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OPTIMAL ERROR BOUNDS FOR THE DERIVATIVES OF TWO POINT HERMITE INTERPOLATION

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Abstract—For the derivatives of the Hermite polynomial interpolation of a function on the interval $[a,b]$ we obtain best possible uniform error estimates. For this, a new representation for the error function is developed.

1. INTRODUCTION

Let $x \in C^{(n)}[a,b]$, $n \geq 2$ be given and let $P_{n-1}(t)$ be the Hermite interpolation polynomial of degree $n-1$, which agrees with $x(t)$ and its first $k-1$, $1 \leq k \leq n-1$ (but fixed) derivatives at the point a, and $x(t)$ and its first $n-k-1$ derivatives at the point b, i.e.,

$$\begin{aligned} P_{n-1}^{(i)}(a) &= x^{(i)}(a), & 0 \leq i \leq k-1, \\ P_{n-1}^{(i)}(b) &= x^{(i)}(b), & 0 \leq i \leq n-k-1. \end{aligned} \tag{1.1}$$

For the associated error $e(t) = x(t) - P_{n-1}(t)$, one of the main results of this paper is contained in the following theorem, the proof of which is given in Section 3.

THEOREM 1.1. *For $2 \leq n \leq 6$ the following inequalities hold:*

$$\|e^{(m)}\| = \max_{a \leq t \leq b} |x^{(m)}(t) - P_{n-1}^{(m)}(t)| \leq C_{n,m}^{k}\,(b-a)^{n-m}\,\|x^{(n)}\|, \qquad 0 \leq m \leq n-1, \tag{1.2}$$

where the constants $C_{n,m}^k$ are given in the following table:

		m					
n	k	0	1	2	3	4	5
2	1	$\frac{1}{8}$	$\frac{1}{2}$				
3	1,2	$\frac{2}{81}$	$\frac{1}{6}$	$\frac{2}{3}$			
4	1,3	$\frac{9}{2048}$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{3}{4}$		
4	2	$\frac{1}{384}$	$\frac{1}{72\sqrt{3}}$	$\frac{1}{12}$	$\frac{1}{2}$		
5	1,4	$\frac{32}{46875}$	$\frac{1}{120}$	$\frac{1}{15}$	$\frac{3}{10}$	$\frac{4}{5}$	
5	2,3	$\frac{9}{31250}$	$\frac{3+8\sqrt{6}}{20000}$	$\frac{1}{60}$	$\frac{3}{20}$	$\frac{3}{5}$	
6	1,5	$\frac{625}{6718464}$	$\frac{1}{720}$	$\frac{1}{72}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{5}{6}$
6	2,4	$\frac{1}{32805}$	$\frac{25+34\sqrt{10}}{911250}$	$\frac{1}{360}$	$\frac{1}{30}$	$\frac{1}{5}$	$\frac{2}{3}$
6	3	$\frac{1}{46080}$	$\frac{\sqrt{5}}{30000}$	$\frac{1}{1920}$	$\frac{1}{120}$	$\frac{1}{10}$	$\frac{1}{2}$

In inequalities (1.2) the constants $C_{n,m}^k$ are optimal, as equality holds for the function $x_1(t) = (t-a)^k(b-t)^{n-k}$, whose Hermite interpolating polynomial $P_{n-1}(t) \equiv 0$, and only for this function up to a constant factor. Although some cases of (1.2) are well known in literature [1–4] and are of immense value [5–10], we give here an entirely different proof, which is based on a new representation of $e(t)$, rather than on the traditionally used integral representations in terms of Peano's kernel (Green's function). The present approach has an extra exciting feature, namely that, at least in all the cases $2 \le n \le 6$, $1 \le k \le n-1$, $0 \le m \le n-1$, computation does not require any computer, compared to as used by Birkhoff and Priver [4] for the case $n = 6$, $k = 3$, $0 \le m \le 5$.

From the transformation $u = a+b-t$, it is clear that for the function $e(a+b-t)$, the conditions (1.1) reduce to

$$\begin{aligned} e^{(i)}(a) &= 0, \qquad 0 \le i \le n-k-1,\\ e^{(i)}(b) &= 0, \qquad 0 \le i \le k-1, \end{aligned} \tag{1.3}$$

and hence, in inequalities (1.2) the constants

$$C^{k}_{n,m} = C^{n-k}_{n,m}. \tag{1.4}$$

Thus, in inequalities (1.2) it is sufficient to consider $1 \le k \le [\frac{n}{2}]$, or $[\frac{n+1}{2}] \le k \le n-1$. Further, if n is an even integer, say $2p$, and $k = p$, then $e(t) = e(a+b-t)$ and hence,

$$\max_{a\le t\le b} |e^{(m)}(t)| = \max_{a\le t\le \frac{a+b}{2}} |e^{(m)}(t)| = \max_{\frac{a+b}{2}\le t\le b} |e^{(m)}(t)|, \qquad 0 \le m \le 2p-1,$$

i.e., it is sufficient to obtain bounds for $|e^{(m)}(t)|$ over $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$.

2. A NEW REPRESENTATION FOR THE ERROR FUNCTION

THEOREM 2.1. *Let the functions $F_j(t)$, $0 \le j \le n-1$ be recursively defined as follows*

$$\begin{aligned} F_0(t) &= \int_b^t (b-t_n)^{n-1}\, x^{(n)}(t_n)\, dt_n,\\ F_j(t) &= \int_b^t (b-t_{n-j})^{-2}\, F_{j-1}(t_{n-j})\, dt_{n-j}, \qquad 1 \le j \le n-k-1,\\ F_{n-k}(t) &= \int_a^t \frac{(a-t_k)^{k-1}}{(b-t_k)^{k+1}}\, F_{n-k-1}(t_k)\, dt_k,\\ F_j(t) &= \int_a^t (a-t_{n-j})^{-2}\, F_{j-1}(t_{n-j})\, dt_{n-j}, \qquad n-k+1 \le j \le n-1. \end{aligned} \tag{2.1}$$

Then, the error function $e(t)$ can be expressed as

$$e(t) = (b-t)^{n-k}(a-t)^{k-1}\, F_{n-1}(t). \tag{2.2}$$

PROOF. Clearly we have $e^{(j)}(a) = 0$, $0 \le j \le k-1$, and $e^{(j)}(b) = 0$, $0 \le j \le n-k-1$. Thus, it remains to show that $e^{(n)}(t) = x^{(n)}(t)$. For this, first inductively, we shall show that

$$\begin{aligned} e^{(j)}(t) = {} & \sum_{i=0}^{j-1} \binom{j}{i} \frac{d^{j-i}}{dt^{j-i}} \left[(b-t)^{n-k}(a-t)^{k-1-i}\right] \frac{F_{n-i-1}(t)}{(a-t)^i}\\ & + (b-t)^{n-k}(a-t)^{k-2j-i}\, F_{n-j-1}(t), \qquad 0 \le j \le k-1. \end{aligned} \tag{2.3}$$

