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Resolvability and the Upper Dimension of Graphs

G. CHARTRAND, C. POISSON AND P. ZHANG*
Western Michigan University
Kalamazoo, MI 49008, U.S.A.

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Abstract—For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the (metric) representation of v with respect to W is the k-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations. A new sharp lower bound for the dimension of a graph G in terms of its maximum degree is presented.

A resolving set of minimum cardinality is a basis for G and the number of vertices in a basis is its (metric) dimension $\dim(G)$. A resolving set S of G is a minimal resolving set if no proper subset of S is a resolving set. The maximum cardinality of a minimal resolving set is the upper dimension $\dim^+(G)$. The resolving number $\text{res}(G)$ of a connected graph G is the minimum k such that every k-set W of vertices of G is also a resolving set of G. Then $1 \le \dim(G) \le \dim^+(G) \le \text{res}(G) \le n - 1$ for every nontrivial connected graph G of order n. It is shown that $\dim^+(G) = \text{res}(G) = n - 1$ if and only if $G = K_n$, while $\dim^+(G) = \text{res}(G) = 2$ if and only if G is a path of order at least 4 or an odd cycle.

The resolving numbers and upper dimensions of some well-known graphs are determined. It is shown that for every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G with $\dim(G) = \dim^+(G) = a$ and $\text{res}(G) = b$. Also, for every positive integer N, there exists a connected graph G with $\text{res}(G) - \dim^+(G) \ge N$ and $\dim^+(G) - \dim(G) \ge N$.

Keywords—Resolving set, Resolving number, Dimension, Upper dimension.

1. INTRODUCTION

A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. As described in [1], the structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1–6].

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$r(v \mid W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

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as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if distinct vertices have distinct representations. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The *(metric) dimension* $\dim(G)$ is the number of vertices in a basis for G. A resolving set W of G is a *minimal resolving set* if no proper subset of W is a resolving set. We refer to the maximum cardinality of a minimal resolving set as the *upper dimension* $\dim^+(G)$ and a minimal resolving set of cardinality $\dim^+(G)$ is called an *upper basis* for G. If G is a nontrivial connected graph, then $\dim(G) \leq \dim^+(G)$.

For example, the graph G of Figure 1 has the basis $W = \{u, z\}$ and so $\dim(G) = 2$. The representations for the vertices of G with respect to W are

$$r(u \mid W) = (0,1), \quad r(v \mid W) = (2,1), \quad r(x \mid W) = (1,2),$$
$$r(y \mid W) = (1,1), \quad r(z \mid W) = (1,0).$$

G:

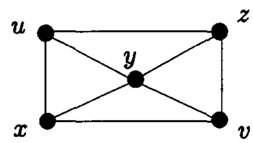

Figure 1. A graph G.

When determining whether a given set W of vertices of a graph G is a resolving set for G, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of G whose distance from w is 0.

Certainly, every minimum resolving set of a graph is a minimal resolving set, but the converse is not true. To illustrate these concepts, consider the graph $G = P_3 \times P_4$ of Figure 2 and let $W = \{u_1, w_1\}$. The representations for the vertices of $V(G) - W$ with respect to W are

$$r(u_2 \mid W) = (1,3), \quad r(u_3 \mid W) = (2,4), \quad r(u_4 \mid W) = (3,5), \quad r(v_1 \mid W) = (1,1),$$
$$r(v_2 \mid W) = (2,2), \quad r(v_3 \mid W) = (3,3), \quad r(v_4 \mid W) = (4,4), \quad r(w_2 \mid W) = (3,1),$$
$$r(w_3 \mid W) = (4,2), \quad r(w_4 \mid W) = (5,3).$$

Since these representations are distinct, W is a resolving set. Moreover, G contains no singleton resolving sets and so $\dim(G) = 2$. Now let $W' = \{v_1, v_3, w_3, w_4\}$. The representations for the vertices of $V(G) - W'$ with respect to W' are

$$r(u_1 \mid W') = (1,3,4,5), \quad r(u_2 \mid W') = (2,2,3,4), \quad r(u_3 \mid W') = (3,1,2,3),$$
$$r(u_4 \mid W') = (4,2,3,2), \quad r(v_2 \mid W') = (1,1,2,3), \quad r(v_4 \mid W') = (3,1,2,1),$$
$$r(w_1 \mid W') = (1,3,2,3), \quad r(w_2 \mid W') = (2,2,1,2).$$

Thus, W' is a resolving set as well. For $W_1 = W' - \{v_1\}$, $W_2 = W' - \{v_3\}$, $W_3 = W' - \{w_3\}$, and $W_4 = W' - \{w_4\}$, we have $r(u_2 \mid W_1) = r(v_1 \mid W_1)$, $r(v_3 \mid W_2) = r(w_2 \mid W_2)$, $r(v_4 \mid W_3) = r(w_3 \mid W_3)$, and $r(u_3 \mid W_4) = r(v_4 \mid W_4)$. Thus, W_i is not a resolving set for $1 \leq i \leq 4$, so W' is a minimal resolving set. Certainly, W' is not a basis as $\dim(G) = 2$. Thus, $\dim^+(G) \geq 4$. By a case-by-case analysis, one can show that there is no minimal resolving sets of cardinality 5. Hence, W' is an upper basis and $\dim^+(G) = 4$.

