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On the generalized reflexive and anti-reflexive solutions to a system of matrix equations

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ABSTRACT

Let P and Q be two generalized reflection matrices, i.e., $P = P^H$, $P^2 = I$ and $Q = Q^H$, $Q^2 = I$. An $n \times n$ matrix A is said to be generalized reflexive (generalized anti-reflexive) with respect to the matrix pair $(P; Q)$ if $A = PAQ$ $(A = -PAQ)$. It is obvious that any $n \times m$ matrix is also a generalized reflexive with respect to the matrix pair $(I_n; I_m)$. By extending the conjugate gradient least square (CGLS) approach, the present paper treats two iterative algorithms to solve the system of matrix equations

$$\begin{cases} \mathcal{F}_1(X) = A_1, \\ \mathcal{F}_2(X) = A_2, \\ \vdots \quad \vdots \ \vdots \\ \mathcal{F}_m(X) = A_m, \end{cases}$$

(including the Sylvester and Lyapunov matrix equations as special cases) over the generalized reflexive and anti-reflexive matrices, where $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ are the linear operators from $\mathbb{C}^{n\times n}$ onto $\mathbb{C}^{r_i \times s_i}$ and $A_i \in \mathbb{C}^{r_i \times s_i}$ for $i = 1, 2, \ldots, m$. When this system is consistent over the generalized reflexive (generalized anti-reflexive) matrix, it is proved that the first (second) iterative algorithm converges to a generalized reflexive (generalized anti-reflexive) solution for any initial generalized reflexive (generalized anti-reflexive)

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matrix. Also the first (second) iterative algorithm can obtain the least Frobenius norm generalized reflexive (generalized anti-reflexive) solution for special initial generalized reflexive (generalized anti-reflexive) matrix. Furthermore, the optimal approximation generalized reflexive (generalized anti-reflexive) solution to a given generalized reflexive (generalized anti-reflexive) matrix can be derived by finding the least Frobenius norm generalized reflexive (generalized anti-reflexive) solution of a new system of matrix equations. Finally, we test the proposed iterative algorithms and show their effectiveness using numerical examples.

1. Introduction

The following notations are adopted in this paper. $\mathbb{C}^n$ ($\mathbb{R}^n$) denotes the complex (real) n-vector space, $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denotes the set of $m \times n$ complex (real) matrices. We denote by I_n the $n \times n$ identity matrix. We also write it as I, when the dimension of this matrix is clear. We use A^T, A^H, $\mathrm{tr}(A)$ and $R(A)$ to denote the transpose, the conjugate transpose, the trace and the column space of the matrix A, respectively. The inner product $\langle .,.\rangle_r$ in $\mathbb{C}^{m\times n}$ is defined as follows:

$$\langle A, B\rangle_r = \mathrm{Re}(\mathrm{tr}(B^H A)) \quad \text{for } A, B \in \mathbb{C}^{m\times n},$$

that is $\langle A, B\rangle_r$ is the real part of the trace of $B^H A$. It is can be shown that $(\mathbb{C}^{m\times n}, \langle .,.\rangle_r)$ is a Hilbert inner product space. The induced matrix norm is $\|A\| = \sqrt{\langle A, A\rangle_r} = \sqrt{\mathrm{Re}(\mathrm{tr}(A^H A))} = \sqrt{\mathrm{tr}(A^H A)}$, which is the Frobenius norm [41,42]. We represent [51] by $\mathcal{F}^*$ the conjugate transpose of the linear operator of $\mathcal{F}$ from $\mathbb{C}^{p\times q}$ to $\mathbb{C}^{r\times s}$ such that

$$\langle \mathcal{F}(X), Y\rangle_r = \langle X, \mathcal{F}^*(Y)\rangle_r \quad \text{where} \quad X \in \mathbb{C}^{r\times s} \quad \text{and} \quad Y \in \mathbb{C}^{p\times q}.$$

For a matrix $A \in \mathbb{C}^{m\times n}$, the so-called stretching function $\mathrm{vec}(A)$ is defined by the following

$$\mathrm{vec}(A) = (\, a_1^T \; a_2^T \; \cdots \; a_n^T \,)^T,$$

where a_k is the k-th column of A. The symbol $A\otimes B$ stands for the Kronecker product of matrices A and B.

The investigation [3–5] indicates that generalized reflexive matrices arise naturally from problems with reflexive symmetry, which account for a great number of real-world scientific and engineering applications. Much of the activity in this field involves exploiting the underlying mathematical or physical problem. These matrices can be defined as follows [5]:

Definition 1.1. A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P = P^H$ and $P^2 = I_n$. Suppose that P and Q are two generalized reflection matrices of dimensions n and m, respectively. A matrix $A \in \mathbb{C}^{n\times m}$ is said to be generalized reflexive (generalized anti-reflexive) with respect to the matrix pair $(P; Q)$ if $A = PAQ$ ($A = -PAQ$). $\mathbb{C}_r^{n\times m}(P, Q)$ ($\mathbb{C}_a^{n\times m}(P, Q)$) denote the set of order $n \times m$ generalized reflexive (generalized anti-reflexive) with respect to the matrix pair $(P; Q)$.

In many areas of principal component analysis, biology, electricity, solid mechanics, automatics control theory and vibration theory, matrix equations can be encountered. In recent years, many new numerical methods have been developed for solving several matrix equations. Several authors have established the problem for determining solutions to various matrix equations [7,13,40,48–51]. The matrix equations

$$AX - XB = C, \tag{1.1}$$

and

$$X - AXB = C, \tag{1.2}$$

play important roles in the theories and applications of stability and control [1,2]. Golub et al. [25] investigated the solution to the matrix equation (1.1) by Hessenberg–Schur method. In [29], Jameson studied the matrix equation (1.1) by the method of characteristic polynomial, and derived explicit solution of this matrix equation. Jiang and Wei [30] obtained explicit solutions of matrix equation (1.2) by the method of characteristic polynomial. The matrix equation

$$AX + X^T C = B, \tag{1.3}$$

plays important roles in system theory, such as eigenstructure assignment [23], observer design [6], control of system with input constraint [22], and fault detection [24]. In [32], the problem of solution to the matrix equation (1.3) was considered by the Moore–Penrose generalized inverse matrix, and a general solution to this equation was obtained. In [34–39], the solutions of several quaternion matrix equations are studied. Wang [33] considered the matrix equations

$$A_1 X B_1 = C_1 \quad \text{and} \quad A_2 X B_2 = C_2, \tag{1.4}$$

over an arbitrary regular ring with identity and derived the necessary and sufficient conditions for the existence and the expression of the general solution to the system. In [14], Dehghan and Hajarian used the principle of hierarchical identification and the Hermitian/skew-Hermitian splitting of the coefficient matrices for solving linear matrix equations. It is well-known that the Sylvester and Lyapunov matrix equations are important equations which play a fundamental role in the various fields of engineering theory, particularly in control systems. Ding and Chen presented the hierarchical gradient-iterative (HGI) algorithms for general matrix equations [16,21] and hierarchical least-squares-iterative (HLSI) algorithms for generalized coupled Sylvester matrix equation and general coupled matrix equations [17,18]. The HGI algorithms [16,21] and HLSI algorithms [20,21,18] for solving general (coupled) matrix equations are innovational and computationally efficient numerical methods and were proposed based on the hierarchical identification principle [17,19] which regards the unknown matrix as the system parameter matrix to be identified. In [15,8–12], some efficient iterative methods were proposed to solve Sylvester and Lyapunov matrix equations. Zhou and Duan [43,44,46] established the solution of the several generalized Sylvester matrix equations. Zhou et al. [45] proposed gradient based iterative algorithms for solving the general coupled Sylvester matrix equations with weighted least squares solutions. In [47], general parametric solution to a family of generalized Sylvester matrix equations arising in linear system theory is presented by using the so-called generalized Sylvester mapping which has some elegant properties.

It is known that solving coupled complex matrix equations can be very difficult and it is sufficiently complicated. The main purpose of the paper is to study the system of matrix equations

$$\begin{cases} \mathcal{F}_1(X) = A_1, \\ \mathcal{F}_2(X) = A_2, \\ \vdots \qquad \vdots \;\; \vdots \\ \mathcal{F}_m(X) = A_m, \end{cases} \tag{1.5}$$

(including the Lyapunov and Sylvester matrix equations as special cases) over the generalized reflexive (generalized anti-reflexive) matrix $X \in \mathrm{C}_r^{n\times n}(P,Q)$ ($X \in \mathrm{C}_a^{n\times n}(P,Q)$), where $\mathcal{F}_i$ are the linear operators from $\mathrm{C}^{n\times n}$ onto $\mathrm{C}^{r_i\times s_i}$ and $A_i \in \mathrm{C}^{r_i\times s_i}$ for $i = 1, 2, \ldots, m$. Also the system of matrix equations (1.5) is quite general and includes many matrix equations such as

$$AX - XB = C, \qquad \text{Continuous–time (CT) Sylvester matrix equation}, \tag{1.6}$$

$$AXB^T - X = C, \qquad \text{Discrete–time (DT) Sylvester matrix equation}, \tag{1.7}$$

$$X - AXB = C, \qquad \text{Kalman–Yakubovich matrix equation}, \tag{1.8}$$

conjugate matrix equations

$$X - A\overline{X}B = C, \tag{1.9}$$

$$XF - A\overline{X} = F, \tag{1.10}$$

and matrix equations (1.1)–(1.4). The present paper considers the following four problems:

Problem 1.1. For given the linear operators $\mathcal{F}_i$ from $\mathbb{C}^{n\times n}$ onto $\mathbb{C}^{r_i\times s_i}$, the generalized reflection matrices P and Q of size n, and $A_i \in \mathbb{C}^{r_i\times s_i}$ for $i = 1, 2, \ldots, m$, find the generalized reflexive matrix $X \in \mathbb{C}_r^{n\times n}(P; Q)$ such that holds (1.5).