For $j = 0$, (2.3) is same as (2.2). Further, from (2.2) we find

$$e'(t) = \frac{d}{dt}\left[(b-t)^{n-k}(a-t)^{k-1}\right]F_{n-1}(t) + (b-t)^{n-k}(a-t)^{k-3}F_{n-2}(t), \tag{2.4}$$

and hence (2.3) holds for $j = 1$. We now assume that (2.3) holds for $j = p$, $1 \le p \le k-2$, then (2.3) provides

$$\begin{aligned}
e^{(p+1)}(t) = {} & \sum_{i=0}^{p-1}\binom{p}{i}\left\{\begin{aligned}
&\frac{d^{p+1-i}}{dt^{p+1-i}}\left[(b-t)^{n-k}(a-t)^{k-1-i}\right]\frac{F_{n-i-1}(t)}{(a-t)^i}\\
&+ i\,\frac{d^{p-i}}{dt^{p-i}}\left[(b-t)^{n-k}(a-t)^{k-1-i}\right]\frac{F_{n-i-1}(t)}{(a-t)^{i+1}}\\
&+ \frac{d^{p-i}}{dt^{p-i}}\left[(b-t)^{n-k}(a-t)^{k-1-i}\right]\frac{F_{n-i-2}(t)}{(a-t)^{i+2}}
\end{aligned}\right\}\\
& + \frac{d}{dt}\left[(b-t)^{n-k}(a-t)^{k-2p-1}\right]F_{n-p-1}(t)\\
& + (b-t)^{n-k}(a-t)^{k-2p-3}F_{n-p-2}(t)\\
= {} & \binom{p+1}{0}\frac{d^{p+1}}{dt^{p+1}}\left[(b-t)^{n-k}(a-t)^{k-1}\right]F_{n-1}(t)\\
& + \sum_{i=1}^{p-1}\left\{\begin{aligned}
&\binom{p}{i}\frac{d^{p+1-i}}{dt^{p+1-i}}\left[(b-t)^{n-k}(a-t)^{k-1-i}\right]\\
&+ i\binom{p}{i}\frac{d^{p-i}}{dt^{p-i}}\left[(b-t)^{n-k}(a-t)^{k-1-i}\right]\frac{1}{(a-t)}\\
&+ \binom{p}{i-1}\frac{d^{p+1-i}}{dt^{p+1-i}}\left[(b-t)^{n-k}(a-t)^{k-i}\right]\frac{1}{(a-t)}
\end{aligned}\right\}\frac{F_{n-i-1}(t)}{(a-t)^i}\\
& + p\,\frac{d}{dt}\left[(b-t)^{n-k}(a-t)^{k-p}\right]\frac{F_{n-p-1}(t)}{(a-t)^{p+1}}\\
& + \frac{d}{dt}\left[(b-t)^{n-k}(a-t)^{k-2p-1}\right]F_{n-p-1}(t)\\
& + (b-t)^{n-k}(a-t)^{k-2p-3}F_{n-p-2}(t)\\
= {} & \binom{p+1}{0}\frac{d^{p+1}}{dt^{p+1}}\left[(b-t)^{n-k}(a-t)^{k-1}\right]F_{n-1}(t)\\
& + \sum_{i=1}^{p-1}(-1)^{p+1-i}\left\{\begin{aligned}
&\binom{p}{i}\sum_{\ell=0}^{p+1-i}\binom{p+1-i}{\ell}\frac{(n-k)!}{(n-k-\ell)!}\,\frac{(k-1-i)!}{(k-p-2+\ell)!}\\
&\qquad\times(b-t)^{n-k-\ell}(a-t)^{k-p-2+\ell}\\
&- i\binom{p}{i}\sum_{\ell=0}^{p-i}\binom{p-i}{\ell}\frac{(n-k)!}{(n-k-\ell)!}\,\frac{(k-1-i)!}{(k-p-1+\ell)!}\\
&\qquad\times(b-t)^{n-k-\ell}(a-t)^{k-p-2+\ell}\\
&+ \binom{p}{i-1}\sum_{\ell=0}^{p+1-i}\binom{p+1-i}{\ell}\frac{(n-k)!}{(n-k-\ell)!}\,\frac{(k-i)!}{(k-p-1+\ell)!}\\
&\qquad\times(b-t)^{n-k-\ell}(a-t)^{k-p-2+\ell}
\end{aligned}\right\}\\
&\qquad\qquad\times\frac{F_{n-i-1}(t)}{(a-t)^i}\\
& + \binom{p+1}{p}\frac{d}{dt}\left[(b-t)^{n-k}(a-t)^{k-p-1}\right]\frac{F_{n-p-1}(t)}{(a-t)^p}\\
& + (b-t)^{n-k}(a-t)^{k-2p-3}F_{n-p-2}(t)
\end{aligned}$$

$$= \binom{p+1}{0} \frac{d^{p+1}}{dt^{p+1}} [(b-t)^{n-k}(a-t)^{k-1}] F_{n-1}(t)$$

$$+ \sum_{i=1}^{p-1}(-1)^{p+1-i} \left\{ \begin{array}{l} \binom{p+1}{i} \frac{(n-k)!}{(n-k-p-1+i)!} (b-t)^{n-k-p-1+i}(a-t)^{k-1-i} \\ + \binom{p+1}{i} \sum_{\ell=0}^{p-i} \binom{p+1-i}{\ell} \frac{(n-k)!}{(n-k-\ell)!} \frac{(k-1-i)!}{(k-p-2+\ell)!} \\ \qquad \times (b-t)^{n-k-\ell}(a-t)^{k-p-2+\ell} \end{array} \right\} \times \frac{F_{n-i-1}(t)}{(a-t)^i}$$

$$+ \binom{p+1}{p} \frac{d}{dt} [(b-t)^{n-k}(a-t)^{k-p-1}] \frac{F_{n-p-1}(t)}{(a-t)^p}$$

$$+ (b-t)^{n-k}(a-t)^{k-2p-3} F_{n-p-2}(t)$$

$$= \binom{p+1}{0} \frac{d^{p+1}}{dt^{p+1}} [(b-t)^{n-k}(a-t)^{k-1}] F_{n-1}(t)$$

$$+ \sum_{i=1}^{p-1} \binom{p+1}{i} \frac{d^{p+1-i}}{dt^{p+1-i}} [(b-t)^{n-k}(a-t)^{k-1-i}] \frac{F_{n-i-1}(t)}{(a-t)^i}$$

$$+ \binom{p+1}{p} \frac{d}{dt} [(b-t)^{n-k}(a-t)^{k-1-p}] \frac{F_{n-p-1}(t)}{(a-t)^p}$$

$$+ (b-t)^{n-k}(a-t)^{k-2p-3} F_{n-p-2}(t)$$

$$= \sum_{i=0}^{p} \binom{p+1}{i} \frac{d^{p+1-i}}{dt^{p+1-i}} [(b-t)^{n-k}(a-t)^{k-1-i}] \frac{F_{n-i-1}(t)}{(a-t)^i}$$

$$+ (b-t)^{n-k}(a-t)^{k-2p-3} F_{n-p-2}(t).$$

This completes the proof of (2.3).