G:

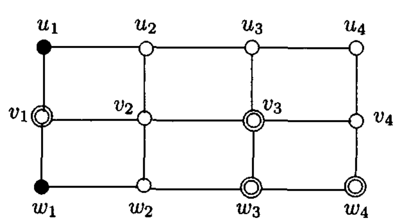

Figure 2. A basis and an upper basis for G.

For a nontrivial connected graph G of order n, the *resolving number* $\mathrm{res}(G)$ of G is the minimum k such that every k-subset W of $V(G)$ is a resolving set of G. Since every $(n-1)$-element subset of $V(G)$ is a resolving set of G and every resolving set contains a minimal resolving set,

$$1 \le \dim(G) \le \dim^+(G) \le \mathrm{res}(G) \le n-1. \tag{1}$$

In this paper, we study the resolving sets described above and investigate the relationships among the corresponding parameters.

The following two theorems (see [1,4–6]) give the dimensions of some well-known classes of graphs.

THEOREM A. *Let G be a connected graph of order $n \ge 2$.*

(a) *Then $\dim(G) = 1$ if and only if $G = P_n$.*
(b) *Then $\dim(G) = n-1$ if and only if $G = K_n$.*
(c) *For $n \ge 3$, $\dim(C_n) = 2$.*
(d) *For $n \ge 4$, $\dim(G) = n-2$ if and only if $G = K_{r,s}$ $(r, s \ge 1)$, $G = K_r + \overline{K_s}$ $(r \ge 1, s \ge 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \ge 1)$.*

A vertex of degree at least 3 in a tree T is called a *major vertex*. An end-vertex u of T is said to be a *terminal vertex of a major vertex* v of T if $d(u,v) < d(u,w)$ for every other major vertex w of T. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of T and let $\mathrm{ex}(T)$ denote the number of exterior major vertices of T.

THEOREM B. *If T is a tree that is not a path, then*

$$\dim(T) = \sigma(T) - \mathrm{ex}(T). \tag{2}$$

Next, we present a lemma that appeared in [1]. The *diameter* of G is the maximum distance between any two vertices of G and is denoted by $\mathrm{diam}\, G$.

LEMMA C. *For positive integers d and n with $d < n$, define $f(n,d)$ as the least positive integer k such that $k + d^k \ge n$. Then for a connected graph G of order $n \ge 2$ and diameter d,*

$$\dim(G) \ge f(n,d).$$

The lower bound in Lemma C is only attainable for graphs of diameter 2 or 3. We now present a sharp lower bound for the dimension of a connected graph G in terms of its maximum degree $\Delta(G)$.

THEOREM 1.1. *Let G be a nontrivial connected graph. Then*

$$\dim(G) \ge \lceil \log_3 (\Delta(G) + 1) \rceil. \tag{3}$$

PROOF. Let $\dim(G) = k$ and $v \in V(G)$ with $\deg v = \Delta(G)$. Moreover, let $N(v)$ be the neighborhood of v and let $B = \{u_1, u_2, \dots, u_k\}$ a basis of G. Observe that if $u \in N(v)$, then $d(u, u_i)$ is one of $d(v, u_i)$, $d(v, u_i) + 1$, or $d(v, u_i) - 1$ for all i with $1 \le i \le k$. Moreover, since B is a basis, $r(u \mid B) \ne r(v \mid B)$ for all $u \in N(v)$. It follows that there are at most $3^k - 1$ distinct representations of the vertices in $N(v)$ with respect to B. Therefore, $|N(v)| = \Delta(G) \le 3^k - 1$, which implies that $\dim(G) \ge \log_3(\Delta(G) + 1)$. ∎

The lower bound in Theorem 1.1 is sharp. In fact, for each pair k, Δ of integers such that $3^k = \Delta + 1$, there exists a connected graph $G_{k,\Delta}$ such that $\dim(G_{k,\Delta}) = k$ and $\Delta(G_{k,\Delta}) = \Delta$, as we show next.

