On the measure of arithmetic sums of Cantor sets

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ABSTRACT

It is shown that the arithmetic sum of middle-\(\alpha\) Cantor sets typically has positive Lebesgue measure whenever the sum of Hausdorff dimensions is greater than one.

1. INTRODUCTION AND STATEMENT OF RESULTS

Motivated by some questions in smooth dynamics, Palis and Takens [5] asked whether the sum of two affine Cantor sets can have positive Lebesgue measure but no interior. This problem remains open, even for the classical middle-\(\alpha\) Cantor sets, but it initiated the investigation of such arithmetic sums.

It is more convenient to deal with the similarity ratio \(\lambda = (1 - \alpha)/2\), rather than \(\alpha\). Then the middle-\(\alpha\) Cantor set can be written as

\[
K_\lambda = \left\{ x = (1 - \lambda) \sum_{i=0}^{\infty} a_i \lambda^i : a_i \in \{0, 1\} \right\}, \quad \text{for } 0 < \lambda < \frac{1}{2},
\]

where \(\alpha = 1 - 2\lambda\). We are interested in the properties of arithmetic sums \(K_\gamma + K_\lambda\). The symbol \(\dim_H\) will denote the Hausdorff dimension and \(|\cdot|\) will be the one-dimensional Lebesgue measure. It is well known that \(\dim_H K_\lambda = \log 2/\log(1/\lambda)\). If

\[
\frac{\log 2}{\log(1/\gamma)} + \frac{\log 2}{\log(1/\lambda)} < 1,
\]

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then \( \dim_H(K_\gamma + K_\lambda) \leq \dim_H(K_\gamma) + \dim_H(K_\lambda) < 1 \), so \(|K_\gamma + K_\lambda| = 0\). This set of points \((\gamma, \lambda)\) is indicated as 'small' in fig. 1. On the other hand, if

\[
\frac{\gamma}{1 - 2\gamma} \frac{\lambda}{1 - 2\lambda} > 1,
\]

then by the Gap lemma (see [5]), the set \(K_\gamma + K_\lambda\) contains an interval. This set of points \((\gamma, \lambda)\) is indicated as 'large' in fig. 1, leaving a 'mysterious region' \(R\) where the morphology of the sum is unclear. We prove that for every vertical line, and by symmetry, every horizontal line intersecting \(R\), for almost every point in this intersection, the arithmetic sum has positive Lebesgue measure.

**Theorem 1.** For any \(\gamma \in (0, \frac{1}{2})\), there exists a set \(E_\gamma \subset (0, \frac{1}{2})\) of zero Lebesgue measure such that

\[
\left( \lambda \notin E_\gamma, \frac{\log 2}{\log(1/\gamma)} + \frac{\log 2}{\log(1/\lambda)} > 1 \right) \Rightarrow |K_\gamma + K_\lambda| > 0.
\]

**Remarks.** 1. Theorem 1 can be compared with the result of [5, Theorem 4.2.1] that if \(K_1\) and \(K_2\) are any Cantor sets in \(\mathbb{R}\) with \(\dim_H K_1 + \dim_H K_2 > 1\), then \(|K_1 + \lambda K_2| > 0\) for a.e. \(\lambda \in \mathbb{R}\). In fact, the proofs have some elements in common. However, the important distinction is that in our case the dependence on \(\lambda\) is non-linear while in [5, Theorem 4.2.1] it is linear.

2. One can consider the 'exceptional set' \(\{(\gamma, \lambda) \in R : |K_\gamma + K_\lambda| = 0\}\). All known exceptions lie on the family of curves \(\lambda = \gamma^\theta\) where \(\theta\) is rational.
The case $\theta = 1$ is easy: the sum $K_\gamma + K_\lambda$ is a Cantor set of Hausdorff dimension less than one for $\gamma \in [\frac{1}{4}, \frac{1}{2})$, that is, on the intersection of the diagonal $\lambda = \gamma$ with the region $R$. More generally, if $\lambda = \gamma^{p/q}$ then $K_\gamma + K_\lambda$ is a union of $2^{p+q} - 1$ translated copies of $\gamma^p(K_\gamma + K_\lambda)$, so $\dim_H(K_\gamma + K_\lambda) \leq \log(2^{p+q} - 1)/\log(1/\gamma^p)$. This implies that $|K_\gamma + K_\lambda| = 0$ for the arc of $\lambda = \gamma^{p/q}$ lying in $R$, corresponding to $\gamma \in [2^{-(p+q)/p}, (2^{p+q} - 1)^{-1/p})$. Keane and Smorodinsky [1] investigated the case $\theta = 2$, which led to the study of 'expansions with deleted digits' in [2] and [7]. Results of [2] suggest that there are exceptions on the curves $\lambda = \gamma^{p/q}$ beyond the arcs described above.