Next let $j = k-1$ in (2.3) and differentiate the resulting equation; we obtain

$$e^{(k)}(t) = \sum_{i=0}^{k-1} \binom{k}{i} \frac{d^{k-i}}{dt^{k-i}} [(b-t)^{n-k}(a-t)^{k-1-i}] \frac{F_{n-i-1}(t)}{(a-t)^i} + (b-t)^{n-2k-1} F_{n-k-1}(t). \quad (2.5)$$

From (2.5) a tedious induction yields

$$e^{(j)}(t) = \sum_{i=0}^{k-1} \binom{j}{i} \frac{d^{j-i}}{dt^{j-i}} [(b-t)^{n-k}(a-t)^{k-1-i}] \frac{F_{n-i-1}(t)}{(a-t)^i}$$

$$+ \sum_{i=k}^{j} \binom{j}{i} \frac{d^{j-i}}{dt^{j-i}} [(b-t)^{n-1-i}] \frac{F_{n-i-1}(t)}{(b-t)^i}, \qquad k \le j \le n-1. \quad (2.6)$$

Finally, in (2.6) let $j = n-1$; then we obtain

$$e^{(n-1)}(t) = (n-1)! \left[\sum_{i=0}^{k-1} \frac{(-1)^{n-1-i}}{i!} \frac{F_{n-i-1}(t)}{(a-t)^i} + \sum_{i=k}^{n-1} \frac{(-1)^{n-1-i}}{i!} \frac{F_{n-i-1}(t)}{(b-t)^i} \right], \quad (2.7)$$

from which it follows that

$$e^{(n)}(t) = (n-1)! \left\{ \begin{array}{l} \displaystyle\sum_{i=0}^{k-2} \frac{(-1)^{n-1-i}}{i!} \left[\frac{F_{n-i-2}(t)}{(a-t)^{i+2}} + i\,\frac{F_{n-i-1}(t)}{(a-t)^{i+1}} \right] \\ \displaystyle + \frac{(-1)^{n-k}}{(k-1)!} \left[(a-t)^{k-1}(b-t)^{-k-1}\,\frac{F_{n-k-1}(t)}{(a-t)^{k-1}} + (k-1)\,\frac{F_{n-k}(t)}{(a-t)^{k}} \right] \\ \displaystyle + \sum_{i=k}^{n-2} \frac{(-1)^{n-1-i}}{i!} \left[\frac{F_{n-i-2}(t)}{(b-t)^{i+2}} + i\,\frac{F_{n-i-1}(t)}{(b-t)^{i+1}} \right] \\ \displaystyle + \frac{1}{(n-1)!} \left[(b-t)^{n-1}\,\frac{x^{(n)}(t)}{(b-t)^{n-1}} + (n-1)\,\frac{F_0(t)}{(b-t)^{n}} \right] \end{array} \right\}$$

$$= (n-1)! \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{k-2} \frac{(-1)^{n-1-i}}{(i-1)!}\,\frac{F_{n-i-1}(t)}{(a-t)^{i+1}} + \frac{(-1)^{n-k+1}}{(k-2)!}\,\frac{F_{n-k}(t)}{(a-t)^{k}} \\ \displaystyle + \sum_{i=1}^{k-2} \frac{(-1)^{n-i}}{(i-1)!}\,\frac{F_{n-i-1}(t)}{(a-t)^{i+1}} + \frac{(-1)^{n-k}}{(k-1)!}\,\frac{F_{n-k-1}(t)}{(b-t)^{k+1}} + \frac{(-1)^{n-k}}{(k-2)!}\,\frac{F_{n-k}(t)}{(a-t)^{k}} \\ \displaystyle + \sum_{i=k+1}^{n-2} \frac{(-1)^{n-1-i}}{(i-1)!}\,\frac{F_{n-i-1}(t)}{(b-t)^{i+1}} + \frac{(-1)^{n-1-k}}{(k-1)!}\,\frac{F_{n-k-1}(t)}{(b-t)^{k+1}} \\ \displaystyle + \sum_{i=k+1}^{n-2} \frac{(-1)^{n-i}}{(i-1)!}\,\frac{F_{n-i-1}(t)}{(b-t)^{i+1}} + \frac{(-1)}{(n-2)!}\,\frac{F_0(t)}{(b-t)^{n}} + \frac{x^{(n)}(t)}{(n-1)!} + \frac{1}{(n-2)!}\,\frac{F_0(t)}{(b-t)^{n}} \end{array} \right\}$$

$$= x^{(n)}(t).$$

THEOREM 2.2. *Let the functions $G_j(t)$, $0 \le j \le n-1$, be recursively defined as follows*

$$\begin{aligned} G_0(t) &= \int_b^t (b-t_n)^{n-1}\, x^{(n)}(a+b-t_n)\, dt_n, \\ G_j(t) &= \int_b^t (b-t_{n-j})^{-2}\, G_{j-1}(t_{n-j})\, dt_{n-j}, \qquad 1 \le j \le k-1, \\ G_k(t) &= \int_a^t \frac{(a-t_{n-k})^{n-k-1}}{(b-t_{n-k})^{n-k+1}}\, G_{k-1}(t_{n-k})\, dt_{n-k}, \\ G_j(t) &= \int_a^t (a-t_{n-j})^{-2}\, G_{j-1}(t_{n-j})\, dt_{n-j}, \qquad k+1 \le j \le n-1. \end{aligned} \tag{2.8}$$

Then, the error function $e(a+b-t)$ can be expressed as

$$e(a+b-t) = (b-t)^{k}(a-t)^{n-k-1}\, G_{n-1}(t). \tag{2.9}$$

PROOF. The proof can be deduced from Theorem 2.1.

LEMMA 2.3. *For the functions $F_j(t)$, $0 \le j \le n-1$, defined in Theorem 2.1, the following inequalities hold:*

$$\begin{aligned} |F_j(t)| &\le \frac{(n-j-1)!}{n!}\,(b-t)^{n-j}\, \|x^{(n)}\|, \qquad 0 \le j \le n-k-1, \\ |F_j(t)| &\le \frac{(n-j-1)!}{n!}\,(t-a)^{n-j}\, \|x^{(n)}\|, \qquad n-k \le j \le n-1. \end{aligned}$$

PROOF. The proof is by direct computation.

LEMMA 2.4. *For the functions $G_j(t)$, $0 \le j \le n-1$, defined in Theorem 2.2, the following inequalities hold:*

$$|G_j(t)| \le \frac{(n-j-1)!}{n!}(b-t)^{n-j}\,\|x^{(n)}\|, \qquad 0 \le j \le k-1,$$
$$|G_j(t)| \le \frac{(n-j-1)!}{n!}(t-a)^{n-j}\,\|x^{(n)}\|, \qquad k \le j \le n-1.$$

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is contained in the following cases:

CASE 1. $m = 0$.

From (2.2) and Lemma 2.3, we find

$$|e(t)| \le \frac{1}{n!}(b-t)^{n-k}(t-a)^k\,\|x^{(n)}\|.$$

Since the function $(b-t)^{n-k}(t-a)^k$ attains its maximum at $t = \frac{1}{n}[k\,b + (n-k)\,a]$, and its maximum value is $\frac{(n-k)^{n-k}\,k^k}{n^n}(b-a)^n$, it follows that

$$|e(t)| \le \frac{k^k\,(n-k)^{n-k}}{n!\,n^n}(b-a)^n\,\|x^{(n)}\|. \tag{3.1}$$

CASE 2. $m = n-1$.

