{Im } d_1 \). So if we let \( u_1 = \varphi_1 \circ \sigma_1(x \otimes y \otimes v_\mu) \), then \( u_1 \) induces an element \( \phi_1 \) in \( \text{Hom}_{U(V)}(X_{1, \mu}, V_1) \) via \( a \otimes b \otimes c v_\mu \rightarrow abc u_1 \), where \( a \in U(G_{-1}) \), \( b \in \lambda^1 \otimes G_{-1}, c \in U(G_0) \). Inductively, we obtain an element \( \phi \in \text{Hom}_{U(V)}(X_{n, \mu}, M) \) such that \( \phi \circ \sigma_{n+1} = 0 \). The cohomology class of \( \phi \) depends only on \( E_n \); we denote this cohomology class by \( (E_n) \). Then \( [E_n] = E_n \) and \( (f) = f \). Hence the theorem follows.

**Definition 5.2.** Let \( \lambda, \mu \in \Sigma^+ \). We define an element \( P_{\lambda, \mu}(q) \) in \( \mathbb{Z}[q] \) by

\[
P_{\lambda, \mu}(q) = \sum_{n \geq 0} \text{dim Ext}^n(K(\lambda), L(\mu)) \cdot q^n.
\]

By using the example in the remark (1) in Section 2, one can see that \( P_{\lambda, \mu}(q) \) is not a polynomial in general. But we do have the following proposition.

**Proposition 5.3.** Assume that for all \( \lambda \in \Sigma^+ \), \( [K(\lambda):L(\lambda)] = 1 \). Then for \( \lambda, \mu \in \Sigma^+ \), we have

(i) \( P_{\lambda, \mu}(q) \in \mathbb{Z}[q] \) with nonnegative coefficients;
(ii) \( P_{\lambda, \lambda}(q) = 1 \);
(iii) \( P_{\lambda, \mu}(q) = 0 \) if \( \lambda \) is not \( \leq \mu \).

**Proof.** Consider the projective resolution for \( K(\lambda): \)

\[
0 \leftarrow K(\lambda) \leftarrow I(\lambda) \leftarrow P_1 \leftarrow \cdots.
\]

Then by Proposition 2.5 and Theorem 2.7, one can see that \( P_1 = \bigoplus I(\lambda^1) \) with all \( \lambda^1 > \lambda \), and in general, if \( P_n = \bigoplus I(\lambda^n) \) and \( P_{n+1} = \bigoplus I(\lambda^{n+1}) \), then each \( \lambda^{n+1} > \lambda^n \) for some index \( s \). Now the proposition follows if one further notes that \( \text{Hom}_{C}(P_n, L(\mu)) \neq 0 \) if and only if \( \mu \) is one of the \( \lambda^n \).

The following proposition shows that under the same condition as that in Proposition 5.3, the characters of the simple objects of \( \mathcal{S} \) can be computed by using the polynomials \( P_{\lambda, \mu}(q) \).

**Proposition 5.4.** Assume that \( [K(\lambda):L(\lambda)] = 1 \) for all \( \lambda \in \Sigma^+ \). Then for \( \lambda \in \Sigma^+ \), we have

\[
\text{ch } L(\lambda) = \sum_{\mu \leq \lambda} P_{\mu, \lambda}(-1)^{\text{ch } K(\mu)}
\]

\[
= \sum_{\mu \leq \lambda} P_{\mu, \lambda}(-1)^{\text{ch } L_0(\mu)} \prod_{a \in R^+} (1 + e^{-a}).
\]
Proof. Let $D(\lambda)$ be the subset of $\Sigma^+$ containing the elements $\leq \lambda$. We order the elements of $D(\lambda)$, say $\lambda_1, \lambda_2, \ldots$, such that if $\lambda_i > \lambda_j$ then $i < j$. Then we have a system of equations

$$\text{ch } K(\lambda_i) = \sum_{\lambda_j \in D(\lambda)} a_{ij} \text{ch } L(\lambda_j),$$

where $a_{ij} = [K(\lambda_i):L(\lambda_j)]$. Since the coefficient matrix is an upper triangular matrix with ones on the diagonal, we can solve the system and get

$$\text{ch } L(\lambda_i) = \sum_{\lambda_j \in D(\lambda)} c_{ij} \text{ch } K(\lambda_j).$$