3. The topological structure of arithmetic sums of Cantor sets was studied by Mendes and Oliveira [3]. They showed that in the 'large' region always $K_\gamma + K_\lambda = [0, 2]$. For $(\gamma, \lambda) \in R$ the set $K_\gamma + K_\lambda$ can be the interval $[0, 2]$, a Cantor set, or an $M$-Cantorval (defined as a perfect subset of $[0, 1]$ such that any gap is accumulated on both sides by infinitely many intervals and gaps). Further, in [3] it is shown that if $\theta = \log \lambda/\log \gamma$ is irrational, then $K_\gamma + K_\lambda \neq [0, 2]$ for $(\gamma, \lambda) \in R$; if $\theta = n$ or $\theta = (n + 1)/n$, then $K_\gamma + K_\lambda = [0, 2]$ if and only if $\lambda \geq 1 - 2\gamma$. Some of the arcs in $R$ on which the morphology of $K_\gamma + K_\lambda$ is known, are shown in fig. 2.

4. It is not known whether positive Lebesgue measure implies nonempty interior for the sums $K_\gamma + K_\lambda$, or even if this is true generically or typically in $R$. For the much larger, infinite-dimensional family of 'dynamically defined

![Fig. 2.](image)

Fig. 2. $|K_\gamma + K_\lambda| = 0$ in the 'small' region and on arcs attached to it; $K_\gamma + K_\lambda = [0, 2]$ in the 'large' region and on arcs attached to it.
Cantor sets' (see [5] for the definition) the answer, generically, is 'yes', as recently shown by Moreira and Yoccoz [4]. On the other hand, Sannami [8] constructed an example of a dynamically defined Cantor set $C$ such that its self-difference $C - C$ has empty interior but positive Lebesgue measure.

2. PROOF OF THEOREM 1

The scheme of the proof is analogous to [9] which in turn relied on [6, 7]. The main new element here is that we have to deal with a two-parameter family $K + K$.  

Let $\Omega = \{0, 1\}^\mathbb{Z}$ be the sequence space with the product topology. Let $u_0 \cdots u_{k-1}$ be a fixed sequence of 0's and 1's. Sets of the form $W = \{s \in \Omega : s_i = u_i, i = 0, \ldots, k - 1\}$ are called cylinder sets of order $k$. Let $\mu$ be the Bernoulli measure on $\Omega$ which assigns probability $\frac{1}{2}$ to each symbol, so that $\mu W = 2^{-k}$ for a cylinder set of order $k$. Consider the map $\Pi : \Omega \to \mathbb{R}$ given by

$$
\Pi(s) = (1 - \lambda) \sum_{i=0}^{\infty} s_i \lambda^i.
$$

Then $K = \Pi(\Omega)$. The measure

$$
(1) \quad \nu = \mu \circ \Pi^{-1}
$$

is called the Cantor measure. (In fact, $\nu$ is the normalized Hausdorff measure of dimension $\dim_H K$ on the Cantor set. The graph of the function $x \mapsto \nu(0, x]$ is sometimes called the 'devil's staircase'.) The way to prove $|K + K| > 0$ will be to show that the convolution measure $\nu \ast \nu$, supported on $K + K$, is absolutely continuous with respect to the Lebesgue measure.

Fix $\gamma \in (0, \frac{1}{2})$. Let $\lambda_0 \in (0, \frac{1}{2})$ be such that

$$
\log 2 \log 2 \log (1/\gamma) + \log (1/\lambda_0) > 1.
$$

It is enough to show that for any $\lambda_1 \in (\lambda_0, \frac{1}{2})$ the measure $\nu \ast \nu$, supported on $K + K$, is absolutely continuous for $a.e. \lambda \in I = [\lambda_0, \lambda_1]$. For the rest of the proof the numbers $\gamma$, $\lambda_0$, and $\lambda_1$ will be fixed. Choose $\delta$ such that $0 < \delta < 1 - 2\lambda_1$. Then for $N$ sufficiently large we have