From (2.7) and Lemma 2.3, we get

$$\begin{aligned}|e^{(n-1)}(t)| &\le \frac{1}{n}\left[\sum_{i=0}^{k-1}(t-a) + \sum_{i=k}^{n-1}(b-t)\right]\|x^{(n)}\| = \frac{1}{n}[k\,(t-a) + (n-k)\,(b-t)]\,\|x^{(n)}\| \\ &\le \frac{1}{n}\max\{k, n-k\}\,(b-a)\,\|x^{(n)}\|. \end{aligned}\tag{3.2}$$

CASE 3. $m = 1$, $k = n-1$.

From (2.4) and Lemma 2.3, we obtain

$$\begin{aligned}|e'(t)| &\le |-(a-t)^{n-2} - (n-2)\,(b-t)\,(a-t)^{n-3}|\,\frac{1}{n!}(t-a)\,\|x^{(n)}\| \\ &\quad + |(b-t)\,(a-t)^{n-4}|\,\frac{1}{n!}(t-a)^2\,\|x^{(n)}\| \\ &= \frac{1}{n!}\left\{\begin{array}{ll}(n-1)\,(t-a)^{n-2}\,(b-t) - (t-a)^{n-1}, & (t-a) \le (n-2)\,(b-t) \\ (t-a)^{n-1} - (n-3)\,(t-a)^{n-2}\,(b-t), & (t-a) \ge (n-2)\,(b-t)\end{array}\right\}\|x^{(n)}\| \\ &\le \frac{1}{n!}(b-a)^{n-1}\left\{\begin{array}{ll}(n-1)\,v^{n-2} - n\,v^{n-1} = \phi_1(v), & 0 \le v \le \frac{n-2}{n-1} \\ 1, & \frac{n-2}{n-1} \le v \le 1\end{array}\right\}\|x^{(n)}\|.\end{aligned}$$

For the function $\phi_1(v)$ we have $\phi_1'(v) = (n-1)\,v^{n-3}\,((n-2) - n\,v)$, and hence, $\phi_1(v) \le \max\{\phi_1(0), \phi_1(\frac{n-2}{n-1}), \phi_1(\frac{n-2}{n})\} < 1$.

Thus, it follows that

$$|e'(t)| \le \frac{1}{n!}(b-a)^{n-1}\,\|x^{(n)}\|. \tag{3.3}$$

CASE 4. $m = 1$, $k = 1$.

From (1.4) and Case 3, it is clear that in this case also inequality (3.3) holds.

CASE 5. $m = 2$, $k = n-1$ $(n \geq 4)$.

From (2.3) and Lemma 2.3, we find

$$\begin{aligned}
|e''(t)| &\leq (n-2)\,|2\,(a-t)^{n-3} + (n-3)\,(b-t)\,(a-t)^{n-4}|\,\frac{1}{n!}\,(t-a)\,\|x^{(n)}\| \\
&\quad + 2\,|-(a-t)^{n-3} - (n-3)\,(b-t)\,(a-t)^{n-4}|\,\frac{1}{n!}\,(t-a)\,\|x^{(n)}\| \\
&\quad + (b-t)\,(t-a)^{n-3}\,\frac{2}{n!}\,\|x^{(n)}\| \\
&= \frac{1}{n!}\,\{(t-a)^{n-3}\,[2\,(b-t) + 2\,|(t-a) - (n-3)\,(b-t)| \\
&\quad + (n-2)\,|(n-3)\,(b-t) - 2\,(t-a)|\,]\,\}\,\|x^{(n)}\| \\
&= \frac{1}{n!}\,(b-a)^{n-2} \\
&\quad \times \left\{ \begin{array}{ll}
v^{n-3}[n(n-3) + 2 - ((n-2)(n+1)+2)v] = \phi_2(v), & 0 \leq v \leq \frac{n-3}{n-1} \\
v^{n-3}[-(n-3)(n-4)+2+((n-2)(n-3)-2)v] = \phi_3(v), & \frac{n-3}{n-1} \leq v \leq \frac{n-3}{n-2} \\
v^{n-3}[-n(n-3) + 2 + ((n+1)(n-2) - 2)v] = \phi_4(v), & \frac{n-3}{n-2} \leq v \leq 1
\end{array} \right\} \|x^{(n)}\|.
\end{aligned}$$

Obviously, $\phi_4(v) \leq \phi_4(1) = 2n-2$, and $\phi_3(v) < [-(n-3)(n-4) + 2 + ((n-2)(n-3)-2)\,\frac{n-3}{n-2}] = 2 + \frac{(n-3)(n-4)}{(n-2)} < 2n-2$. Further, since $\phi_2'(v_1) = 0$, where $v_1 = \frac{(n-3)(n(n-3)+2)}{(n-2)((n-2)(n+1)+2)}$, $\phi_2(v) \leq \max\{\phi_2(0), \phi_2(v_1), \phi_2(\frac{n-3}{n-1})\} < 2n-2$. Therefore, we obtain

$$|e''(t)| \leq \frac{2\,(n-1)}{n!}\,(b-a)^{n-2}\,\|x^{(n)}\|. \tag{3.4}$$

CASE 6. $m = 2$, $k = 1$ $(n \geq 4)$.

From (1.4) and Case 5, it is clear that in this case also inequality (3.4) holds.

CASE 7. $n = 2p$, $k = p$, $m = 1$.

For this case recently in [3], it has been shown that

$$|e'(t)| \leq \frac{1}{2^p\,\sqrt{2p-1}\,(2p-1)!}\left(\frac{p-1}{2p-1}\right)^{p-1}(b-a)^{2p-1}\,\|x^{(2p)}\|. \tag{3.5}$$

CASE 8. $n = 2p$, $k = p$, $m = 2p-2$ $(p \geq 2)$.

From (2.6) and Lemma 2.3, we have

$$\begin{aligned}
|e^{(2p-2)}(t)| &\leq \sum_{i=0}^{p-1} \binom{2p-2}{i} \left| \frac{d^{2p-2-i}}{dt^{2p-2-i}}\,[(b-t)^p (a-t)^{p-1-i}] \right| \frac{i!}{(2p)!}\,(t-a)\,\|x^{(2p)}\| \\
&\quad + \sum_{i=p}^{2p-2} \binom{2p-2}{i} \left| \frac{d^{2p-2-i}}{dt^{2p-2-i}}\,[(b-t)^{2p-1-i}] \right| \frac{i!}{(2p)!}\,(b-t)\,\|x^{(2p)}\|
\end{aligned}$$

$$= \frac{1}{2p\,(2p-1)}\left\{\sum_{i=0}^{p-1}|p\,(b-a)-(2p-1-i)\,(t-a)|\,(t-a) + \sum_{i=p}^{2p-2}(2p-1-i)\,(b-t)^2\right\}\|x^{(2p)}\|$$

$$\le \frac{1}{2p\,(2p-1)}\,(b-a)^2 \max_{\frac{1}{2}\le v\le 1}\left\{\sum_{i=0}^{p-1}|p-(2p-1-i)\,v|\,v+\frac{1}{2}p\,(p-1)(1-v)^2\right\}\|x^{(2p)}\|$$

$$= \frac{1}{2p\,(2p-1)}\,(b-a)^2 \max_{0\le j\le p-1}\ \max_{v_j\le v\le v_{j+1}}\left\{\sum_{i=0}^{j-1}[(2p-1-i)\,v-p]\,v + \sum_{i=j}^{p-1}[p-(2p-1-i)\,v]\,v+\frac{1}{2}p\,(p-1)\,(1-v)^2\right\}\|x^{(2p)}\|$$