The coefficient matrix of this system is also upper triangular. We want to show that $c_{ij} = P_{\lambda_i, \lambda_j}(-1)$. Consider the projective resolution of $K(\lambda_i)$:

$$0 \leftarrow K(\lambda_i) \leftarrow I(\lambda_i) \leftarrow P_1 \leftarrow \cdots.$$  

If we write $P_n = \bigoplus P_n(\lambda_i, \mu_{n,i})I(\mu_{n,i})$, and let

$$d_{ij} = \sum_{n \geq 0} (-1)^n \dim \text{Ext}^n(K(\lambda_i), L(\lambda_j)),$$

then

$$d_{ij} = \sum_{n \geq 0} (-1)^n \dim \text{Hom}_G(P_n, L(\lambda_j)) = \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_j).$$

By the duality theorem and by using introduction on $n$, we see that

$$\sum_{i < k < j} \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_k)[K(\lambda_k):L(\lambda_j)] = \sum_{n \geq 0} (-1)^n b_{n+1}(\lambda_i, \lambda_j),$$

and hence we have

$$\sum_{i \leq k \leq j} d_{ik} a_{kj} = \sum_{i \leq k \leq j} \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_k)[K(\lambda_k):L(\lambda_j)]$$

$$= \sum_{i < k < j} \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_k)[K(\lambda_k):L(\lambda_j)]$$

$$+ \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_j)[K(\lambda_j):L(\lambda_j)]$$

$$= \sum_{n \geq 0} (-1)^n b_{n+1}(\lambda_i, \lambda_j) + \sum_{n \geq 0} (-1)^n b_n(\lambda_i, \lambda_j)$$

$$= b_0(\lambda_i, \lambda_j) = \delta_{ij}.$$  

Hence $c_{ij} = d_{ij} = P_{\lambda_i, \lambda_j}(-1)$ and the proof of the proposition is complete. □
In this section we assume that the Lie superalgebra $G$ is one of $\mathfrak{sl}(1, n)$ $(n \geq 2)$ and $C(n)$.

**Lemma 6.1.** An element $\lambda \in \Sigma^+$ is either typical or singly atypical.

**Proof.** We will treat the case $G = \mathfrak{sl}(1, n)$; and the other cases can be treated similarly. By using linear functions $e_1, \delta_1, \ldots, \delta_n$ to express an element $\lambda \in H^*$ as $a e_1 + \sum_{1 \leq i \leq n} b_i \delta_i$, we see that the condition $(\lambda + \rho, \beta_{ii}) = 0$ (where $\beta_{ii} = e_1 - \delta_i$) is equivalent to

$$a + b_i + 1 - i = 0. \quad (6.1)$$

If $\lambda \in \Sigma^+$ has more than one atypical root, let $\beta_{ii}, \beta_{ij}$ be two of them and assume that $i < j$. Then by (6.1) we have

$$a + b_i + 1 - i = 0 \quad \text{and} \quad a + b_j + 1 - j = 0,$$

and hence $b_i - b_j = i - j$. But this is impossible since $\lambda \in \Sigma^+$ implies $b_i - b_j \geq 0$. $\blacksquare$

The following theorem is due to Van der Jeugt, Hughes, King, and Thierry-Mieg in the case $\mathfrak{sl}(1, n)$, and due to Van der Jeugt in the case $C(n)$ (see [VHKT1], [V]).

**Theorem 6.2.** Let $\lambda \in \Sigma^+$ be atypical. Then the maximal submodule of $K(\lambda)$ is simple with atypical highest weight $\mu \neq \lambda$.

By this theorem, $[K(\lambda); L(\lambda)] = 1$ for all $\lambda \in \Sigma^+$. We make the following definition for our convenience.

**Definition 6.3.** If $\lambda, \mu \in \Sigma^+$, we use the notation $\lambda \leftarrow \mu$ to indicate the fact that $L(\lambda)$ is the maximal submodule of $K(\mu)$. We say that $\mu$ is $k$-step up-linked to $\lambda$ if there are elements $\lambda_1, \ldots, \lambda_k \in \Sigma^+$ such that $\lambda \leftarrow \lambda_1 \leftarrow \cdots \leftarrow \lambda_k = \mu$. If $\mu$ is $k$-step up-linked to $\lambda$, we also say that $\lambda$ is $k$-step down-linked to $\mu$.