Problem 1.2. When Problem 1.1 is consistent, let its solution set be denoted by S_r. For a given generalized reflexive matrix $\widehat{X} \in \mathbb{C}_r^{n\times n}(P; Q)$, find $\widetilde{X} \in \mathbb{C}_r^{n\times n}(P; Q)$ such that

$$\|\widetilde{X} - \widehat{X}\|^2 = \min_{X\in S_r} \|X - \widehat{X}\|^2. \tag{1.11}$$

Problem 1.3. For given the linear operators $\mathcal{F}_i$ from $\mathbb{C}^{n\times n}$ onto $\mathbb{C}^{r_i\times s_i}$, the generalized reflection matrices P and Q of size n ($P \neq I,\ Q \neq I$) and $A_i \in \mathbb{C}^{r_i\times s_i}$ for $i = 1, 2, \ldots, m$, find the generalized anti-reflexive matrix $X \in \mathbb{C}_a^{n\times n}(P; Q)$ such that holds (1.5).

Problem 1.4. When Problem 1.3 is consistent, let its solution set be denoted by S_r. For a given generalized anti-reflexive matrix $\widehat{X} \in \mathbb{C}_a^{n\times n}(P; Q)$, find $\widetilde{X} \in \mathbb{C}_a^{n\times n}(P; Q)$ such that hold (1.11)

The remainder of this paper is organized as follows. In Section 2, by extending the CGLS scheme we first construct two iterative algorithms to solve Problems 1.1–1.4, then we present some basic properties of the iterative algorithms. For any initial generalized reflexive (generalized anti-reflexive) matrix $X(1)$, a generalized reflexive (generalized anti-reflexive) solution can be obtained within a finite number of iterations in the absence of roundoff errors, and the least Frobenius norm generalized reflexive (generalized anti-reflexive) solution can be obtained by choosing a special kind of initial generalized reflexive (generalized anti-reflexive) matrix. The generalized reflexive (generalized anti-reflexive) solution of Problem 1.2 (1.4) is obtained by finding the least Frobenius norm generalized reflexive (generalized anti-reflexive) solution of a new system of matrix equations in Section 3. In Section 4 we present two examples to illustrate the effectiveness of the proposed algorithms. Finally, we offer some concluding remarks in Section 5.

2. The solution of Problems 1.1 and 1.3

It is well-known that iterative algorithms are common in the areas of matrix algebra and system identification. If the conjugate-gradient method for symmetric positive definite systems is applied naively to the normal equations $A^TAx = A^Tb$, the method does not perform well on ill-conditioned systems. An algorithm with better numerical properties is easily derived by a slight algebraic rearrangement, making use of the intermediate vector Ap_i [26]. This approach is named the conjugate gradient least square (CGLS) and is usually stated in notation similar to the following [31].

Algorithm CGLS

Set $r_0 = b,\ s_0 = A^Tb,\ p_1 = s_0,\ \gamma_0 = \|s_0\|^2,\ x_0 = 0$

For $i = 1, 2, 3, \ldots$ repeat the following:

$q_i = Ap_i$

$\alpha_i = \gamma_{i-1}/\|q_i\|^2$

$x_i = x_{i-1} + \alpha_i p_i$

$r_i = r_{i-1} - \alpha_i q_i$
$s_i = A^T r_i$
$\gamma_i = \|s_i\|^2$
$\beta_i = \gamma_i / \gamma_{i-1}$
$p_{i+1} = s_i + \beta_i p_i.$

It is obvious that the above algorithm can not be used directly to solve Problems 1.1 and 1.3 . In this work, we extend Algorithm CGLS for solving Problems 1.1 and 1.3. By extending CGLS method, we construct two iterative algorithms to solve Problems 1.1 and 1.3 as:

Algorithm 1.

Step 1. Given the linear operators $\mathcal{F}_i$ from $\mathrm{C}^{n\times n}$ onto $\mathrm{C}^{r_i\times s_i}$, the generalized reflection matrices P, Q of size n, $A_i \in \mathrm{C}^{r_i\times s_i}$ for $i = 1, 2, \ldots, m$ and $X(1) \in \mathrm{C}_r^{n\times n}(P; Q)$;

Step 2. Compute

$$R_i(1) = A_i - \mathcal{F}_i(X(1)), \quad \text{for } i = 1, 2, \ldots, m;$$
$$S(1) = \frac{1}{2}\left[\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(1)) + \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(1))Q\right];$$
$$k := 1;$$

Step 3. If $\sum_{i=1}^{m} \|R_i(k)\|^2 = 0$ or $\sum_{i=1}^{m} \|R_i(k)\|^2 \neq 0, S(k) = 0$ then stop; else $k = k + 1$;

Step 4. Compute

$$X(k) = X(k-1) + \frac{\sum_{t=1}^{m} \|R_t(k-1)\|^2}{\|S(k-1)\|^2} S(k-1);$$
$$R_i(k) = A_i - \mathcal{F}_i(X(k))$$
$$= R_i(k-1) - \frac{\sum_{t=1}^{m} \|R_t(k-1)\|^2}{\|S(k-1)\|^2} \mathcal{F}_i(S(k-1)), \quad \text{for } i = 1, 2, \ldots, m;$$
$$S(k) = \frac{1}{2}\left[\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(k)) + \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(k))Q\right] + \frac{\sum_{t=1}^{m} \|R_t(k)\|^2}{\sum_{t=1}^{m} \|R_t(k-1)\|^2} S(k-1);$$

Step 5. Go to Step 3.

Algorithm 2.

Step 1. Given the linear operators $\mathcal{F}_i$ from $\mathrm{C}^{n\times n}$ onto $\mathrm{C}^{r_i\times s_i}$, the generalized reflection matrices P, Q of size n $(P \neq I)$, $A_i \in \mathrm{C}^{r_i\times s_i}$ for $i = 1, 2, \ldots, m$ and $X(1) \in \mathrm{C}_a^{n\times n}(P; Q)$;

Step 2. Compute

$$R_i(1) = A_i - \mathcal{F}_i(X(1)), \quad \text{for } i = 1, 2, \ldots, m;$$
$$S(1) = \frac{1}{2}\left[\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(1)) - \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(1))Q\right];$$
$$k := 1;$$

Step 3. If $\sum_{i=1}^{m} \|R_i(k)\|^2 = 0$ or $\sum_{i=1}^{m} \|R_i(k)\|^2 \neq 0, S(k) = 0$ then stop; else $k = k + 1$;

Step 4. Compute

$$X(k) = X(k-1) + \frac{\sum_{t=1}^{m} \|R_t(k-1)\|^2}{\|S(k-1)\|^2} S(k-1);$$

$$R_i(k) = A_i - \mathcal{F}_i(X(k)) = R_i(k-1) - \frac{\sum_{t=1}^{m} \|R_t(k-1)\|^2}{\|S(k-1)\|^2} \mathcal{F}_i(S(k-1)), \quad \text{for } i = 1, 2, \ldots, m;$$

$$S(k) = \frac{1}{2}\left[\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(k)) - \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(k))Q\right] + \frac{\sum_{t=1}^{m} \|R_t(k)\|^2}{\sum_{t=1}^{m} \|R_t(k-1)\|^2} S(k-1);$$

Step 5. Go to Step 3.

Remark 2.1. From Algorithm 1 (2), it is obvious that $X(k) \in \mathbb{C}_r^{n\times n}(P; Q)$ and $S(k) \in \mathbb{C}_r^{n\times n}(P; Q)$ ($X(k) \in \mathbb{C}_a^{n\times n}(P; Q)$ and $S(k) \in \mathbb{C}_a^{n\times n}(P; Q)$) for all $k = 1, 2, \ldots$ Also Algorithm 1 (2) implies that if $\sum_{i=1}^{m} \|R_i(k)\|^2$, then $X(k) \in \mathbb{C}_r^{n\times n}(P; Q)$ ($X(k) \in \mathbb{C}_a^{n\times n}(P; Q)$) is the generalized reflexive (generalized anti-reflexive) solution of (1.5).

Remark 2.2. Because of the influence of the error of calculation, the residual R_k ($k = 1, 2, \ldots$) is usually unequal to zero exactly in the process of the iteration. We regard the matrix $\sum_{i=1}^{m} \|R_i(k)\|^2$ as a zero matrix if $\sum_{i=1}^{m} \|R_i(k)\|^2 < \varepsilon$ where ε is a small positive number. In Algorithms 1 and 2, the iteration will be stopped whenever $\sum_{i=1}^{m} \|R_i(k)\|^2 < \varepsilon$.