For $(k,\Delta) = (1,2)$, the graph $G = P_n$ has the desired properties by Theorem A(a). For $(k,\Delta) = (2,8)$, we consider the graph $G_{2,8}$ of Figure 3. The maximum degree of $G_{2,8}$ is 8 with $\deg u_0 = 8$ and $N(u_0) = \{u_1, u_2, \ldots, u_8\}$. Let $W = \{v_1, v_2\}$. Then the representations of vertices of $V(G_{2,8}) - W$ with respect to W are

$$\begin{array}{lll} r(u_1 \mid W) = (1,1), & r(u_2 \mid W) = (1,2), & r(u_3 \mid W) = (1,3), \\ r(u_4 \mid W) = (2,3), & r(u_5 \mid W) = (2,1), & r(u_6 \mid W) = (3,3), \\ r(u_7 \mid W) = (3,2), & r(u_8 \mid W) = (3,1), & r(u_0 \mid W) = (2,2), \end{array}$$

which are distinct. Therefore, W is a resolving set of $G_{2,8}$ and so $\dim(G_{2,8}) = 2$ by Theorem 1.1.

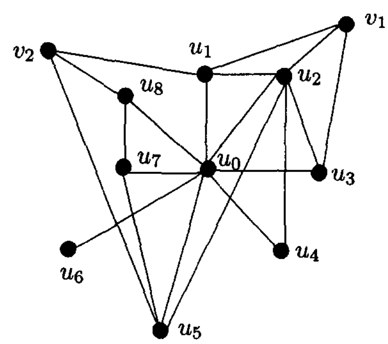

Figure 3. The graph $G_{2,8}$.

For $(k,\Delta) = (3,26)$, we construct the graph $G_{3,26}$ based on $G_{2,8}$ in two steps.

Step 1. Replace each vertex u_i by a path u_{i_1}, u_i, u_{i_2}, where $0 \le i \le 8$, such that
1. u_0 is adjacent to all vertices u_{i_1} and u_{i_2} with $0 \le i \le 8$ and all u_j with $1 \le j \le 8$,
2. u_{0_1} and u_{0_2} are adjacent, respectively, to all vertices u_{i_1}, u_{i_2}, where $1 \le i \le 8$,
3. v_j is adjacent to u_i, u_{i_1}, and u_{i_2} if and only if v_j is adjacent to u_i in $G_{2,8}$, where $0 \le i \le 8$ and $j = 1, 2$.

Step 2. Add a new vertex v_3 such that v_3 is adjacent to every vertex u_{i_1} for all $1 \le i \le 8$.

Certainly, $G_{2,8}$ is a subgraph of $G_{3,26}$. Observe that $\Delta(G_{3,26}) = \deg u_0 = 26$ as

$$N(u_0) = \{u_{0_1}, u_{0_2}\} \cup \{u_{i_1}, u_i, u_{i_2} : 1 \le i \le 8\}.$$

Next we show that $\dim(G_{3,26}) = 3$. Let $W = \{v_1, v_2, v_3\}$. By the structure of G, we have that the representations of u_{i_1}, u_i, u_{i_2} with respect to the subset $\{v_1, v_2\}$ of W are distinct from those of u_{j_1}, u_j, u_{j_2} for all i, j with $0 \le i \ne j \le 8$. Hence, the representations of u_{i_1}, u_i, u_{i_2} with respect to W are distinct from those of u_{j_1}, u_j, u_{j_2} $(0 \le i \ne j \le 8)$ as well. Moreover, since

$$r(u_{i_1} \mid W) = (*,*,1), \qquad r(u_i \mid W) = (*,*,2), \qquad r(u_{i_2} \mid W) = (*,*,3),$$

for all $0 \le i \le 8$, it follows that the representations u_{i_1}, u_i, and u_{i_2} with respect to W are distinct. So the representations of the vertices of $G_{3,26}$ with respect to W are distinct. Therefore, W is a resolving set of $G_{3,26}$. Since there are no 2-element resolving sets in $G_{3,26}$ by Theorem 1.1, it follows that $\dim(G_{3,26}) = 3$. Repeating this procedure, we have the desired result.

As a result of Theorem 1.1, we can now add another inequality to (1) for a nontrivial connected graph G of order n:

$$\lceil \log_3(\Delta(G)+1) \rceil \le \dim(G) \le \dim^+(G) \le \text{res}(G) \le n-1. \tag{4}$$

2. THE UPPER DIMENSION OF A GRAPH

Since $\dim(G) \leq \dim^+(G) \leq \text{res}(G)$ for every nontrivial connected graph G, it follows that $\dim^+(G)$ is bounded above by $\text{res}(G)$ and below by $\dim(G)$. Thus, to determine the upper dimension of a graph G, it is useful to know its resolving number. First, we present the resolving numbers of some well-known graphs in the next two propositions. Since the proof of the first proposition is routine, we omit it.