$$= \frac{1}{2p\,(2p-1)}\,(b-a)^2 \max_{0\le j\le p-1}\ \max_{v_j\le v\le v_{j+1}} \left\{\frac{1}{2}p\,(p-1)+(p-2pj)\,v+(4pj-j^2-j-p^2)\,v^2=\phi_5^j(v)\right\}\|x^{(2p)}\|,$$

where $v_j = \frac{p}{2p-j}$. We shall show that $\max_{0\le j\le p-1}\max_{v_j\le v\le v_{j+1}}\phi_5^j(v) = \frac{1}{2}p\,(p-1)$. For this, if $j=0$, then we have $\phi_5^0(v)=\frac{1}{2}p\,(p-1)+pv\,(1-pv)$, however, since $pv\ge 1$, for all $v\in[v_0,v_1]$, it is obvious that $\phi_5^0(v)\le\frac{1}{2}p\,(p-1)$. If $1\le j\le p-1$ and $4pj-j^2-j-p^2\le 0$, then for all $v\in[v_j,v_{j+1}]$, $\phi_5^j(v)\le\frac{1}{2}p\,(p-1)$ is immediate. Also, if $1\le j\le p-1$ and $4pj-j^2-j-p^2>0$, then $\phi_5^{j\prime}(v^*)=0$, where $v^*=-\frac{(p-2pj)}{2(4pj-j^2-j-p^2)}\ge\frac{1}{2}$, and since $\phi_5^j(v^*)=\frac{1}{2}p\,(p-1)+v_1\,\frac{1}{2}\,(p-2pj)\le\frac{1}{2}p\,(p-1)$, it follows that, for all $v\in[v_j,v_{j+1}]$, $\phi_5^j(v)\le\max\{\phi_5^j(v_j),\phi_5^j(v_{j+1})\}$. However,

$$\phi_5^j(v_j)=\frac{1}{2}p\,(p-1)+v_j^2\,(p+j-2)\,(j-p)\le\frac{1}{2}p\,(p-1),$$

and

$$\phi_5^j(v_{j+1})=\frac{1}{2}p\,(p-1)+v_{j+1}^2\,[j^2-(p-1)^2]\le\frac{1}{2}p\,(p-1)\quad\text{(with equality only for } j=p-1\text{)},$$

imply that, for all $v\in[v_j,v_{j+1}]$, $\phi_5^j(v)\le\frac{1}{2}p(p-1)$. Finally, it remains to note that $\phi_5^{p-1}(1)=\frac{1}{2}p\,(p-1)$.

Therefore, we obtain

$$|e^{(2p-2)}(t)|\le\frac{1}{4}\,\frac{(p-1)}{(2p-1)}\,(b-a)^2\,\|x^{(2p)}\|. \tag{3.6}$$

CASE 9. $m=n-2$, $k=n-1$ $(n\ge 5)$.

From (2.3) and Lemma 2.3, we get

$$|e^{(n-2)}(t)|\le\sum_{i=0}^{n-3}\binom{n-2}{i}\left|\frac{d^{n-2-i}}{dt^{n-2-i}}[(b-t)(a-t)^{n-2-i}]\right|\frac{i!}{n!}\,(t-a)\,\|x^{(n)}\| + (b-t)\,\frac{(n-2)!}{n!}\,(t-a)\,\|x^{(n)}\|$$

$$=\frac{1}{n\,(n-1)}\left\{\sum_{i=0}^{n-3}|(b-a)+(n-1-i)\,(a-t)|\,(t-a)+(t-a)\,(b-t)\right\}\|x^{(n)}\|$$

$$\le\frac{(b-a)^2}{n\,(n-1)}\max_{0\le v\le 1}v\left\{\sum_{i=0}^{n-3}|1-(n-1-i)\,v|+(1-v)\right\}\|x^{(n)}\|$$

$$= \frac{(b-a)^2}{n(n-1)} \left\{ \begin{array}{l} \max\limits_{0\le v\le \frac{1}{n-1}} v\left[(n-1)-\frac{1}{2}n(n-1)v\right] \\ \max\limits_{\frac{1}{n-j-1}\le v\le \frac{1}{n-j-2}} v\left[n-2j-3+\frac{1}{2}\left((j+1)(2n-j-2)\right.\right. \\ \qquad \left.\left. -(n-j)(n-j-3)-2\right)v=\phi_6(v)\right], \\ \qquad\qquad\qquad 0\le j\le n-4 \\ \max\limits_{\frac{1}{2}\le v\le 1} v\left[\frac{(n^2-n-4)}{2}v-(n-3)\right] \end{array} \right\} \|x^{(n)}\|$$

$$\le \frac{(b-a)^2}{n(n-1)} \left\{ \begin{array}{l} 1 \\ \max\limits_{0\le j\le n-4} \frac{1}{n-j-2}\max\left\{\phi_6^j\left(\frac{1}{n-j-1}\right), \phi_6^j\left(\frac{1}{n-j-2}\right)\right\} \\ \frac{1}{2}(n-1)(n-2) \qquad \text{(equality at } v=1) \end{array} \right\} \|x^{(n)}\|$$

$$= \frac{(b-a)^2}{n(n-1)}\,\frac{1}{2}(n-1)(n-2)\,\|x^{(n)}\|,$$

and, hence,

$$|e^{(n-2)}(t)| \le \frac{(n-2)}{2n}(b-a)^2\,\|x^{(n)}\|. \tag{3.7}$$

Case 10. $m=n-2,\ k=1\ (n\ge 5)$.

From (1.4) and Case 9, it follows that in this case also inequality (3.7) holds.

Case 11. $m=1,\ k=n-2\ (n\ge 5)$.

From (2.9) and (2.8), we have

$$\begin{aligned} -e'(a+b-t) =& \left[-(b-t)^{n-2}-(n-2)(b-t)^{n-3}(a-t)\right]G_{n-1}(t) \\ & + (b-t)^{n-2}(a-t)\frac{1}{(a-t)^2}G_{n-2}(t) \\ =& \left[-(b-t)^{n-2}-(n-2)(b-t)^{n-3}(a-t)\right] \\ & \times\left\{\frac{1}{(a-t)}G_{n-2}(t)-\int_a^t\frac{1}{(a-t_1)}G'_{n-2}(t_1)\,dt_1\right\} \\ & + \frac{(b-t)^{n-2}}{(a-t)}G_{n-2}(t) \\ =& \left[(b-t)^{n-2}+(n-2)(b-t)^{n-3}(a-t)\right]\int_a^t\frac{1}{(b-t_1)^3}G_{n-3}(t_1)\,dt_1 \\ & -(n-2)(b-t)^{n-3}\int_a^t\frac{(a-t_1)}{(b-t_1)^3}G_{n-3}(t_1)\,dt_1. \end{aligned}$$