We begin with the following useful lemmas about Algorithms 1 and 2 to be used in the next results.

Lemma 2.1. *For any initial generalized reflexive matrix $X(1)$, the sequences $\{R(i)\}$ ($R(i) \neq 0$) and $\{S(i)\}$ generated by Algorithm 1 satisfy*

$$\sum_{r=1}^{m} \langle R_r(i), R_r(j)\rangle_r = 0, \quad \langle S(i), S(j)\rangle_r = 0 \quad \text{for } i, j = 1, 2, \ldots, v\ (i \neq j). \tag{2.1}$$

The proof of Lemma 2.1 is given in the Appendix.
Similarly to the proof of Lemma 2.1, we can prove the following lemma.

Lemma 2.2. *For any initial generalized anti-reflexive matrix $X(1)$, the sequences $\{R(i)\}$ ($R(i) \neq 0$) and $\{S(i)\}$ generated by Algorithm 2 satisfy*

$$\sum_{r=1}^{m} \langle R_r(i), R_r(j)\rangle_r = 0, \quad \langle S(i), S(j)\rangle_r = 0 \quad \text{for } i, j = 1, 2, \ldots, v\ (i \neq j). \tag{2.2}$$

Lemma 2.3. *Suppose that the system of matrix equations (1.5) is consistent over the generalized reflexive matrices and X^* is an arbitrary generalized reflexive solution of (1.5), then for any initial generalized reflexive matrix $X(1)$, the sequences $\{X(i)\}$, $\{R(i)\}$ and $\{S(i)\}$ generated by Algorithm 1 satisfy*

$$\langle X^* - X(i), S(j)\rangle_r = \sum_{r=1}^{m} \|R_r(j)\|^2 \quad \text{for } j \geqslant i, \tag{2.3}$$

$$\langle X^* - X(i), S(j)\rangle_r = 0 \quad \text{for } j < i. \tag{2.4}$$

The proof of Lemma 2.3 is presented in the Appendix.
Similarly to the proof of Lemma 2.3, we can prove the following lemma.

Lemma 2.4. *Suppose that the system of matrix equations (1.5) is consistent over the generalized anti-reflexive matrices and X^* is an arbitrary generalized anti-reflexive solution of (1.5), then for any initial generalized anti-reflexive matrix $X(1)$, the sequences $\{X(i)\}$, $\{R(i)\}$ and $\{S(i)\}$ generated by Algorithm 2 satisfy*

$$\langle X^* - X(i), S(j)\rangle_r = \sum_{r=1}^{m} \|R_r(j)\|^2 \quad \text{for } j \geqslant i, \tag{2.5}$$

$$\langle X^* - X(i), S(j)\rangle_r = 0 \quad \text{for } j < i. \tag{2.6}$$

Remark 2.3. If there exists a positive number k such that $S(k) = 0$ but $R_t(k) \neq 0$ for some $t \in 1, 2, .., m$, then by considering Lemma 2.3 (2.4), we have that the matrix equations (1.5) are not consistent over the generalized reflexive (generalized anti-reflexive) matrices. Hence, the solvability of the matrix equations (1.5) over the generalized reflexive (generalized anti-reflexive) matrices can be determined by Algorithm 1 (2) in the absence of roundoff errors.

Theorem 2.1. *Suppose that Problem 1.1 is consistent, then by Algorithm 1 with any initial generalized reflexive matrix $X(1)$, a generalized reflexive solution of Problem 1.1 can be obtained within a finite number of iterations in the absence of roundoff errors.*

Proof. From Lemma 2.3, it is no difficult to obtain that $S(1), S(2), \ldots$ are orthogonal to each other in finite dimension matrix space $\mathbb{C}^{n\times n}$, therefore there exists a positive number k such that $S(k) = 0$. This implies that $\sum_{t=1}^{m} R_t(k) = 0$. The proof is completed. □

Similarly to the proof of the above theorem, we can prove the following theorem.

Theorem 2.2. *Suppose that Problem 1.3 is consistent, then by Algorithm 2 with any initial generalized reflexive matrix $X(1)$, a generalized anti-reflexive solution of Problem 1.3 can be obtained within a finite number of iterations in the absence of roundoff errors.*

Lemma 2.5 [27]. *Let $\mathcal{F}$ be a given linear operator from $\mathbb{C}^{p\times q}$ to $\mathbb{C}^{r\times s}$ and $\mathcal{F}^*$ be the conjugate transpose of the linear operator $\mathcal{F}$. Then there exists a unique matrix $H \in \mathbb{C}^{rs\times pq}$ such that $\text{vec}(\mathcal{F}(X)) = M\text{vec}(X)$ and $\text{vec}(\mathcal{F}^*(Y)) = M^H\text{vec}(Y)$ for all $X \in \mathbb{C}^{p\times q}$ and $Y \in \mathbb{C}^{r\times s}$.*

Lemma 2.6 [28]. *Assume that the consistent system of linear equations $Ay = b$ has a solution $y^* \in R(A^H)$. Then y^* is an unique least Frobenius norm solution of the system of linear equations.*

Theorem 2.3. *Assume that Problem 1.1 is consistent. If we take the initial generalized reflexive matrix*

$$X(1) = \mathcal{F}_1^*(K_1) + P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) + P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) + P\mathcal{F}_m^*(K_m)Q, \tag{2.7}$$

where $K_1, K_2, \ldots, K_m$ are arbitrary, or more especially $X(1) = 0$, then the generalized reflexive solution X^* obtained by Algorithm 1 is the least Frobenius generalized reflexive solution of Problem 1.1.

Proof. The solvability of linear matrix equation (1.5) over the generalized reflexive matrix X is equivalent to

$$\begin{cases} \mathcal{F}_1(X) = A_1, \\ \mathcal{F}_1(PXQ) = A_1, \\ \mathcal{F}_2(X) = A_2, \\ \mathcal{F}_2(PXQ) = A_2, \\ \vdots \quad \vdots \; \vdots \\ \mathcal{F}_m(X) = A_m, \\ \mathcal{F}_m(PXQ) = A_m. \end{cases} \tag{2.8}$$

By considering Lemma 2.5, there exist matrices $H_1, H_2, \ldots, H_{2m}$ such that the system of matrix equations (2.8) is equivalent to the following system:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{2m} \end{pmatrix} \mathrm{vec}(X) = \begin{pmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \\ \mathrm{vec}(A_2) \\ \vdots \\ \mathrm{vec}(A_m) \end{pmatrix} \tag{2.9}$$

Now let $K_1, K_2, \ldots, K_m$ be arbitrary matrices, we can write

$$\mathrm{vec}(\mathcal{F}_1^*(K_1) + P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) + P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) + P\mathcal{F}_m^*(K_m)Q)$$

$$= \begin{pmatrix} H_1^H & H_2^H & H_3^H & H_4^H & \ldots & H_{2m}^H \end{pmatrix} \begin{pmatrix} \mathrm{vec}(K_1) \\ \mathrm{vec}(K_1) \\ \mathrm{vec}(K_2) \\ \mathrm{vec}(K_2) \\ \cdots \\ \mathrm{vec}(K_m) \end{pmatrix}$$

$$\in R\left(\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{2m} \end{pmatrix}^H \right).$$

It is obvious that if we consider

$$X(1) = \mathcal{F}_1^*(K_1) + P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) + P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) + P\mathcal{F}_m^*(K_m)Q, \tag{2.10}$$

then all $X(k)$, generated by Algorithm 1 satisfy

$$\mathrm{vec}(X(k)) \in R\left(\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{2m} \end{pmatrix}^H \right).$$

From Lemma 2.6, the solution X^* obtained by Algorithm 1 with such initial matrix $X(1)$ (2.10) is the least Frobenius norm generalized reflexive solution. □

Similarly to the above results, we can obtain the following theorem.

Theorem 2.4. *Assume that Problem 1.3 is consistent. If we take the initial generalized anti-reflexive matrix*

$$X(1) = \mathcal{F}_1^*(K_1) - P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) - P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) - P\mathcal{F}_m^*(K_m)Q, \quad (2.11)$$

where $K_1, K_2, \ldots, K_m$ are arbitrary, or more especially $X(1) = 0$, then the generalized anti-reflexive solution X^* obtained by Algorithm 2 is the least Frobenius generalized anti-reflexive solution of Problem 1.3.