PROPOSITION 2.1. *Let $n \geq 3$ be an integer. Then*

(a) $\text{res}(K_n) = n - 1$ *and* $\text{res}(P_n) = 2$*;*

(b) $\text{res}(C_n) = 2$ *if n is odd,* $\text{res}(C_n) = 3$ *if n is even;*

(c) *for integers $k, n_1, n_2, \ldots, n_k$ with $k \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$,*

$$\text{res}(K_{n_1,n_2,\ldots,n_k}) = (n_1 + n_2 + \cdots + n_k) - 1.$$

The following theorem was presented in [7].

THEOREM D. *Let T be a nonpath tree of order $n \geq 3$ having p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leq j \leq k_i)$. Suppose that W is a set of vertices of T. Then W is a basis of T if and only if W contains exactly one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i$ and $1 \leq i \leq p)$ with exactly one exception for each i with $1 \leq i \leq p$ and W contains no other vertices of T.*

PROPOSITION 2.2. *Let T be a nonpath tree of order $n \geq 3$ having p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i,1}, u_{i,2}, \ldots, u_{i,k_i}$ be the terminal vertices of v_i and let P_{ij} be the $v_i - u_{ij}$ path of length ℓ_{ij} $(1 \leq j \leq k_i)$ with $\ell_{i1} \leq \ell_{i2} \leq \cdots \leq \ell_{i,k_i}$. For $1 \leq i \leq p$, let $\ell_i = \ell_{i1} + \ell_{i2}$, and let $\ell = \min\{\ell_i : 1 \leq i \leq p\}$. Then*

$$\text{res}(T) = n - \ell + 1.$$

PROOF. We first show that $\text{res}(T) \geq n - \ell + 1$. Assume, without loss of generality, that $\ell = \ell_1$. Let

$$W_0 = \{V(T) - [(V(P_{11}) \cup V(P_{11})]\} \cup \{v_1\}.$$

Then $|W_0| = n - \ell$. Since W_0 contains neither a vertex of $V(P_{11}) - \{v_1\}$ nor a vertex of $V(P_{12}) - \{v_1\}$, it follows by Theorem D that W_0 is not a resolving set and so $\text{res}(T) \geq n - \ell + 1$. Next we show that $\text{res}(T) \leq n - \ell + 1$. Let $W \subseteq V(T)$ with $|W| \geq n - \ell + 1$. Then W contains at least one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i$ and $1 \leq i \leq p)$ with at most one exception for some P_{ij}. Again, it follows by Theorem D that S is a resolving set of T. Therefore, $\text{res}(T) = n - \ell + 1$. ∎

Since nontrivial paths are the only connected graphs with dimension 1, it follows by Theorem A(a) and Proposition 2.1(a) that there are no graphs of order $n \geq 3$ with $\dim(G) = 1$ and $\text{res}(G) \geq 3$. Hence, for integers a and b with $a = 1$ and $b \geq 3$, there is no connected graph with dimension a and resolving number b. However, we show that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the dimension and resolving number, respectively, of some connected graph.

THEOREM 2.3. *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G with $\dim(G) = a$ and $\text{res}(G) = b$.*

PROOF. For $a = b$, let $G = K_{a+1}$, which has the desired properties by Theorem A(b) and Proposition 2.1(a). So we may assume that $2 \leq a < b$. Let G be the graph obtained from the path $P : u_1, u_2, \ldots, u_{b-a+1}$ by adding a new vertices $v_1, v_2, \ldots, v_a$ and the a edges $v_i u_1$ $(1 \leq i \leq a)$. Then G is a tree of order $b + 1$. Since G has only one exterior vertex, namely u_1, it follows that $\text{ex}(G) = 1$ and $\sigma(G) = \text{ter}(u_1) = a + 1$. By Theorem B, $\dim(G) = (a + 1) - 1 = a$. Since the order of G is $b+1$ and $\ell = 2$, it follows by Proposition 2.2 that $\text{res}(G) = (b+1)-2+1 = b$, as desired. ∎