Thus, from Lemma 2.4 it follows that

$$\begin{aligned} |e'(a+b-t)| \le& (b-t)^{n-3}\int_a^t\frac{1}{(b-t_1)^3}\left|(b-t)+(n-2)(a-t)-(n-2)(a-t_1)\right| \\ & \qquad \times\frac{2}{n!}(b-t_1)^3\,\|x^{(n)}\|\,dt_1 \\ =& \frac{2}{n!}(b-a)^{n-1}(1-v)^{n-3}\left[\int_0^v\left|(n-2)v_1+1-(n-1)v\right|dv_1\right]\|x^{(n)}\| \end{aligned}$$

$$=\frac{1}{n!}(b-a)^{n-1}\left\{\begin{array}{ll}(1-v)^{n-3}v(2-nv)=\phi_7(v), & 0\le v\le \frac{1}{n-1}\\ (1-v)^{n-3}[v(nv-2)+\frac{2}{n-2}(1-v)^2] & \\ \qquad =\phi_8(v), & \frac{1}{n-1}\le v\le 1\end{array}\right\}\|x^{(n)}\|$$

$$\le\frac{1}{n!}(b-a)^{n-1}\phi_7\left(\frac{2(n-1)-\sqrt{2(n-1)(n-2)}}{n(n-1)}\right)\|x^{(n)}\|.$$

Therefore, we have

$$|e'(t)|\le\frac{1}{n^{n-2}(n-1)^{n-2}n!}[(n-1)(n-2)+\sqrt{2(n-1)(n-2)}]^{n-3}$$
$$\times[2(n-1)-\sqrt{2(n-1)(n-2)}]\sqrt{\frac{2(n-2)}{(n-1)}}(b-a)^{n-1}\|x^{(n)}\|. \quad (3.8)$$

CASE 12. $m=1$, $k=2$ $(n\ge 5)$.

From (1.4) and Case 11, the inequality (3.8) holds in this case also.

CASE 13. $m=n-3$, $k=\left\{\begin{array}{ll}n-2, & if\, n=5\\ 2, & if\, n>5\end{array}\right\}$.

For this case we shall show that

$$|e^{(n-3)}(t)|\le\frac{1}{6}\frac{(n-3)(n-4)}{n(n-1)}(b-a)^3\|x^{(n)}\|. \quad (3.9)$$

If $n=5$, then from (2.3) and Lemma 2.3, we have

$$|e''(t)|\le\sum_{i=0}^{1}\binom{2}{i}\left|\frac{d^{2-i}}{dt^{2-i}}[(b-t)^2(a-t)^{2-i}]\right|\frac{i!}{5!}(t-a)\|x^{(5)}\|+(b-t)^2(t-a)\frac{1}{60}\|x^{(5)}\|$$
$$=\frac{1}{60}(b-a)^3v[|1-6v+6v^2|+|1-4v+3v^2|+(1-v)^2]\|x^{(5)}\|$$

$$=\frac{1}{60}(b-a)^3v\left\{\begin{array}{ll}10v^2-12v+3, & 0\le v\le\frac{3-\sqrt{3}}{6}\\ -2v^2+1, & \frac{3-\sqrt{3}}{6}\le v\le\frac{1}{3}\\ -8v^2+8v-1, & \frac{1}{3}\le v\le\frac{3+\sqrt{3}}{6}\\ 4v^2-4v+1, & \frac{3+\sqrt{3}}{6}\le v\le 1\end{array}\right\}\|x^{(5)}\|$$

$$\le\frac{1}{60}(b-a)^3\|x^{(5)}\|.$$

If $n>5$, then from (2.6) and Lemma 2.3, we find

$$|e^{(n-3)}(t)|\le\frac{1}{n(n-1)}\left[\left|\frac{(n-1)}{2}(b-t)^2-(b-t)(b-a)\right|(t-a)+\frac{(n-3)}{2}(b-t)^2(t-a)\right.$$
$$\left.+\frac{1}{2(n-2)}\sum_{i=2}^{n-3}(n-1-i)(n-2-i)(b-t)^3\right]\|x^{(n)}\|$$

$$=\frac{1}{n(n-1)}(b-a)^3$$
$$\times\left\{\begin{array}{ll}\frac{1}{6}(n-3)(n-4)(1-v)^3+(n-3)v(1-v)^2-v^2(1-v), & 0\le v\le\frac{n-3}{n-1}\\ \frac{1}{6}(n-3)(n-4)(1-v)^3+v^2(1-v), & \frac{n-3}{n-1}\le v\le 1\end{array}\right\}\|x^{(n)}\|$$

$$\le\frac{1}{6}\frac{(n-3)(n-4)}{n(n-1)}(b-a)^3\|x^{(n)}\|.$$

CASE 14. $m = n-3,\ k = \begin{Bmatrix} 2, & if\, n = 5 \\ n-2, & if\, n > 5 \end{Bmatrix}.$

From (1.4) and Case 13, the inequality (3.9) holds for this case also.

CASE 15. $m = n-2,\ k = n-2\ (n \geq 5)$.

From (2.6) and Lemma 2.3, we find

$$\begin{aligned}|e^{(n-2)}(t)| \leq& \frac{1}{n\,(n-1)}\left[\sum_{i=0}^{n-3}(t-a)\,|2(b-a)-(n-1-i)\,(t-a)| + (b-t)^2\right]\|x^{(n)}\| \\ =& \frac{1}{n\,(n-1)}\,(b-a)^2 \\ &\times \left\{\begin{array}{ll} 1+2v\,(n-3)-\frac{1}{2}v^2\,(n^2-n-4), & 0 \leq v \leq \frac{2}{n-1} \\ 1+2v\,(n-3-2j)+\frac{1}{2}v^2\,(j(2n-j-1) & \\ \quad -(n-2-j)\,(n-j+1)+2), & \frac{2}{n-j} \leq v \leq \frac{2}{n-j-1}, \\ & 1 \leq j \leq n-3 \end{array}\right\}\|x^{(n)}\|.\end{aligned}$$

Therefore, it follows that

$$|e^{(n-2)}(t)| \leq \frac{(n-2)\,(n-3)}{2n\,(n-1)}\,(b-a)^2\,\|x^{(n)}\|. \tag{3.10}$$

CASE 16. $m = n-2,\ k = 2\ (n \geq 5)$.

From (1.4) and Case 15, the inequality (3.10) holds in this case also.

CASE 17. $m = n-3,\ k = n-1\ (n \geq 6)$.

From (2.3) and Lemma 2.3, it follows that

$$\begin{aligned}|e^{(n-3)}(t)| \leq& \frac{1}{n\,(n-1)\,(n-2)}\left[\sum_{i=0}^{n-4}(n-2-i)\left|(b-a)-\frac{1}{2}\,(n-1-i)\,(t-a)\right| + (b-t)\right] \\ &\times (t-a)^2\,\|x^{(n)}\| \\ =& \frac{1}{n\,(n-1)\,(n-2)}\,(b-a)^3 \\ &\times \left\{\begin{array}{ll} v^2\,[1-\frac{1}{6}n(n-1)(n-2)v+\frac{1}{2}n(n-3)] = \phi_9(v), & 0 \leq v \leq \frac{2}{n-1} \\ v^2\,[\frac{1}{6}(n(n-1)(n-2)-2(n-j)(n-j-1)(n-j-2))v & \\ \quad +\frac{1}{2}(n^2-3n+2+2j^2-4nj+6j)] = \phi_{10}(v), & \\ & \frac{2}{n-j} \leq v \leq \frac{2}{n-j-1}, \\ & 1 \leq j \leq n-4 \\ v^2\,[1-2v+\frac{1}{6}n(n-1)(n-2)v-\frac{1}{2}n(n-3)] = \phi_{11}(v), & \frac{2}{3} \leq v \leq 1 \end{array}\right\}\|x^{(n)}\|.\end{aligned}$$