3. The solution of Problems 1.2 and 1.4

In this section, we study Problems 1.2 and 1.4. Now suppose that the system of linear matrix equations (1.5) is consistent over the generalized reflexive (generalized anti-reflexive) matrix X. Obviously, the solution set S_r is nonempty. For a given generalized reflexive generalized anti-reflexive matrix $\widehat{X} \in \mathbb{C}_r^{n\times n}(P;Q)$ ($\widehat{X} \in \mathbb{C}_a^{n\times n}(P;Q)$), we can get

$$\begin{cases} \mathcal{F}_1(X) = A_1, \\ \mathcal{F}_2(X) = A_2, \\ \vdots \quad \vdots \ \vdots \\ \mathcal{F}_m(X) = A_m, \end{cases} \Leftrightarrow \begin{cases} \mathcal{F}_1(X-\widehat{X}) = A_1 - \mathcal{F}_1(\widehat{X}), \\ \mathcal{F}_2(X-\widehat{X}) = A_2 - \mathcal{F}_2(\widehat{X}), \\ \vdots \quad \vdots \ \vdots \\ \mathcal{F}_m(X-\widehat{X}) = A_m - \mathcal{F}_m(\widehat{X}). \end{cases} \quad . \quad (3.1)$$

Set $X_1 = X - \widehat{X}$ and $\widehat{A}_i = A_i - \mathcal{F}_i(\widehat{X})$ for $i = 1, 2, \ldots, m$, then the matrix nearness problem 1.2 (1.4) is equivalent to find the least Frobenius norm generalized reflexive (generalized anti-reflexive) solution X_1^* of the following system of matrix equations

$$\begin{cases} \mathcal{F}_1(X_1) = \widehat{A}_1, \\ \mathcal{F}_2(X_1) = \widehat{A}_2, \\ \vdots \quad \vdots \ \vdots \\ \mathcal{F}_m(X_1) = \widehat{A}_m, \end{cases} \quad (3.2)$$

which can be obtained using Algorithm 1 (2) and the initial the generalized reflexive (generalized anti-reflexive) matrix

$$X_1(1) = \mathcal{F}_1^*(K_1) + P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) + P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) + P\mathcal{F}_m^*(K_m)Q,$$
$$\left(X_1(1) = \mathcal{F}_1^*(K_1) - P\mathcal{F}_1^*(K_1)Q + \mathcal{F}_2^*(K_2) - P\mathcal{F}_2^*(K_2)Q + \cdots + \mathcal{F}_m^*(K_m) - P\mathcal{F}_m^*(K_m)Q\right),$$

where $K_1, K_2, \ldots, K_m$ are arbitrary matrices, and the solution of the matrix nearness problem 1.2 (1.4) can be obtained as

$$\widetilde{X} = X_1^* + \widehat{X}.$$

4. Illustrative examples

In this section, two numerical examples are presented to illustrate the validity and the merits of the presented methods. We have implemented the algorithms in MATLAB and run the programs on a Pentium IV.

Example 4.1. In this example, we consider the pair of Sylvester matrix equations

$$A_1XB_1 + A_2XB_2 = C, \quad \text{and} \quad D_1XE_1 + D_2XE_2 = F, \quad (4.1)$$

where

$$A_1 = \begin{bmatrix} 1.0000 + 1.0000i & 2.0000 - 1.0000i & 3.0000 & 1.0000 & 2.0000 & 1.0000 \\ 1.0000 & -1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 2.0000 + 4.0000i & 3.0000 & 1.0000 & -1.0000 & -2.0000 & 1.0000 \\ 3.0000 & 2.0000 & 4.0000 & 3.0000 & 1.0000 & -1.0000 \\ 0 + 1.0000i & 2.0000 & 2.0000 & 3.0000 & 5.0000 + 1.0000i & 3.0000 \\ 1.0000 & 2.0000 & 3.0000 & -1.0000 - 1.0000i & -2.0000 & 0 + 1.0000i \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 4.0000 - 2.0000i & 3.0000 + 2.0000i & -1.0000 & 6.0000 & 5.0000 & 3.0000 - 1.0000i \\ 1.0000 + 1.0000i & 2.0000 & 2.0000 & 2.0000 & 4.0000 & 4.0000 \\ -1.0000 - 1.0000i & -2.0000 & -3.0000 & 1.0000 & 2.0000 & 2.0000 \\ 1.0000 & 2.0000 - 1.0000i & 3.0000 & 0 - 1.0000i & 1.0000 & 2.0000 \\ 3.0000 & 5.0000 & 4.0000 + 1.0000i & 2.0000 & 1.0000 & 2.0000 \\ -1.0000 & -1.0000 & -1.0000 + 1.0000i & 2.0000 & 1.0000 & 1.0000 + 1.0000i \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 + 1.0000i & 1.0000 & 0 + 1.0000i & 0 + 1.0000i & 0 + 1.0000i & 1.0000 \\ 1.0000 & -1.0000 & 1.0000 & -1.0000 & 2.0000 & 2.0000 \\ 2.0000 + 1.0000i & 2.0000 + 1.0000i & 2.0000 - 1.0000i & 3.0000 - 1.0000i & 2.0000 & 1.0000 \\ 1.0000 + 1.0000i & 1.0000 + 1.0000i & 2.0000 & 2.0000 & 0 & 3.0000 \\ 3.0000 + 3.0000i & 2.0000 & 2.0000 & 2.0000 & 1.0000 & 1.0000 \\ 4.0000 & 3.0000 & 4.0000 + 1.0000i & 3.0000 - 1.0000i & 3.0000 & 3.0000 + 1.0000i \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1.0000 + 2.0000i & 2.0000 - 1.0000i & 1.0000 & 2.0000 & 1.0000 & 2.0000 \\ -1.0000 & -1.0000 & -1.0000 + 2.0000i & 2.0000 & 2.0000 & 2.0000 \\ 3.0000 - 2.0000i & 4.0000 & 1.0000 & 2.0000 & 1.0000 & 2.0000 \\ -1.0000 & -1.0000 + 4.0000i & -1.0000 & -1.0000 & 2.0000 & 1.0000 \\ 1.0000 & 2.0000 & 1.0000 & 2.0000 - 5.0000i & 3.0000 & 1.0000 \\ 0 + 1.0000i & 1.0000 & 2.0000 & 0 + 1.0000i & 1.0000 & 2.0000 - 1.0000i \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 2.0000 + 2.0000i & 2.0000 + 1.0000i & 1.0000 & 2.0000 & 0 & 0 \\ 0 + 1.0000i & 0 + 1.0000i & 0 + 1.0000i & 2.0000 & 2.0000 & 2.0000 \\ 1.0000 & 2.0000 & 3.0000 + 1.0000i & 0 + 1.0000i & 0 & 0 \\ 2.0000 & 3.0000 & 4.0000 & 2.0000 + 2.0000i & 0 & 2.0000 \\ 3.0000 & 4.0000 + 1.0000i & 4.0000 & 3.0000 & 2.0000 & 2.0000 \\ 0 & 1.0000 & 0 + 1.0000i & 1.0000 & 0 & 2.0000 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1.0000 + 1.0000i & 2.0000 & 1.0000 & 2.0000 & 3.0000 & 4.0000 \\ 1.0000 - 1.0000i & 1.0000 & 1.0000 & 2.0000 + 4.0000i & 1.0000 & 2.0000 \\ -1.0000 & -1.0000 + 1.0000i & 2.0000 & 1.0000 & 2.0000 & 1.0000 \\ 2.0000 & 3.0000 & 1.0000 & 5.0000 + 1.0000i & 3.0000 & 1.0000 \\ 1.0000 & 2.0000 & -2.0000 & 1.0000 - 1.0000i & 2.0000 & 3.0000 \\ 4.0000 + 4.0000i & 1.0000 & 2.0000 - 1.0000i & 3.0000 & 2.0000 - 1.0000i & 3.0000 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 2.0000 + 1.0000i & 1.0000 & 2.0000 & 1.0000 & 2.0000 - 1.0000i & 1.0000 \\ 1.0000 - 1.0000i & 2.0000 & 3.0000 & 1.0000 & 2.0000 - 1.0000i & 1.0000 \\ 0 + 1.0000i & 1.0000 + 1.0000i & 1.0000 & 2.0000 & 1.0000 & 2.0000 \\ 1.0000 & 2.0000 & 1.0000 - 1.0000i & 2.0000 & 1.0000 + 1.0000i & -1.0000 \\ 0 + 1.0000i & 1.0000 & 1.0000 + 1.0000i & 2.0000 & 1.0000 + 1.0000i & 2.0000 \\ 1.0000 + 2.0000i & 2.0000 & 1.0000 & 2.0000 & 1.0000 & 1.0000 \end{bmatrix},$$

$$E_2=\begin{bmatrix} 3.0000+3.0000i & 2.0000 & 1.0000 & 2.0000+1.0000i & 1.0000 & 2.0000 \\ -1.0000-1.0000i & 2.0000 & -1.0000-1.0000i & 2.0000 & 1.0000 & 2.0000 \\ 2.0000+3.0000i & 3.0000 & 3.0000 & 3.0000-1.0000i & 1.0000 & 2.0000 \\ 4.0000+1.0000i & 2.0000+1.0000i & 1.0000 & 2.0000 & 4.0000 & 4.0000 \\ 1.0000 & 2.0000-1.0000i & 3.0000-1.0000i & 1.0000 & 3.0000 & 1.0000 \\ -1.0000+1.0000i & -1.0000+1.0000i & -1.0000 & 2.0000+1.0000i & 2.0000 & 2.0000+1.0000i \end{bmatrix},$$