We now determine the upper dimensions of some well-known graphs, starting with paths and cycles. It was shown in [7] that a vertex v_i of the path $P_n : v_1, v_2, \ldots, v_n$ is a basis for P_n if and only if v_i is an end-vertex of P_n and so $\dim^+(P_2) = 1$. For $n \geq 3$, since $\text{res}(P_n) = 2$ by Proposition 2.1(a), it follows that $\dim^+(P_n) \leq 2$ for all $n \geq 3$ by (1). If $n = 3$, then every two-element subset S of $V(P_3)$ contains an end-vertex of P_3 and so S contains a proper resolving subset, implying that S is not a minimal resolving set. Hence, $\dim^+(P_3) = 1$. For $n \geq 4$, the two-element resolving set $\{v_2, v_3\}$ contains no proper resolving subset and so $\dim^+(P_n) = 2$ for all $n \geq 4$. Therefore,

$$\dim^+(P_n) = \begin{cases} 1, & n \leq 3, \\ 2, & \text{otherwise}. \end{cases} \tag{5}$$

Since nontrivial paths are the only graphs with dimension 1, it follows by (5) that $\dim^+(G) \geq 2$ for all connected graphs of order $n \geq 4$.

Let C_n be a cycle of order $n \geq 3$. If n is odd, then $\dim^+(C_n) = 2$ by (1), Theorem A(c), and Proposition 2.1(b). If n is even, then every three-element subset S of $V(C_n)$ contains two vertices that are not antipodal. Hence, these two vertices form a proper resolving subset of S. This implies that S is not minimal and so $\dim^+(C_n) \leq 2$. Therefore,

$$\dim^+(C_n) = 2, \qquad \text{for } n \geq 3. \tag{6}$$

We have seen for paths of order at least 4 and all odd cycles that the upper dimension and resolving number are both 2. In fact, these are the only connected graphs having this property.

THEOREM 2.4. *For a nontrivial connected graph G, $\dim^+(G) = \text{res}(G) = 2$ if and only if G is a path of order at least 4 or G is an odd cycle.*

PROOF. It suffices to show that if $\dim^+(G) = \text{res}(G) = 2$, then G is a path of order at least 4 or G is an odd cycle. First, we verify that $\Delta(G) \leq 2$. Assume, to the contrary, that there exists a vertex v of G such that $\deg v \geq 3$. Let v_1, v_2, v_3 be three neighbors of v. We consider four cases according to the possible number of adjacencies of these vertices. These cases are shown in Figure 4. In each of the Figures 4a–4d, the solid vertices indicate a set S of two vertices that cannot be a resolving set for G as the remaining two vertices have the same representation with respect to S. Therefore, $\Delta(G) \leq 2$, as claimed.

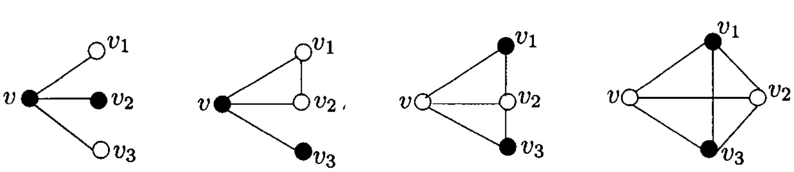

Figure 4. The possible number adjacencies of three neighbors of v.

Since G is connected, G is either a cycle or a path. By (5), (6), and Proposition 2.1(b), G is a path of order at least 4 or an odd cycle. ∎

By Theorem A(b) and (1), we have $\dim^+(K_n) = n - 1$ for $n \geq 2$. In fact, if $n \geq 2$, then the complete graph K_n is the only connected graph of order n with upper dimension $n - 1$.

PROPOSITION 2.5. *Let G be a nontrivial connected graph. Then $\dim^+(G) = n - 1$ if and only if $G = K_n$.*

PROOF. Assume, to the contrary, that there exists a connected graph $G \neq K_n$ with $\dim^+(G) = n - 1$. Let $W = V(G) - \{x\}$ be a minimal resolving set of G, where $x \in V(G)$. Since G is not complete, there exists an induced path $P_3 : u, v, w$ of length 2 in G. First, assume that $x \in \{u, v, w\}$. If $x = u$, then $d(v, w) = 1$ and $d(x, w) = 2$ and so $W - \{v\}$ is also a resolving set, which is a contradiction. Similarly, $x \neq w$. If $x = v$, then $W - \{u\}$ is also a resolving set. Hence, $x \notin \{u, v, w\}$. If $d(x, u) = 2$, then $W - \{v\}$ is a resolving set as $d(v, u) = 1$. Otherwise, $W - \{w\}$ is a resolving set as $d(w, u) = 2$. Therefore, $W = V(G) - \{x\}$ is not a minimal resolving set, a contradiction. ∎

Note that by (1), if G is a nontrivial connected graph of order n with $\dim^+(G) = n - 1$, then $\mathrm{res}(G) = n - 1$. So Proposition 2.5 implies that if G is a nontrivial connected graph of order n, then

$$\dim^+(G) = \mathrm{res}(G) = n - 1 \text{ if and only if } G = K_n.$$

However, the complete graph K_n is not the only connected graph of order n with resolving number $n - 1$ as we have seen in Proposition 2.1(c). By Theorem A(d) and (1), we are able to present a characterization of graphs G of order $n \geq 4$ with $\dim^+(G) = \dim(G) = n - 2$.