Clearly, $\phi_{11}(v) \leq \phi_{11}(1) = \frac{1}{6}(n-1)(n-2)(n-3)$. Further, since for $n \geq 6$, $\phi_9(v) \leq \left(\frac{2}{n-1}\right)^2 [1+\frac{1}{2}\,n\,(n-3)] = \frac{2(n-2)}{(n-1)} < 2$, and $\phi_{11}(1) \geq 10$, we find that $\phi_9(v) < \phi_{11}(1)$. Next, since $(n^2 - 3n + 2 + 2j^2 - 4nj + 6j) \leq (n^2 - 7n + 10)$, it follows that $\phi_{10}(v) \leq (\frac{2}{3})^2\,[\frac{1}{6}\,n\,(n-1)\,(n-2)+\frac{1}{2}\,(n^2-7n+10)] < \phi_{11}(1)$. Therefore, we get

$$|e^{(n-3)}(t)| \leq \frac{(n-3)}{6n}\,(b-a)^3\,\|x^{(n)}\|. \tag{3.11}$$

CASE 18. $m = n - 3$, $k = 1$ $(n \geq 6)$.

From (1.4) and Case 17, the inequality (3.11) holds in this case as well.

CASE 19. $n = 2p$, $k = 2p - 2$, $m = 2p - 4$ $(p \geq 3)$.

From (2.3) and Lemma 2.3, we obtain

$$\begin{aligned}
|e^{(2p-4)}(t)| \leq & \frac{1}{2p\,(2p-1)\,(2p-2)\,(2p-3)} \\
& \times \left[\sum_{i=0}^{2p-5} |(b-a)^2\,(2p-3-i)\,(a-t) + (b-a)\,(2p-2-i)\,(2p-3-i)\,(a-t)^2 \right. \\
& \left. + \frac{1}{6}\,(2p-1-i)\,(2p-2-i)\,(2p-3-i)\,(a-t)^3|\,(t-a) + (b-t)^2\,(t-a)^2 \right] \|x^{(2p)}\| \\
= & \frac{1}{2p\,(2p-1)\,(2p-2)\,(2p-3)}\,(b-a)^4\,v^2 \\
& \times \left[\sum_{i=0}^{2p-5} (2p-3-i)\,|-1 + (2p-2-i)\,v - \frac{1}{6}\,(2p-1-i)\,(2p-2-i)\,v^2| \right. \\
& \left. + (1-v)^2 \right] \|x^{(2p)}\|, \quad 0 \leq v \leq 1.
\end{aligned} \tag{3.12}$$

In particular, for $p = 3$, (3.12) reduces to

$$\begin{aligned}
|e''(t)| \leq & \frac{1}{360}\,(b-a)^4\,v^2 \left[|10v^2 - 12v + 3| + |4v^2 - 6v + 2| + (1-v)^2 \right] \|x^{(6)}\|, \qquad 0 \leq v \leq 1 \\
= & \frac{1}{360}\,(b-a)^4\,v^2 \left\{ \begin{array}{ll} 6 - 20v + 15v^2, & 0 \leq v \leq \frac{6-\sqrt{6}}{10} \\ 4v - 5v^2, & \frac{6-\sqrt{6}}{10} \leq v \leq \frac{1}{2} \\ -4 + 16v - 13v^2, & \frac{1}{2} \leq v \leq \frac{6+\sqrt{6}}{10} \\ 2 - 8v + 7v^2, & \frac{6+\sqrt{6}}{10} \leq v \leq 1 \end{array} \right\} \|x^{(6)}\| \\
\leq & \frac{1}{360}\,(b-a)^4\,\|x^{(6)}\|.
\end{aligned} \tag{3.13}$$

CASE 20. $n = 2p$, $k = p$, $m = 2$ $(p \geq 3)$.

From (2.3), we have

$$\begin{aligned}
e''(t) = & \Big[p\,(p-1)\,(b-t)^{p-2}\,(a-t)^{p-1} + 2p\,(p-1)\,(b-t)^{p-1}\,(a-t)^{p-2} \\
& + (p-1)\,(p-2)\,(b-t)^p\,(a-t)^{p-3} \Big] \left[\frac{F_{2p-2}(t_1)}{(a-t_1)} \Big|_a^t - \int_a^t \frac{F'_{2p-2}(t_1)}{(a-t_1)}\,dt_1 \right] \\
& + 2\left[-p\,(b-t)^{p-1}\,(a-t)^{p-2} - (p-2)\,(b-t)^p\,(a-t)^{p-3} \right] \frac{F_{2p-2}(t)}{(a-t)} \\
& + (b-t)^p\,(a-t)^{p-5}\,F_{2p-3}(t)
\end{aligned}$$

$$
\begin{aligned}
&= \Big[p(p-1)(b-t)^{p-2}(a-t)^{p-2} + 2p(p-2)(b-t)^{p-1}(a-t)^{p-3} \\
&\qquad + (p-2)(p-3)(b-t)^{p}(a-t)^{p-4}\Big] F_{2p-2}(t) \\
&\quad - \Big[p(p-1)(b-t)^{p-2}(a-t)^{p-1} + 2p(p-1)(b-t)^{p-1}(a-t)^{p-2} \\
&\qquad + (p-1)(p-2)(b-t)^{p}(a-t)^{p-3}\Big] \int_a^t \frac{F_{2p-3}(t_1)}{(a-t_1)^3}\, dt_1 \\
&\quad + (b-t)^{p}(a-t)^{p-5} F_{2p-3}(t).
\end{aligned} \tag{3.14}
$$