$$C=10^3\begin{bmatrix} 0.2400-0.0160i & 0.2500+0.0640i & 0.2060+0.0800i & 0.5180+0.0940i & 0.4140+0.1940i & 0.4840+0.1540i \\ 0.1460+0.0220i & 0.2480-0.0080i & 0.1980+0.0880i & 0.3440-0.0760i & 0.4280-0.0100i & 0.4780-0.0540i \\ -0.0160+0.0060i & -0.1800+0.3000i & -0.2840-0.0720i & 0.1560-0.0700i & 0.3640+0.1640i & 0.2220+0.1060i \\ 0.3740-0.0620i & 0.4620+0.1320i & 0.2760+0.1500i & 0.5540+0.0020i & 0.6300-0.0080i & 0.7380-0.0780i \\ 0.6020+0.1960i & 1.0200+0.3400i & 0.7440+0.3760i & 0.8540-0.0660i & 0.9940+0.0780i & 1.1300+0.2280i \\ 0.0600-0.1220i & 0.1740+0.2520i & -0.1680+0.0280i & 0.3240-0.2360i & 0.6040-0.0580i & 0.4780-0.1320i \end{bmatrix},$$

$$F=10^3\begin{bmatrix} 0.2500+0.2380i & 0.4160+0.0440i & 0.2580-0.0120i & 0.6740+0.0980i & 0.6560-0.0060i & 0.6060+0.0480i \\ 0.6080+0.3080i & 0.5600-0.0440i & 0.2900-0.1420i & 0.8940+0.2060i & 0.7760-0.0680i & 0.8060-0.0160i \\ 0.1040+0.2260i & 0.2200+0.1560i & 0.0960+0.0220i & 0.3460+0.1420i & 0.4700+0.1240i & 0.4800+0.1400i \\ 0.5220+0.3160i & 0.4500+0.1720i & 0.4660-0.0360i & 0.5620+0.3020i & 0.7200+0.1800i & 0.6620+0.1120i \\ 0.3640+0.4520i & 0.5860+0.2520i & 0.3100+0.0160i & 1.0660+0.4120i & 0.9780+0.1720i & 0.9720+0.2080i \\ 0.1820+0.3060i & 0.2700+0.1360i & 0.2220+0.0360i & 0.4500+0.2360i & 0.4260+0.0660i & 0.4340+0.1320i \end{bmatrix},$$

We define two operators $\mathcal{F}_1 : X \to A_1XB_1 + A_2XB_2$ and $\mathcal{F}_2 : X \to D_1XE_1 + D_2XE_2$, so the pair of Sylvester matrix equations (4.1) is equivalent to pair of matrix equations

$$\mathcal{F}_1(X) = E, \quad \text{and} \quad \mathcal{F}_2(X) = F. \tag{4.2}$$

We can also define $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ as $\mathcal{F}_1^* : Y \to A_1^TYB_1^T + A_2^TYB_2^T$ and $\mathcal{F}_2^* : Y \to D_1^TYE_1^T + D_2^TYE_2^T$ respectively. We can verify that the pair of Sylvester matrix equations (4.1) is consistent over generalized reflexive matrices and has a generalized reflexive solution as follows:

$$X^*=\begin{bmatrix} 4.0000+2.0000i & 0 & 12.0000 & 0 & 0 & 0 \\ 0 & -4.0000 & 0 & 4.0000 & 4.0000 & 4.0000 \\ 8.0000-2.0000i & 0 & 4.0000 & 0 & 0 & 0 \\ 0 & 8.0000-2.0000i & 0 & 12.0000 & 4.0000 & -4.0000 \\ 0 & 8.0000-2.0000i & 0 & 12.0000+2.0000i & 20.0000 & 12.0000 \\ 0 & 8.0000-2.0000i & 0 & -4.0000+2.0000i & -8.0000 & 12.0000+2.0000i \end{bmatrix} \in \mathbb{C}_r^{6\times6}(P,P), \tag{4.3}$$

where

$$P=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

If we let the initial matrix $X(1) = 0$, applying Algorithm 1, we obtain the sequence $\{X(k)\}$ that after 35 steps, we have

$$\sqrt{\|R_1(35)\|^2 + \|R_2(35)\|^2} = 4.8609 \times 10^{-13}.$$

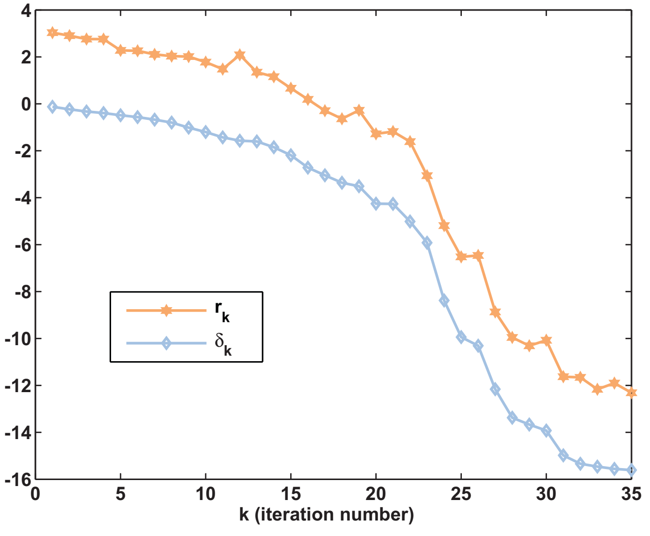

Fig. 1. The obtained results for Example 4.1.

The obtained results are presented in Fig. 1, where

$$\delta_k = \log_{10}\left(\frac{\|X(k) - X^*\|}{\|X^*\|}\right) \quad \text{and} \quad r_k = \log_{10}\sqrt{(\|R_1(k)\|^2 + \|R_2(k)\|^2)}.$$

From Fig. 1, we can see that Algorithm 1 is effective.

Example 4.2. Assume that S_r denote the set of generalized reflexive solutions to the pair of Sylvester matrix equations (4.1), where the matrices $A_1, B_1, A_2, B_2, D_1, E_1, D_2, E_2, C$ and F are mentioned in Example 4.1. For

$$\widehat{X} = \begin{bmatrix} 0 + 2.0000i & 0 & 0 + 2.0000i & 0 & 0 & 0 \\ 0 & -2.0000 & 0 & -2.0000 & 4.0000 & 4.0000 \\ 4.0000 + 2.0000i & 0 & 4.0000 - 2.0000i & 0 & 0 & 0 \\ 0 & 2.0000 + 2.0000i & 0 & 4.0000 & 0 & 6.0000 \\ 0 & 4.0000 & 0 & 4.0000 & 2.0000 & 2.0000 \\ 0 & 6.0000 & 0 & 6.0000 - 2.0000i & 6.0000 & 6.0000 + 2.0000i \end{bmatrix} \in \mathbb{C}_r^{6\times 6}(P, P),$$

we find solution of Problem 1.2. By computing $\widehat{C} = C - A_1\widehat{X}B_1 - A_2\widehat{X}B_2$ and $\widehat{F} = F - D_1\widehat{X}E_1 - D_2\widehat{X}E_2$, we can get the least-norm generalized reflexive solution $\widehat{X}^*$ of new matrix equations

$$A_1X_1B_1 + A_2X_1B_2 = \widehat{C}, \quad \text{and} \quad D_1X_1E_1 + D_2X_1E_2 = \widehat{F}. \tag{4.4}$$

By Algorithm 1 with $X_1(1) = 0$, we can obtain

$$X_1^* = X_1(35) = \begin{bmatrix} 4.0000 + 0.0000i & 0 & 12.0000 - 2.0000i & 0 & 0 & 0 \\ 0 & -2.0000 + 0.0000i & 0 & 6.0000 + 0.0000i & -0.0000 + 0.0000i & -0.0000 - 0.0000i \\ 4.0000 - 4.0000i & 0 & -0.0000 + 2.0000i & 0 & 0 & 0 \\ 0 & 6.0000 - 4.0000i & 0 & 8.0000 + 0.0000i & 4.0000 + 0.0000i & -10.0000 + 0.0000i \\ 0 & 4.0000 - 2.0000i & 0 & 8.0000 + 2.0000i & 18.0000 + 0.0000i & 10.0000 + 0.0000i \\ 0 & 2.0000 - 2.0000i & 0 & -10.0000 + 4.0000i & -14.0000 + 0.0000i & 6.0000 - 0.0000i \end{bmatrix},$$

$$\sqrt{\|R_1(35)\|^2 + \|R_2(35)\|^2} = 4.6637 \times 10^{-13},$$

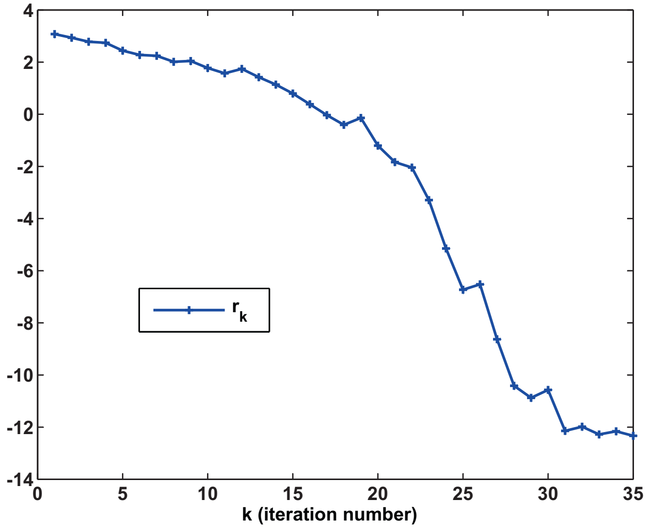

Fig. 2. The results obtained for Examples 4.2.

and

$$\widetilde{X} = X_1^* + \widehat{X} = \begin{bmatrix} 1.0000 - 1.0000i & 1.0000 - 0.0000i & 0 & 0 & 0 \\ 2.0000 + 2.0000i & 1.0000 + 0.0000i & 0 & 0 & 0 \\ 0 & 0 & 1.0000 + 1.0000i & 2.0000 - 0.0000i & 1.0000 + 0.0000i \\ 0 & 0 & 2.0000 - 0.0000i & 2.0000 + 1.0000i & 1.0000 - 0.0000i \\ 0 & 0 & 1.0000 - 0.0000i & 1.0000 + 2.0000i & 1.0000 - 5.0000i \end{bmatrix}.$$

The obtained results are presented in Fig. 2. The results show that Algorithm 1 is quite efficient.