COROLLARY 2.6. *Let G be connected graph of order $n \geq 4$. Then $\dim^+(G) = \dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geq 1)$, $G = K_r + \overline{K_s}$ $(r \geq 1, s \geq 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geq 1)$.*

We now determine the upper dimension of all complete multipartite graphs and all trees that are not paths.

PROPOSITION 2.7. *For integers $k \geq 2$ and $n_1, n_2, \ldots, n_k$ with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $n = n_1 + n_2 + \cdots + n_k$,*

$$\dim^+\left(K_{n_1, n_2, \ldots, n_k}\right) = n - k.$$

PROOF. Let $G = K_{n_1, n_2, \ldots, n_k}$ whose partite sets are V_i with $|V_i| = n_i$ and $1 \leq i \leq k$. If W is a resolving set for G, then W must contains at least $n_i - 1$ vertices from each set V_i for $1 \leq i \leq k$. Hence, $\dim^+(G) \geq n - k$. Next we show that $\dim^+(G) \leq n - k$. Assume, to the contrary, that there exists a minimal resolving set S of G with $|S| \geq n - k + 1$. Since S is a resolving set, S contains at least $n_i - 1$ vertices from each set V_i for $1 \leq i \leq k$. Since $|S| \geq n - k + 1$, there exists an integer i with $1 \leq i \leq k$ such that $V_i \subseteq S$. Let $u \in V_i$. Then $S - \{u\}$ contains at least $n_i - 1$ vertices from V_i for $1 \leq i \leq k$. Hence, $S - \{u\}$ is a proper resolving subset of G, which is a contradiction. ∎

PROPOSITION 2.8. *If T is a tree that is not a path, then*

$$\dim^+(T) = \dim(T) = \sigma(T) - \mathrm{ex}(T). \tag{7}$$

PROOF. Assume, to the contrary, that there exists a tree T that is not a path such $\dim^+(T) > \dim(T)$. Then let S be a minimal resolving set of T with $|S| = \dim^+(T)$. Since S is minimal, S does not contain any basis of T as a proper subset. By Theorem D, there exists an exterior major vertex v of T and two terminal vertices u_1 and u_2 of v such that S contains no vertex from the paths $P_i - v$, where P_i is the $v - u_i$ path for $i = 1, 2$ in T. Let u_1' and u_2' be the vertices adjacent to v on P_1 and P_2, respectively. Since neither $P_1 - v$ nor $P_2 - v$ contains a vertex of S, it follows that $r(u_1' \mid S) = r(u_2' \mid S)$, contradicting the fact that S is a resolving set for T. Therefore, $\dim^+(T) = \dim(T)$ for all trees T that are not paths. ∎

By Propositions 2.5 and 2.8 and the proof to Theorem 2.3, we have the following.

COROLLARY 2.9. *For every pair a, b of integers with $2 \leq a \leq b$, there exists a graph G with $\dim(G) = \dim^+(G) = a$ and $\mathrm{res}(G) = b$.*

Every graph G encountered thus far has the property that $\dim^+(G) - \dim(G) \leq 2$. This might lead one to believe that there is a constant K such that $\dim^+(G) - \dim(G) \leq K$ for every connected graph G. However, this is not the case. Indeed, as we next show, every pair of the parameters $\dim(G)$, $\dim^+(G)$, $\mathrm{res}(G)$ can differ by an arbitrarily large number.

THEOREM 2.10. *For every positive integer N, there exists a connected graph G with*

$$\operatorname{res}(G) - \dim^+(G) \geq N \quad \text{and} \quad \dim^+(G) - \dim(G) \geq N.$$

PROOF. For each positive integer N, choose $k \geq \max\{4, N+1\}$. This implies that $2^k - k \geq 2^{k-1} + 2$ and $k \geq \log_2 N + 1$. We construct a graph G of order $2^k + k$ such that

$$\dim(G) = k, \qquad \dim^+(G) = 2^k - 1, \qquad \text{and} \qquad \operatorname{res}(G) = 2^k + (k-2),$$

which implies that $\operatorname{res}(G) - \dim^+(G) = k - 1 \geq N$ and $\dim^+(G) - \dim(G) = (2^k - 1) - k \geq 2^{k-1} \geq N$, as desired.