For $p = 3$, (3.14) reduces to

$$
\begin{aligned}
e''(t) &= (b-a)^4 \Big\{ [-6(1-v)\,v + 6(1-v)^2]\, F_4(v) \\
&\qquad + [6(1-v)\,v^2 - 12(1-v)^2 v + 2(1-v)^3] \int_0^v \frac{F_3(v_1)}{v_1^3}\, dv_1 + \frac{(1-v)^3}{v^2} F_3(v) \Big\} \\
&= (b-a)^4 \Big\{ (1-v) \int_0^v \left(\frac{6-12v}{v_1^2} + \frac{20v^2-16v+2}{v_1^3} \right) F_3(v_1)\, dv_1 + \frac{(1-v)^3}{v^2} F_3(v) \Big\} \\
&= (b-a)^4 \Big\{ (1-v) \int_0^v \left(\frac{6-12v}{v_1^2} + \frac{20v^2-16v+2}{v_1^3} \right) \left(\int_0^{v_1} v_2^2 (1-v_2)^{-4} F_2(v_2)\, dv_2 \right) dv_1 \\
&\qquad + \frac{(1-v)^3}{v^2} F_3(v) \Big\} \\
&= (b-a)^4 \Big\{ (1-v) \int_0^v \left(\int_{v_2}^v \left(\frac{6-12v}{v_1^2} + \frac{20v^2-16v+2}{v_1^3} \right) dv_1 \right) v_2^2 (1-v_2)^{-4} F_2(v_2)\, dv_2 \\
&\qquad + \frac{(1-v)^3}{v^2} F_3(v) \Big\} \\
&= (b-a)^4 \Big\{ (1-v) \int_0^v \left(\frac{12v-6}{v} - \frac{10v^2-8v+1}{v^2} + \frac{6-12v}{v_2} + \frac{10v^2-8v+1}{v_2^2} \right) \\
&\qquad \times v_2^2 (1-v_2)^{-4} F_2(v_2)\, dv_2 + \frac{(1-v)^3}{v^2} \int_0^v v_2^2 (1-v_2)^{-4} F_2(v_2)\, dv_2 \Big\} \\
&= (b-a)^4 (1-v) \int_0^v \left(3 + \frac{6-12v}{v_2} + \frac{10v^2-8v+1}{v_2^2} \right) v_2^2 (1-v_2)^{-4} F_2(v_2)\, dv_2 \\
&= -(b-a)^4 (1-v) \int_0^v \left(3 + \frac{6-12v}{v_2} + \frac{10v^2-8v+1}{v_2^2} \right) v_2^2 (1-v_2)^{-4} \\
&\qquad \times \left(\int_{v_2}^v (1-v_3)^{-2} F_1(v_3)\, dv_3 + \int_v^1 (1-v_3)^{-2} F_1(v_3)\, dv_3 \right) dv_2 \\
&= -(b-a)^4 (1-v) \Big\{ \int_0^v \left[\int_0^{v_3} \left(3 + \frac{6-12v}{v_2} + \frac{10v^2-8v+1}{v_2^2} \right) v_2^2 (1-v_2)^{-4}\, dv_2 \right] \\
&\qquad \times (1-v_3)^{-2} F_1(v_3)\, dv_3 \\
&\qquad - F_2(v) \int_0^v \left(3 + \frac{6-12v}{v_2} + \frac{10v^2-8v+1}{v_2^2} \right) v_2^2 (1-v_2)^{-4}\, dv_2 \Big\} \\
&= -(b-a)^4 \Big\{ (1-v) \int_0^v \left[\frac{10}{3} \frac{(1-v)^2}{(1-v_3)^3} - \frac{6(1-v)}{(1-v_3)^2} + \frac{3}{(1-v_3)} - \frac{1}{3}(10v^2-2v+1) \right] \\
&\qquad \times (1-v_3)^{-2} F_1(v_3)\, dv_3 - \frac{1}{3}(3v - 12v^2 + 10v^3) F_2(v) \Big\},
\end{aligned}
$$

which, on using Lemma 2.3, gives

$$|e''(t)| \le (b-a)^4 \left\{ \frac{1}{90}(1-v) \int_0^v |v_3^3(10v^2-2v+1) + v_3^2(-30v^2+6v+6) + v_3(30v_3^2-24v_3+3)|\, dv_3 \right.$$
$$\left. + \frac{1}{360}\, |3v-12v^2+10v^3|\, (1-v)^4 \right\} \|x^{(6)}\|, \qquad 0 \le v \le \frac{1}{2}. \tag{3.15}$$

For $v \in [0, \frac{4-\sqrt{6}}{10}]$, (3.15) becomes

$$\begin{aligned} |e''(t)| &\le \frac{1}{120}\,(b-a)^4\, v\,(1-v)\,(5v^2-5v+1)\, \|x^{(6)}\| \\ &\le \frac{1}{2400}\,(b-a)^4\, \|x^{(6)}\|. \end{aligned} \tag{3.16}$$

For $v \in [\frac{4-\sqrt{6}}{10}, \frac{6-\sqrt{6}}{10}]$, (3.15) is the same as

$$|e''(t)| \le (b-a)^4 \left\{ -\frac{1}{180}\,(1-v)\,[\alpha^2(60v^2-48v+6) + \alpha^3(-40v^2+8v+8) + \alpha^4(10v^2-2v+1)] \right.$$
$$\left. + \frac{1}{120}\, v\,(1-v)\,(5v^2-5v+1) = \phi_{12}(v) \right\} \|x^{(6)}\|,$$

where $\alpha = \frac{15v^2-3v-3+(1-v)\sqrt{-75v^2+60v+6}}{10v^2-2v+1}$. The maximum of $\phi_{12}(v)$ is attained when $t = \frac{4-\sqrt{6}}{10}$, and the maximum value is $\frac{1}{2655.3468}$. Thus, in this interval,

$$|e''(t)| \le \frac{1}{2655.3468}\,(b-a)^4\, \|x^{(6)}\|. \tag{3.17}$$

For $v \in [\frac{6-\sqrt{6}}{10}, \frac{1}{2}]$, (3.15) reduces to

$$\begin{aligned} |e''(t)| &\le \frac{1}{120}\,(b-a)^4\, v(1-v)\,(-5v^2+5v-1)\, \|x^{(6)}\| \\ &\le \frac{1}{1920}\,(b-a)^4\, \|x^{(6)}\|. \end{aligned} \tag{3.18}$$

Combining (3.16)–(3.18), we find that

$$|e''(t)| \le \frac{1}{1920}\,(b-a)^4\, \|x^{(6)}\|. \tag{3.19}$$

CASE 21. *$n = 2p$, $k = p$, $m = 2p-3$ ($p \ge 3$).*

From (2.6) and Lemma 2.3, we find

$$\begin{aligned} |e^{(2p-3)}(t)| \le& \frac{1}{2p\,(2p-1)\,(2p-2)} \left[\sum_{i=0}^{p-1} \left| \frac{1}{2}\,(2p-1-i)\,(2p-2-i)\,(a-t)^2 \right.\right. \\ &\left. + p\,(2p-2-i)\,(a-t)\,(b-a) + \frac{1}{2}\,p\,(p-1)\,(b-a)^2 \right| (t-a) \\ &\left. + \sum_{i=p}^{2p-3} \frac{(2p-1-i)\,(2p-2-i)}{2}\,(b-t)^3 \right] \|x^{(2p)}\| \\ =& \frac{1}{2p\,(2p-1)\,(2p-2)}\,(b-a)^3 \\ &\times \left[\sum_{i=0}^{p-1} \left| \frac{1}{2}\,(2p-1-i)\,(2p-2-i)\,v^2 - p\,(2p-2-i)\,v + \frac{1}{2}\,p\,(p-1) \right| v \right. \\ &\left. + \frac{1}{6}\,p\,(p-1)\,(p-2)\,(1-v)^3 \right] \|x^{(2p)}\|, \qquad \frac{1}{2} \le v \le 1. \end{aligned} \tag{3.20}$$

For $p = 3$, (3.20) is the same as

$$|e''(t)| \le \frac{1}{120}(b-a)^3 \left[|10v^2 - 12v + 3|\, v + |6v^2 - 9v + 3|\, v + |3v^2 - 6v + 3|\, v + (1-v)^3 \right] \times \|x^{(6)}\|, \quad \frac{1}{2} \le v \le 1$$

$$= \frac{1}{120}(b-a)^3 \left\{ \begin{array}{ll} 1 - 6v + 18v^2 - 14v^3, & \frac{1}{2} \le v \le \frac{6+\sqrt{6}}{10} \\ 1 - 6v^2 + 6v^3, & \frac{6+\sqrt{6}}{10} \le v \le 1 \end{array} \right\} \|x^{(6)}\|$$

$$\le \frac{1}{120}(b-a)^3 \|x^{(6)}\|. \tag{3.21}$$

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