5. Concluding remarks

The linear matrix equations have numerous applications in control theory, signal processing, decoupling techniques for ordinary and partial differential equations. By extending the CGLS method, two iterative algorithms have been constructed to solve the system of matrix equations (1.5) over the generalized reflexive (generalized anti-reflexive) matrix X. With Algorithm 1 (2), the solvability of Problem 1.1 (1.3) can be judged automatically. When Problem 1.1 (1.3) is consistent, its generalized reflexive (generalized anti-reflexive) solution can be obtained within a finite number of iterations in the absence of roundoff errors, and its least Frobenius norm generalized reflexive (generalized anti-reflexive) solution can be obtained by choosing a suitable initial generalized reflexive (generalized anti-reflexive) matrix. In addition, by Algorithm 1 (2) we have obtained the generalized reflexive (generalized anti-reflexive) solution of Problem 1.2 (1.4). The numerical results in Section 4 show that the proposed algorithms may be applied to solve several matrix equations. It is interesting to develop the introduced algorithms for solving another matrix equation. We leave it as a topic for further research.

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Appendix.

The proof of Lemma 2.1

Because $\langle R(i), R(j)\rangle_r = \langle R(j), R(i)\rangle_r$ and $\langle S(i), S(j)\rangle_r = \langle S(j), S(i)\rangle_r$, we only need to prove

$$\sum_{r=1}^{m}\langle R_r(i), R_r(j)\rangle_r = 0, \quad \langle S(i), S(j)\rangle_r = 0 \quad \text{for } 1 \leqslant i < j \leqslant v. \tag{5.1}$$

We prove (5.1) by induction. Also we do it in two steps.

Step 1. We show that

$$\sum_{r=1}^{m}\langle R_r(i), R_r(i+1)\rangle_r = 0, \quad \langle S(i), S(i+1)\rangle_r = 0 \quad \text{for } i = 1, 2, \ldots, v. \tag{5.2}$$

Let $i = 1$, we can write

$$\begin{aligned}
\sum_{r=1}^{m}\langle R_r(1), R_r(2)\rangle_r &= \sum_{r=1}^{m}\left\langle R_r(1), R_r(1) - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\mathcal{F}_r(S(1))\right\rangle_r \\
&= \sum_{r=1}^{m}\langle R_r(1), R_r(1)\rangle_r - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\sum_{r=1}^{m}\langle R_r(1), \mathcal{F}_r(S(1))\rangle_r \\
&= \sum_{r=1}^{m}\|R_r(1)\|^2 - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\left\langle \sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)), S(1)\right\rangle_r \\
&= \sum_{r=1}^{m}\|R_r(1)\|^2 - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\left[\left\langle \frac{\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)) + \sum_{r=1}^{m}P\mathcal{F}_r^*(R_r(1))Q}{2}, S(1)\right\rangle_r\right. \\
&\quad \left. + \left\langle \frac{\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)) - \sum_{r=1}^{m}P\mathcal{F}_r^*(R_r(1))Q}{2}, S(1)\right\rangle_r\right] \\
&= \sum_{r=1}^{m}\|R_r(1)\|^2 - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\left[\left\langle \frac{\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)) + \sum_{r=1}^{m}P\mathcal{F}_r^*(R_r(1))Q}{2}, S(1)\right\rangle_r\right. \\
&\quad \left. + \frac{1}{2}\left\langle \sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)), S(1)\right\rangle_r - \frac{1}{2}\left\langle \sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)), PS(1)Q\right\rangle_r\right] \\
&= \sum_{r=1}^{m}\|R_r(1)\|^2 - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\left\langle \frac{\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(1)) + \sum_{r=1}^{m}P\mathcal{F}_r^*(R_r(1))Q}{2}, S(1)\right\rangle_r \\
&= \sum_{r=1}^{m}\|R_r(1)\|^2 - \frac{\sum_{t=1}^{m}\|R_t(1)\|^2}{\|S(1)\|^2}\langle S(1), S(1)\rangle_r \\
&= 0.
\end{aligned} \tag{5.3}$$

Also it is not difficult to get

$$\langle S(1),S(2)\rangle_r=\left\langle S(1),\ \frac{\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(2))+\sum_{t=1}^{m}P\mathcal{F}_t^*(R_t(2))Q}{2}+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}S(1)\right\rangle_r$$
$$=\left\langle S(1),\ \frac{\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(2))+\sum_{t=1}^{m}P\mathcal{F}_t^*(R_t(2))Q}{2}\right\rangle_r+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\|S(1)\|^2$$
$$=\frac{1}{2}\left\langle S(1),\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(2))\right\rangle_r+\frac{1}{2}\left\langle PS(1)Q,\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(2))\right\rangle_r+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\|S(1)\|^2$$
$$=\sum_{t=1}^{m}\langle\mathcal{F}_t(S(1)),R_t(2)\rangle_r+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\|S(1)\|^2$$
$$=\frac{\|S(1)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\sum_{t=1}^{m}\langle R_t(1)-R_t(2),R_t(2)\rangle_r+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\|S(1)\|^2$$
$$=\frac{\|S(1)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\sum_{t=1}^{m}\langle R_t(1),R_t(2)\rangle_r-\frac{\|S(1)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\sum_{t=1}^{m}\|R_t(2)\|^2$$
$$+\frac{\sum_{t=1}^{m}\|R_t(2)\|^2}{\sum_{t=1}^{m}\|R_t(1)\|^2}\|S(1)\|^2=0. \tag{5.4}$$

Now assume that conclusion (5.2) holds for $0<i\leqslant l-1<v$, then

$$\sum_{r=1}^{m}\langle R_r(l),R_r(l+1)\rangle_r$$
$$=\sum_{r=1}^{m}\left\langle R_r(l),R_r(l)-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\|S(l)\|^2}\mathcal{F}_r(S(l))\right\rangle_r$$
$$=\sum_{r=1}^{m}\|R_r(l)\|^2-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\|S(l)\|^2}\langle\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(l)),S(l)\rangle_r$$
$$=\sum_{r=1}^{m}\|R_r(l)\|^2-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\|S(l)\|^2}\left\langle\frac{\sum_{r=1}^{m}\mathcal{F}_r^*(R_r(l))+\sum_{r=1}^{m}P\mathcal{F}_r^*(R_r(l))Q}{2},S(l)\right\rangle_r$$
$$=\sum_{r=1}^{m}\|R_r(l)\|^2-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\|S(l)\|^2}\left\langle S(l)-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\sum_{t=1}^{m}\|R_t(l-1)\|^2}S(l-1),S(l)\right\rangle_r$$
$$=\sum_{r=1}^{m}\|R_r(l)\|^2-\frac{\sum_{t=1}^{m}\|R_t(l)\|^2}{\|S(l)\|^2}\langle S(l),S(l)\rangle_r$$
$$+\frac{(\sum_{t=1}^{m}\|R_t(l)\|^2)(\sum_{t=1}^{m}\|R_t(l)\|^2)}{\|S(l)\|^2\sum_{t=1}^{m}\|R_t(l-1)\|^2}\langle S(l-1),S(l)\rangle_r=0. \tag{5.5}$$

Also we have

$$\langle S(l),S(l+1)\rangle_r$$
$$=\left\langle S(l),\ \frac{\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(l+1))+\sum_{t=1}^{m}P\mathcal{F}_t^*(R_t(l+1))Q}{2}+\frac{\sum_{t=1}^{m}\|R_t(l+1)\|^2}{\sum_{t=1}^{m}\|R_t(l)\|^2}S(l)\right\rangle_r$$
$$=\left\langle S(l),\ \frac{\sum_{t=1}^{m}\mathcal{F}_t^*(R_t(l+1))+\sum_{t=1}^{m}P\mathcal{F}_t^*(R_t(l+1))Q}{2}\right\rangle_r+\frac{\sum_{t=1}^{m}\|R_t(l+1)\|^2}{\sum_{t=1}^{m}\|R_t(l)\|^2}\|S(l)\|^2$$

$$= \frac{\|S(l)\|^2}{\sum_{t=1}^m \|R_t(l)\|^2} \sum_{t=1}^m \langle R_t(l), R_t(l+1)\rangle_r - \frac{\|S(l)\|^2}{\sum_{t=1}^m \|R_t(l)\|^2} \sum_{t=1}^m \|R_t(l+1)\|^2 + \frac{\sum_{t=1}^m \|R_t(l+1)\|^2}{\sum_{t=1}^m \|R_t(l)\|^2} \|S(l)\|^2 = 0. \tag{5.6}$$

Hence, (5.2) holds by the principle of induction.