Let $V(G) = U \cup W$, where $U = \{u_0, u_1, \ldots, u_{2^k-1}\}$ and the ordered set $W = \{w_{k-1}, w_{k-2}, \ldots, w_0\}$ are disjoint. The subgraph $\langle U \rangle$ of G is complete, while W is independent. It remains to define the adjacencies between W and U. Let each integer j $(0 \leq j \leq 2^k - 1)$ be expressed in its base 2 (binary) representation. Thus, each such j can be expressed as a sequence of k coordinates, that is, a k-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j, with $0 \leq i \leq k-1$ and $0 \leq j \leq 2^k - 1$, we join w_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. For each j with $0 \leq j \leq 2^k - 1$, we also denote u_j by $(a_{j,k-1}, a_{j,k-2}, \ldots, a_{j0})$, where $a_{j,m} = a_{jm}$ $(0 \leq m \leq k-1)$ is the value in the 2^m position of the binary representation of j. This completes the construction of the graph G. For $k = 3$, the edges joining W and U in the graph G just constructed are shown in Figure 5.

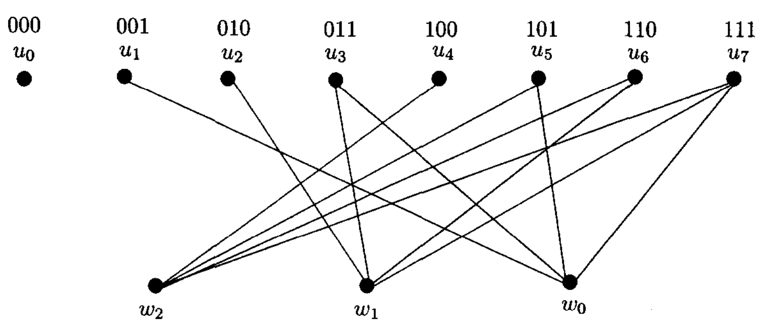

Figure 5. The edges joining W and U for $k = 3$.

We now show that $\dim(G) = k$. Since G has diameter 2 and order $k + 2^k$, it follows by Lemma C that $\dim(G) \geq k$. Next we show that W is a resolving set for G. To do this we need only show that the vertices of U have distinct metric representations with respect to W. The metric representation for each $u_j = (a_{j,k-1}, a_{j,k-2}, \ldots, a_{j0})$ $(0 \leq j \leq 2^k - 1)$ can be expressed as

$$r(u_j \mid W) = (2 - a_{j,k-1}, 2 - a_{j,k-2}, \ldots, 2 - a_{j0}).$$

Since the binary representations $a_{j,k-1}a_{j,k-2} \cdots a_{j1}a_{j0}$ are distinct for the vertices of U, their metric representations $(2 - a_{j,k-1}, 2 - a_{j,k-2}, \ldots, 2 - a_{j0})$ are distinct as well. Hence, W is a resolving set of G and $\dim(G) \leq |W| = k$. Thus, $\dim(G) = k$.

Next, we show that $\dim^+(G) = 2^k - 1$. Let $S = U - \{u_0\}$. We show that S is a minimal resolving set. By the construction of G, we have $r(u_0 \mid S) = (1, 1, \ldots, 1)$. Since at least one of the coordinates of each $r(w_i \mid S)$ is 2, we have $r(u_0 \mid S) \neq r(w_i \mid S)$ for all i with $0 \leq i \leq k-1$. Next, we show that $r(w_i \mid S) \neq r(w_j \mid S)$ for $0 \leq i \neq j \leq k-1$. Recall that w_i and u_j are adjacent if and only if the value in the 2^i position in the binary representation of j is 1. Hence, there are exactly 2^{k-2} vertices in U that are adjacent to both w_i and w_j and exactly 2^{k-2} vertices in U that are adjacent to neither w_i nor w_j. Since $k \geq 4$, it follows that $|S| = 2^k - 1 > 2^{k-1} = 2(2^{k-2})$. Hence, there exists $u_\ell \in S$ such that u_ℓ is adjacent to w_i and not adjacent to w_j. Hence, $r(w_i \mid S) \neq$

$r(w_j \mid S)$ and S is a resolving set. On the other hand, for each $u_i \in S$, where $1 \leq i \leq 2^k - 1$, we have $r(u_i \mid S - \{u_i\}) = r(u_0 \mid S - \{u_i\}) = (1, 1, \ldots, 1)$, implying that S has no proper resolving subsets. Therefore, S is minimal resolving set of G and so $\dim^+(G) \geq |S| = 2^k - 1$. A similar argument will show that $U - \{u_j\}$ is a resolving set of G for all j with $0 \leq j \leq 2^k - 1$.