**Corollary 6.4.** Let $\lambda, \mu \in \Sigma^+$. Then

$$P_{\lambda, \mu}(q) = \begin{cases} q^\delta, & \text{if } \mu \text{ is } k\text{-step up-linked to } \lambda; \\
0, & \text{otherwise.} \end{cases}$$

**Proof.** By the duality theorem and Theorem 6.2, for any atypical $\lambda \in \Sigma^+$, $I(\lambda)$ has a filtration

$$I(\lambda) \supseteq I^1 \supseteq 0$$
such that $I(\lambda)/I^1 = K(\lambda)$, $I^1 = K(\lambda_1)$, and $\lambda \leftarrow \lambda_1$. Thus the projective resolution of $K(\lambda)$ is given by

$$0 \leftarrow K(\lambda) \leftarrow I(\lambda) \leftarrow I(\lambda_1) \leftarrow \cdots \leftarrow I(\lambda_k) \leftarrow \cdots,$$

with $\lambda \leftarrow \lambda_1 \leftarrow \cdots \leftarrow \lambda_k \leftarrow \cdots$. If one applies $\text{Hom}_G(I(\lambda), L(\mu))$ to the resolution, the resulting complex collapses to zero unless $\mu = \lambda_k$ for some $k$. If $\mu = \lambda_k$ for some $k \geq 1$, then $\text{Hom}_G(I(\lambda_k), L(\mu)) = 0$ for $i \neq k$ and $\text{Hom}_G(I(\lambda_k), L(\mu)) = \mathbb{C}$. Thus the corollary follows.

The following result was first obtained by Bernstein and Leites in [BL] by a different method.

**Corollary 6.5.** If $\lambda \in \Sigma^+$ is atypical, then

$$\text{ch} \ L(\lambda) = \sum_{k \geq 0} (-1)^k \text{ch} \ K(\lambda_k),$$

where $\lambda_k$ is $k$-step down-linked to $\lambda$ and $\lambda_0 = \lambda$.

**Proof.** Use Proposition 5.4 and Corollary 6.4. □

7. **The case $G = gl(m, n)$**

In this section, we discuss the relationship between the polynomials that we defined in Section 5 and the polynomials $K(\lambda, \mu)$ defined in [Ser].

First let us recall the definition of $K(\lambda, \mu)$ in [Ser]. Let $G = gl(m, n)$ and let $G = G_{-1} \oplus G_0 \oplus G_{+1}$ be the natural consistent $\mathbb{Z}$-grading with $G_0 = gl(m) \oplus gl(n)$. Fix a Cartan subalgebra $H$ of $G$ and a triangular decomposition $G = N^- \oplus H \oplus N^+$ such that $G_{-1} \subset N^-$, $G_{+1} \subset N^+$. Since $G_{-1}$ is a $G_0$-module, for $\lambda \in H^*$, the homology group $H_i(G_{-1}, L(\lambda))$ has a natural $G_0$-module structure. For $\lambda, \mu \in H^*$, we define the generating function [Ser, Sect. 2]

$$K(\lambda, \mu) = \sum_{i \geq 0} \left[ H_i(G_{-1}, L(\lambda)) : L_0(\mu) \right] q^i.$$
Proof. We define

\[ \phi : \text{Hom}_C(U \otimes V, W) \rightarrow \text{Hom}_C(U, \text{Hom}_C(V, W)) \]

by

\[ (\phi(\varphi))(v) = \varphi(u \otimes v), \]

where \( \varphi \in \text{Hom}_C(U \otimes V, W), u \in U, v \in V \) and define

\[ \phi' : \text{Hom}_C(U, \text{Hom}_C(V, W)) \rightarrow \text{Hom}_C(U \otimes V, W) \]

by

\[ \phi'(\tau)(\sum u_i \otimes v_i) = \sum \tau(u_i)(v_i), \]

where \( \tau \in \text{Hom}_C(U, \text{Hom}_C(V, W)), u_i \in U, v_i \in V \). Then \( \phi \) and \( \phi' \) are inverses of each other. To see that \( \phi \) is a \( G \)-module homomorphism, let \( x \in G_a, \varphi \in \text{Hom}_C(U \otimes V, W) \), \( u \in U_c \), and \( v \in V \), where \( a, b, c \in \mathbb{Z}_2 \). Then

\[ (\phi(\varphi))(u)(v) = (x \varphi)(u \otimes v) \]

\[ = (x \circ \varphi)(u \otimes v) - (-1)^{ab}(\varphi \circ x)(u \otimes v) \]

\[ = x(\varphi(u \otimes v)) - (-1)^{ab}\varphi(xu \otimes v + (-1)^{ac}u \otimes xv)), \]