Step 2. Suppose that

$$\sum_{r=1}^m \langle R_r(i), R_r(i+l)\rangle = 0, \quad \langle S(i), S(i+l)\rangle = 0 \quad \text{for } 1 \leqslant i \leqslant v \quad \text{and} \quad 1 < l < v.$$

Now we will show

$$\sum_{r=1}^m \langle R_r(i), R_r(i+l+1)\rangle_r = 0, \quad \langle S(i), S(i+l+1)\rangle_r = 0.$$

By using the above results, we can get

$$\langle S(1), S(l+2)\rangle = 0.$$

We can write

$$\begin{aligned}
&\sum_{r=1}^m \langle R_r(i), R_r(i+l+1)\rangle_r \\
&= \sum_{r=1}^m \left\langle R_r(i), R_r(i+l) - \frac{\sum_{t=1}^m \|R_t(i+l)\|^2}{\|S(i+l)\|^2} \mathcal{F}_r(S(i+l)) \right\rangle_r \\
&= \sum_{r=1}^m \langle R_r(i), R_r(i+l)\rangle_r - \frac{\sum_{t=1}^m \|R_t(i+l)\|^2}{\|S(i+l)\|^2} \langle \sum_{r=1}^m \mathcal{F}_r^*(R_r(i)), S(i+l)\rangle_r \\
&= -\frac{\sum_{t=1}^m \|R_t(i+l)\|^2}{\|S(i+l)\|^2} \left\langle \frac{\sum_{r=1}^m \mathcal{F}_r^*(R_r(i)) + \sum_{r=1}^m P\mathcal{F}_r^*(R_r(i))Q}{2}, S(i+l) \right\rangle_r \\
&= -\frac{\sum_{t=1}^m \|R_t(i+l)\|^2}{\|S(i+l)\|^2} \left\langle S(i) - \frac{\sum_{t=1}^m \|R_t(i)\|^2}{\sum_{t=1}^m \|R_t(i-1)\|^2} S(i-1), S(i+l) \right\rangle_r \\
&= -\frac{\sum_{t=1}^m \|R_t(i+l)\|^2}{\|S(i+l)\|^2} \langle S(i), S(i+l)\rangle_r \\
&\quad + \frac{(\sum_{t=1}^m \|R_t(l)\|^2)(\sum_{t=1}^m \|R_t(i+l)\|^2)}{\|S(i+l)\|^2 \sum_{t=1}^m \|R_t(l-1)\|^2} \langle S(i-1), S(i+l)\rangle_r \\
&= \cdots = \alpha \langle S(1), S(l+2)\rangle = 0,
\end{aligned} \tag{5.7}$$

for certain α. From the above results, we can obtain

$$\sum_{r=1}^m \langle R_r(i), R_r(i+l+1)\rangle_r = 0 \quad \text{and} \quad \sum_{r=1}^m \langle R_r(i+1), R_r(i+l+1)\rangle_r = 0.$$

Therefore

$$\langle S(i), S(i+l+1)\rangle_r = \left\langle S(i), \frac{\sum_{t=1}^m \mathcal{F}_t^*(R_t(i+l+1)) + \sum_{t=1}^m P\mathcal{F}_t^*(R_t(i+l+1))Q}{2} + \frac{\sum_{t=1}^m \|R_t(i+l+1)\|^2}{\sum_{t=1}^m \|R_t(i+l)\|^2} S(i+l) \right\rangle_r$$

$$= \left\langle S(i), \frac{\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(i+l+1)) + \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(i+l+1))Q}{2} \right\rangle_r$$

$$+ \frac{\sum_{t=1}^{m} \|R_t(i+l+1)\|^2}{\sum_{t=1}^{m} \|R_t(i+l)\|^2} \langle S(i), S(i+l) \rangle_r$$

$$= \frac{\|S(i)\|^2}{\sum_{t=1}^{m} \|R_t(i)\|^2} \sum_{t=1}^{m} \langle R_t(i), R_t(i+l+1) \rangle_r$$

$$- \frac{\|S(i)\|^2}{\sum_{t=1}^{m} \|R_t(i)\|^2} \sum_{t=1}^{m} \langle R_t(i+1), R_t(i+l+1) \rangle_r$$

$$= \cdots = \beta \langle S(1), S(l+2) \rangle = 0, \tag{5.8}$$

for certain β. From Steps 1 and 2, conclusion (5.1) holds by the principle of induction.

The proof of Lemma 2.3

First, we show that

$$\langle X^* - X(i), S(i) \rangle_r = \sum_{r=1}^{m} \|R_r(i)\|^2, \quad i = 1, 2, \ldots. \tag{5.9}$$

We prove conclusion (5.9) by induction. If $i = 1$, we have

$$\langle X^* - X(1), S(1) \rangle_r = \left\langle X^* - X(1), \frac{\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(1)) + \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(1))Q}{2} \right\rangle_r$$

$$= \frac{1}{2} \left\langle X^* - X(1), \sum_{t=1}^{m} \mathcal{F}_t^*(R_t(1)) \right\rangle_r + \frac{1}{2} \left\langle P(X^* - X(1))Q, \sum_{t=1}^{m} \mathcal{F}_t^*(R_t(1)) \right\rangle_r$$

$$= \sum_{t=1}^{m} \langle \mathcal{F}_t(X^* - X(1)), R_t(1) \rangle_r = \sum_{t=1}^{m} \langle A_t - \mathcal{F}_t(X(1)), R_t(1) \rangle_r$$

$$= \sum_{t=1}^{m} \|R_t(1)\|^2. \tag{5.10}$$

Now assume conclusion (5.9) holds for $i = v$. For $i = v + 1$, similarly to the proof of (5.10) we can obtain

$$\langle X^* - X(v+1), S(v+1) \rangle_r$$

$$= \left\langle X^* - X(v+1), \frac{\sum_{t=1}^{m} \mathcal{F}_t^*(R_t(v+1)) + \sum_{t=1}^{m} P\mathcal{F}_t^*(R_t(v+1))Q}{2} \right.$$

$$\left. + \frac{\sum_{t=1}^{m} \|R_t(v+1)\|^2}{\sum_{t=1}^{m} \|R_t(v)\|^2} S(v) \right\rangle_r$$

$$= \left\langle X^* - X(v+1), \sum_{t=1}^{m} \mathcal{F}_t^*(R_t(v+1)) + \frac{\sum_{t=1}^{m} \|R_t(v+1)\|^2}{\sum_{t=1}^{m} \|R_t(v)\|^2} S(v) \right\rangle_r$$

$$= \sum_{t=1}^{m} \langle A_t - \mathcal{F}_t(X(v+1)), R_t(v+1)\rangle_r + \frac{\sum_{t=1}^{m} \|R_t(v+1)\|^2}{\sum_{t=1}^{m} \|R_t(v)\|^2} \langle X^* - X(v), S(v)\rangle_r$$

$$- \frac{\sum_{t=1}^{m} \|R_t(v+1)\|^2}{\|S(v)\|^2} \|S(v)\|^2$$

$$= \sum_{t=1}^{m} \|R_t(v+1)\|^2. \tag{5.11}$$

By the principle of induction, conclusion (5.9) holds for all $i = 1, 2, \ldots$..

We suppose that

$$\langle X^* - X(i), S(i+r)\rangle_r = \sum_{t=1}^{m} \|R_t(i+r)\|^2 \quad \text{for} \quad r = 0, 1, \ldots, k. \tag{5.12}$$

It is not difficult to get

$$\langle X^* - X(i), S(i+r+1)\rangle_r$$

$$= \left\langle X^* - X(i), \sum_{t=1}^{m} \mathcal{F}_t^*(R_t(i+r+1)) + \frac{\sum_{t=1}^{m} \|R_t(i+r+1)\|^2}{\sum_{t=1}^{m} \|R_t(i+r)\|^2} S(i+r) \right\rangle_r$$

$$= \sum_{t=1}^{m} \langle A_t - \mathcal{F}_t(X(i)), R_t(i+r+1)\rangle_r + \frac{\sum_{t=1}^{m} \|R_t(i+r+1)\|^2}{\sum_{t=1}^{m} \|R_t(i+r)\|^2} \langle X^* - X(i), S(i+r)\rangle_r$$

$$= \sum_{t=1}^{m} \|R_t(i+r+1)\|^2. \tag{5.13}$$

By the principle of induction, the conclusion (2.3) holds.
It follows from (2.3) that

$$\langle X^* - X(i+1), S(i)\rangle_r = \langle X^* - X(i) - \frac{\sum_{t=1}^{m} \|R_t(i)\|^2}{\|S(i)\|^2} S(i), S(i)\rangle_r$$

$$= \langle X^* - X(i), S(i)\rangle_r - \frac{\sum_{t=1}^{m} \|R_t(i)\|^2}{\|S(i)\|^2} \|S(i)\| = 0. \tag{5.14}$$

Now we suppose that

$$\langle X^* - X(i+r), S(i)\rangle_r = 0 \quad \text{for} \quad r = 1, 2, \ldots \tag{5.15}$$

By applying (5.1) and (5.15), we can obtain

$$\langle X^* - X(i+r+1), S(i)\rangle_r = \langle X^* - X(i+r) - \frac{\sum_{t=1}^{m} \|R_t(i+r)\|^2}{\|S(i+r)\|^2} S(i+r), S(i)\rangle_r$$

$$= \langle X^* - X(i+r), S(i)\rangle_r - \frac{\sum_{t=1}^{m} \|R_t(i+r)\|^2}{\|S(i+r)\|^2} \langle S(i+r), S(i)\rangle_r$$

$$= 0. \tag{5.16}$$

Therefore the conclusion (2.4) holds by the principle of induction.