To show that $\dim^+(G) \leq 2^k - 1$, it suffices to show that if S' is a resolving set of G with $|S'| \geq 2^k$, then S' is not minimal. Let $S' = W' \cup U'$, where $W' \subseteq W$ and $U' \subseteq U$. Since for each $u_j \in U$ $(0 \leq j \leq 2^k - 1)$, the set $U - \{u_j\}$ is a resolving set of G, it follows that $|U'| \leq 2^k - 2$ and $|W'| \geq 2$. Also, $|U'| \geq 2^k - k \geq 2^{k-1} + 2$. Let $u' \in U'$ and let

$$S^* = S' - \{u'\} = W' \cup (U' - \{u'\}).$$

We show that S^* is a resolving set. For $x, y \in V(G) - S^*$, we consider three cases.

CASE 1. $x, y \in U$. Since $r(x \mid S') \neq r(y \mid S')$ and $d(x, u) = d(y, u) = 1$ for all $u \in U'$, it follows that $r(x \mid W') \neq r(y \mid W')$ and so $r(x \mid S^*) \neq r(y \mid S^*)$.

CASE 2. $x, y \in W$. As before, there are exactly 2^{k-2} vertices in U that are adjacent to both x and y and exactly 2^{k-2} vertices in U that are adjacent to neither x nor y. Hence, there are exactly $2(2^{k-2}) = 2^{k-1}$ vertices u in U such that $d(x, u) = d(y, u)$. Since $|U' - \{u'\}| = |U'| - 1 \geq 2^{k-1} + 1$, it follows that there exists $u^* \in U' - \{u'\}$ such that $d(x, u^*) \neq d(y, u^*)$ and so $r(x \mid S^*) \neq r(y \mid S^*)$.

CASE 3. *One of x and y is in U and the other is in W, say $x \in U$ and $y \in W$.* Then $d(x, u) = 1$ for all $u \in U'$. Since there are exactly 2^{k-1} vertices in U that are adjacent to y and $|U' - \{u'\}| \geq 2^{k-1} + 1$, there exists $u'' \in U' - \{u'\} \subseteq S^*$ such that $d(y, u'') = 2$. So $r(x \mid S^*) \neq r(y \mid S^*)$.

Thus, S^* is a proper resolving subset of S' and so S' is not minimal. Therefore, $\dim^+(G) = 2^k - 1$.

Finally, we show that $\text{res}(G) = n - 2 = 2^k + k - 2$. Let

$$S_0 = \{w_{k-1}, w_{k-2}, \ldots, w_1, u_2, u_3, \ldots, u_{2^k-1}\} = V(G) - \{w_0, u_0, u_1\},$$

where $u_0 = (0, 0, \ldots, 0)$ and $u_1 = (0, 0, \ldots, 0, 1)$. Since

$$r(u_0 \mid S_0) = (2, 2, \ldots, 2, 1, 1, \ldots, 1) = r(u_1 \mid S_0),$$

where the first $k-1$ coordinates are 2 and the remaining $2^k - 2$ coordinates are 1, it follows that S_0 is not a resolving set and so $\text{res}(G) \geq |S_0| + 1 = n - 2$. On the other hand, let $S = V(G) - \{x, y\}$, where $x, y \in V(G)$. We consider three cases depending on the location of x and y in $W \cup U$.

(1) If $x, y \in W$, then $U \subseteq S$. Since U is resolving set, S is a resolving set.
(2) If $x, y \in U$, then $W \subseteq S$. Because W is a basis, S is a resolving set.
(3) Let one of x and y be in U and the other in W, say $x \in U$ and $y \in W$. But then $U - \{x\}$ is a resolving set in S and so S is a resolving set.

Since every set of $n - 2$ of vertices in G is a resolving set, $\text{res}(G) \leq n - 2$. Therefore, $\text{res}(G) = n - 2 = 2^k + k - 2$. ∎

There is reason to believe that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the dimension and upper dimension, respectively, of some connected graph, but this remains an open question. We close with this conjecture.

CONJECTURE 2.11. *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G with $\dim(G) = a$ and $\dim^+(G) = b$.*

REFERENCES

1. G. Chartrand, L. Eroh, M. Johnson and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, (Preprint).
2. G. Chartrand, M. Raines and P. Zhang, The directed distance dimension of oriented graphs, *Math. Bohem.* (to appear).

3. G. Chartrand, E. Salehi and P. Zhang, On the partition dimension of a graph, *Congres. Numer.* **131**, 55–66, (1998).
4. F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars. Combin.* **2**, 191–195, (1976).
5. P.J. Slater, Leaves of trees, *Congress. Numer.* **14**, 549–559, (1975).
6. P.J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.* **22**, 445–455, (1988).
7. C. Poisson and P. Zhang, The dimension of unicyclic graphs, (Preprint).