and

\[ (x(\phi(\varphi)))(u)(v) \]

\[ = (x \circ \varphi(u \otimes v)) - (-1)^{ab}(\phi(\varphi) \circ x)(u)(v) \]

\[ = (x(\phi(\varphi))(u))(v) - (-1)^{ab}\phi(\varphi)(xu)(v) \]

\[ = (x \circ (\phi(\varphi))(u))(v) - (-1)^{ab+c}(\phi(\varphi)(u))(x)(v) \]

\[ - (-1)^{ac}\varphi(xu \otimes v) \]

\[ = x(\varphi(u \otimes v)) - (-1)^{ab+c}\varphi(u \otimes xv) - (-1)^{ab}\varphi(xu \otimes v). \]

Hence \( \phi \) is a \( G \)-module homomorphism.

**Proposition 7.2. (Poincaré Duality)** Let \( G \) be a finite dimensional Lie superalgebra, and let \( V \) be a \( G \)-module. Then

\[ H^n(G, V^*) \cong H_n(G, V)^* \]

as \( G \)-modules.
**Proof.** By Lemma 7.1, we have a $G$-module isomorphism
\[ \text{Hom}_c(U \otimes_c V, C) \cong \text{Hom}_c(U, \text{Hom}_c(V, C)), \]
for any $G$-modules $U, V$ and the trivial $G$-module $C$. Hence we have a $G$-module isomorphism of complexes
\[ \text{Hom}_c(P \otimes_c V, C) \cong \text{Hom}_c(P, \text{Hom}_c(V, C)), \]
where $P$ is a projective resolution of $G$. Since $H_n$ commutes with $\text{Hom}_c(-, C)$, by taking homology we obtain the isomorphism in the proposition.

**Corollary 7.3.** Let $G$ be $\mathfrak{gl}(m, n)$ or a type I classical Lie superalgebra and assume that for any $\lambda \in \Sigma^+$, $[K(\lambda)L(\lambda)] = 1$. Let $\lambda, \mu \in \Sigma^+$. Then
\[ P_{\mu, \lambda}(q) = \sum_{i \geq 0} \left[ H^i(G_{+1}, L(\lambda))^L_0(\mu) \right] q^i = \sum_{i \geq 0} [H_i(G_{+1}, L(\lambda)^*)^L_0(\mu)] q^i. \]

**Proof.** By Theorem 5.1 and Proposition 7.2, we have
\[ \text{Ext}^n(K(\mu), L(\lambda)) \cong \text{Hom}_{G_0}(L_0(\mu), H^n(G_{+1}, L(\lambda))) = H^n(G_{+1}, L_0(\mu)). \]
Since $G_{+1}$ and $L(\lambda)$ are finite dimensional $G_0$-modules and all $G_0$-modules involved are completely reducible, we have
\[ \dim(\text{Ext}^n(K(\mu), L(\lambda))) = [H^n(G_{+1}, L(\lambda))^L_0(\mu)] = [H_n(G_{+1}, L(\lambda)^*)^L_0(\mu)]. \]
Thus the corollary follows.

Now we come back to $G = \mathfrak{gl}(m, n)$. Define an involution $\sigma$ of $G$ by $\sigma(e_+) = f_+$, $\sigma(f_+) = e_+$, and $\sigma(h) = h$ for $h \in H$. The category $\mathcal{F}$ is self-dual with respect to the functor $\sigma$ as described below.

For $M \in \mathcal{F}$, we define a $G$-module structure on $M^*$ by
\[(x\varphi)(m) = \varphi(\sigma(x)m), \quad \text{for } \varphi \in M^*, x \in G, m \in M, \]
and denote the resulting $G$-module by $M^{\sigma}$. The proofs of Propositions 4.6 and 4.7 in [DGK] work in the case of a contragredient Lie superalgebra without much change; hence we have similar results. We state these results in Lemma 7.4 and Proposition 7.5 below without proof.
**Lemma 7.4.** Let $G$ be a contragredient Lie superalgebra and let $M \in \mathcal{F}$. Then

(i) $M^\sigma \in \mathcal{F}$,

(ii) $\text{ch} \ M = \text{ch} \ M^\sigma$,

(iii) $M \cong (M^\sigma)^\sigma$,

(iv) $M \rightarrow M^\sigma$ is a contravariant functor which is exact,

(v) $L(\lambda)^\sigma \cong L(\lambda)$ for $\forall \lambda \in \Sigma^+$. $lacksquare$

**Proposition 7.5.** Let $G = \mathfrak{gl}(m, n)$. Then for $M \in \mathcal{F}$, there is a natural isomorphism

$$H^i(G_{+1}, M^\sigma)_\mu \cong (H_i(G_{-1}, M)_\mu)^*, \quad \text{for any } i. \quad \blacksquare$$

Now we can prove the following theorem.