References

[1] S. Barnett, C. Storey, Matrix Methods in Stability Theory, Nelson, London, 1970.
[2] S. Barnett, Matrices in Control Theory with Applications to Linear Programming, Van Nostrand Reinhold, New York, 1971.
[3] H.C. Chen, Generalized reflexive matrices: special properties and applications, SIAM J. Matrix Anal. Appl. 19 (1998) 140–153.
[4] H.C. Chen, A. Sameh, Numerical linear algebra algorithms on the ceder system, in: A.K. Noor (Ed.), Parrallel computations and their impact on mechanics, AMD-vol. 86, The American Society of Mechanical Engineers, 1987, pp. 101–125.
[5] H.C. Chen, The SAS domain decomposition method for structural analysis, CSRD Teach, report 754, Center for Supercomputing Research and Development, University of Illinois, Urbana, IL, 1988.
[6] L. Dai, Singular control systems, Springer-Verlag, New York, Inc, 1989.
[7] M. Dehghan, M. Hajarian, The reflexive and anti-reflexive solutions of a linear matrix equation and systems of matrix equations, Rocky Mountain J. Math. 40 (2010) 1–23.
[8] M. Dehghan, M. Hajarian, On the generalized bisymmetric and skew-symmetric solutions of the system of generalized Sylvester matrix equations, Linear Multilinear Algebra 59 (2011) 1281–1309.
[9] M. Hajarian, M. Dehghan, The generalized centro-symmetric and least squares generalized centro-symmetric solutions of the matrix equation $AYB + CY^TD = E$, Math. Meth. Appl. Sci. 34 (2011) 1562–1579.
[10] M. Dehghan, M. Hajarian, Solving the generalized Sylvester matrix equation $\sum_{i=1}^{p} A_iXB_i + \sum_{j=1}^{q} C_jYD_j = E$ over reflexive and anti-reflexive matrices, Int. J. Contr. Automat. Syst. 9 (2011) 118–124.
[11] M. Dehghan, M. Hajarian, Analysis of an iterative algorithm to solve the generalized coupled Sylvester matrix equations, Appl. Math. Modell. 35 (2011) 3285–3300.
[12] M. Dehghan, M. Hajarian, Two algorithms for the Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations, Appl. Math. Lett. 24 (2011) 444–449.
[13] M. Dehghan, M. Hajarian, The (R,S)-symmetric and (R,S)-skew symmetric solutions of the pair of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$, Bull. Iranian Math. Soc. 37 (2011) 269–279.
[14] M. Dehghan, M. Hajarian, SSHI methods for solving general linear matrix equations, Eng. Comput. 28 (2012) 1028–1043.
[15] M. Dehghan, M. Hajarian, The generalised Sylvester matrix equations over the generalised bisymmetric and skew-symmetric matrices, Int. J. Syst. Sci. 43 (2012) 1580–1590.
[16] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Autom. Contr. 50 (2005) 1216–1221.
[17] F. Ding, T. Chen, Hierarchical gradient-based identification of multivariable discrete-time systems, Automatica 41 (2005) 315–325.
[18] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Syst. Contr. Lett. 54 (2005) 95–107.
[19] F. Ding, T. Chen, Hierarchical least squares identification methods for multivariable systems, IEEE Trans. Autom. Contr. 50 (2005) 397–402.
[20] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM J. Contr. Optim. 44 (2006) 2269–2284.
[21] F. Ding, P.X. Liu, J. Ding, Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Appl. Math. Comput. 197 (2008) 41–50.
[22] G.R. Duan, The solution to the matrix equation $AV + BW + EVJ + R$, Appl. Math. Lett. 17 (2004) 1197–1202.
[23] L.R. Fletcher, J. Kuatsky, N.K. Nichols, Eigenstructure assignment in descriptor systems, IEEE Trans. Auto. Contr. 31 (12) (1986) 1138–1141.
[24] P.M. Frank, Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy-a survey and some new results, Automatica 26 (1990) 459–474.
[25] G.H. Golub, S. Nash, C.F. Van Loan, A Hessenberg–Schur method for the matrix problem $AX + XB = C$, IEEE Trans. Auto. Contr. 24 (1979) 909–913.
[26] M.R. Hestenes, E.L. Stiefel, Methods of conjugate gradients for solving linear systems, J. Natl. Bur. Stand 49 (1952) 409–436.
[27] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, NewYork, 1991.
[28] A.B. Israel, T.N.E. Greville, Generalized inverses theory and applications, 2nd ed., Springer-Verlag, New York, 2003.
[29] A. Jameson, Solution of the equation $AX - XB = C$ by inversion of an $M \times M$ or $N \times N$ matrix, SIAM J. Appl. Math. 16 (1968) 1020–1023.
[30] T. Jiang, M. Wei, On solutions of the matrix equations $X - AXB = C$ and $X - A\overline{X}B = C$, Linear Algebra Appl. 367 (2003) 429–436.
[31] C.C. Paige, M.A. Saunders, LSQR: an algorithm for sparse linear equations and sparse least squares, ACM Trans. Math. Softw. 8 (1982) 43–71.
[32] F. Piao, Q. Zhang, Z. Wang, The solution to matrix equation $AX + X^TC = B$, J. Franklin Inst. 344 (2007) 1056–1062.
[33] Q.W. Wang, A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity, Linear Algebra Appl. 384 (2004) 43–54.
[34] Q.W. Wang, C.K. Li, Ranks and the least-norm of the general solution to a system of quaternion matrix equations, Linear Algebra Appl. 430 (2009) 1626–1640.
[35] Q.W. Wang, H.X. Chang, Q. Ning, The common solution to six quaternion matrix equations with applications, Appl. Math. Comput. 198 (2008) 209–226.
[36] Q.W. Wang, F. Zhang, The reflexive re-nonnegative definite solution to a quaternion matrix equation, Electron. J. Linear Algebra 17 (2008) 88–101.
[37] Q.W. Wang, H.S. Zhang, S.W. Yu, On solutions to the quaternion matrix equation $AXB + CYD = E$, Electron. J. Linear Algebra 17 (2008) 343–358.
[38] Q.W. Wang, J.W. Woude, H.X. Chang, A system of real quaternion matrix equations with applications, Linear Algebra Appl. 431 (2009) 2291–2303.
[39] Q.W. Wang, C.K. Li, Ranks and the least-norm of the general solution to a system of quaternion matrix equations, Linear Algebra Appl. 430 (2009) 1626–1640.
[40] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, Comput. Math. Appl. 58 (2009) 1441–1448.

[41] K. Ziętak, On approximation problems with zero-trace matrices, Linear Algebra Appl. 247 (1996) 169–183.
[42] L. Zhao, X. Hu, L. Zhang, Linear restriction problem of Hermitian reflexive matrices and its approximation, Appl. Math. Comput. 200 (2008) 341–351.
[43] B. Zhou, G.R. Duan, A new solution to the generalized Sylvester matrix equation $AV - EVF = BW$, Syst. Contr. Lett. 55 (2006) 193–198.
[44] B. Zhou, G.R. Duan, Solutions to generalized Sylvester matrix equation by Schur decomposition, Int. J. Syst. Sci. 38 (2007) 369–375.
[45] B. Zhou, Z.Y. Li, G.R. Duan, Y. Wang, Weighted least squares solutions to general coupled Sylvester matrix equations, J. Comput. Appl. Math. 224 (2009) 759–776.
[46] B. Zhou, Z.B. Yan, Solutions to right coprime factorizations and generalized Sylvester matrix equations, Trans. Inst. Measur. Contr. 30 (2008) 397–426.
[47] B. Zhou, G.R. Duan, On the generalized Sylvester mapping and matrix equations, Syst. Contr. Lett. 57 (2008) 200–208.
[48] B. Zhou, J. Lam, G.R. Duan, On Smith-type iterative algorithms for the Stein matrix equation, Appl. Math. Lett. 22 (2009) 1038–1044.
[49] B. Zhou, G.R. Duan, Z.Y. Li, Gradient based iterative algorithm for solving coupled matrix equations, Syst. Contr. Lett. 58 (2009) 327–333.
[50] B. Zhou, Z.Y. Li, G.R. Duan, Y. Wang, Solutions to a family of matrix equations by using the Kronecker matrix polynomials, Appl. Math. Comput. 212 (2009) 327–336.
[51] Y. Su, G. Chen, Iterative methods for solving linear matrix equation and linear matrix system, Int. J. Comput. Math. 87 (2010) 763–774.