**Theorem 7.6.** Let $G = \mathfrak{gl}(m, n)$, and let $\lambda, \mu \in \Sigma^+$. Then $K(\lambda, \mu) = P_{\mu, \lambda}(q)$.

**Proof.** By Theorem 5.1, we have

$$\text{Ext}^i(K(\lambda), L(\lambda)) \cong \text{Hom}_{G_0}(L_\mu(\lambda), H^i(G_{+1}, L(\lambda))),$$

and hence

$$\dim \text{Ext}^i(K(\lambda), L(\lambda)) = \left[ H^i(G_{+1}, L(\lambda)):L_\mu(\lambda) \right].$$

By Lemma 7.1 and Proposition 7.2, we have

$$H^i(G_{+1}, L(\lambda))_\mu \cong (H_i(G_{-1}, L(\lambda))_\mu)^*.$$ 

Therefore

$$\dim H^i(G_{+1}, L(\lambda))_\mu = \dim H_i(G_{-1}, L(\lambda))_\mu,$$

and thus

$$\text{ch} \ H^i(G_{+1}, L(\lambda)) = \text{ch} \ H_i(G_{-1}, L(\lambda)).$$

Hence

$$\dim \text{Ext}^i(K(\lambda), L(\lambda)) = \left[ H_i(G_{-1}, L(\lambda)):L_\mu(\lambda) \right],$$

which proves the theorem. $lacksquare$
8. THE FUNCTOR Ext¹

We discuss Ext¹ briefly in this section.

Lemma 8.1. (i) Let $\lambda, \mu \in \Sigma^+$ be such that $\lambda < \mu$, and suppose that there is no element $\nu \in \Sigma^+$ such that $\lambda < \nu < \mu$ and $[K(\mu): L(\nu)][K(\nu): L(\lambda)] \neq 0$. Then $[K(\mu): L(\lambda)] \neq 0$ if and only if Ext¹(K(\lambda), L(\mu)) ≠ 0.

(ii) Let $\lambda, \mu \in \Sigma^+$. Then $P_{\lambda, \mu}(q) \neq 0$ only if there are $\lambda_i \in \Sigma^+$, $i = 0, \ldots, k$, such that $\lambda = \lambda_0 < \lambda_1 < \cdots < \lambda_k = \mu$ and Ext¹(K(\lambda_i), L(\lambda_{i+1})) ≠ 0, $0 \leq i < k$.

Proof. Consider the projective resolution of $K(\lambda)$:

$$0 \leftarrow K(\lambda) \leftarrow I(\lambda) \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

If $I_0 = I(\lambda) \supset I_1 \supset \cdots \supset I_k = 0$ is a filtration of $I(\lambda)$ such that $I_i/I_{i+1} \cong K(\lambda_i)$, $0 \leq i < k$, where $\lambda_0 = \lambda$, then $P_1$ can be taken to be $\bigoplus_{1 \leq i < k} I(\lambda_i)$. Hence Ext¹(K(\lambda), L(\mu)) ≠ 0 implies that $P_1$ has at least one copy of $I(\mu)$ as a summand, so by the duality theorem, $[K(\mu): L(\lambda)] \neq 0$. If $[K(\mu): L(\lambda)] \neq 0$, then $P_1 = \bigoplus_{1 \leq i < k} I(\lambda_i)$ has at least a copy of $I(\mu)$ as a summand. Note that $P_2$ is a direct sum of some $I(\nu)$ such that for each $\nu$, there is a $\lambda_i$ satisfying $\lambda < \lambda_i < \nu$, and $[K(\nu): L(\lambda_i)][K(\lambda_i): L(\lambda)] \neq 0$. So under the assumption of the lemma, $I(\mu)$ does not appear in $P_2$, and therefore Ext¹(K(\lambda), L(\mu)) ≠ 0. Hence part (i) is proved. Part (ii) follows from part (i) and a similar consideration.

Question. Let $G = sl(m, n)$ $(m \neq n)$. Is it true that $\dim \text{Ext}^1(K(\lambda), L(\mu)) = 0$ or 1 for $\lambda, \mu \in \Sigma^+$?

References