$$
(2) \quad 1 + \frac{1}{N} < \frac{1 - \delta}{2\lambda_1}.
$$

For technical reasons which will become apparent in Section 3, it is convenient to restrict the measure $\mu$ to a cylinder set $W$ of order $N$ where $N$ satisfies (2). Fix such a cylinder set and let

$$
(3) \quad \sigma = \mu \big|_W \circ \Pi^{-1}.
$$

Observe that $\nu$ is a sum of $2^N$ translated copies of the measure $\sigma$, so the absolute continuity of $\nu \ast \sigma$ implies the absolute continuity of $\nu \ast \nu$. 

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For a measure \( \eta \) on \( \mathbb{R} \) its Fourier transform is \( \hat{\eta}(p) = \int_{\mathbb{R}} e^{ipx} d\eta(x) \). We are going to show that

\[
\int_{L} \|\nu_\gamma \ast \sigma_\lambda\|_2^2 d\lambda < \infty.
\]

Then \( \nu_\gamma \ast \sigma_\lambda \in L^2(\mathbb{R}) \) for a.e. \( \lambda \in I \), and by Plancherel's theorem, for all such \( \lambda \) the measure \( \nu_\gamma \ast \sigma_\lambda \) is absolutely continuous with a density in \( L^2 \). Clearly, it is enough to check that

\[
(4) \quad \mathcal{J}(p_0) = \int_{L} \int_{-p_0}^{p_0} |(\nu_\gamma \ast \sigma_\lambda)(p)|^2 dp d\lambda < C,
\]

with a constant \( C \) independent of \( p_0 > 0 \).

By the definition of Fourier transform and convolution,

\[
|(\nu_\gamma \ast \sigma_\lambda)(p)|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y)p} d(\nu_\gamma \ast \sigma_\lambda)(x) d(\nu_\gamma \ast \sigma_\lambda)(y)
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(u+v-u'-v')p} d\nu_\gamma(u) d\sigma_\lambda(v) d\nu_\gamma(u') d\sigma_\lambda(v').
\]

Then we make a change of variables \( u = \Pi_\gamma(s), u' = \Pi_\gamma(s'), \) and \( v = \Pi_\lambda(t), v' = \Pi_\lambda(t') \), using (1) and (3):

\[
|(\nu_\gamma \ast \sigma_\lambda)(p)|^2 = \int_{W} \int_{W} \int_{W} e^{i[p_0(\Pi_\gamma(s) + \Pi_\lambda(t) - \Pi_\lambda(t')) - \Pi_\gamma(s') - \Pi_\lambda(t')] \mu(s) d\mu(t) d\mu(s') d\mu(t')}
\]

Substitute this into (4), use Fubini's theorem, and integrate with respect to \( p \) to obtain

\[
(5) \quad \mathcal{J}(p_0) = \int_{W} \int_{W} \int_{W} I_{s,t,s',t'}(p_0) d\mu(t) d\mu(t') d\mu(s) d\mu(s'),
\]

where

\[
I_{s,t,s',t'}(p_0) = \int_{L} \frac{\sin[p_0(\Pi_\gamma(s) - \Pi_\gamma(s') + \Pi_\lambda(t) - \Pi_\lambda(t'))]}{\Pi_\gamma(s) - \Pi_\gamma(s') + \Pi_\lambda(t) - \Pi_\lambda(t')} d\lambda.
\]

The next proposition contains the key estimate; it is proved in Section 3. Let \( |s \wedge s'| = \min\{|i : s_i \neq s_i'|\} \) for \( s, s' \in \Omega \).

**Proposition 2.** If \( |t \wedge t'| \geq N \), where \( N \geq 2 \) satisfies (2), then

\[
|I_{s,t,s',t'}(p_0)| \leq \frac{\text{const}}{\max\{|\gamma^{s \wedge s'}|, \lambda_0^{t \wedge t'}\}}
\]

with a constant independent of \( p_0 \).

Using that \( (\mu \times \mu)\{(s, s') : |s \wedge s'| = k\} < 2^{-k} \), we obtain from (5) and Proposition 2:

\[
(7) \quad \mathcal{J}(p_0) \leq \text{const} \sum_{k=0}^{\infty} \sum_{l=N}^{\infty} \frac{2^{k+l}}{\max\{|\gamma^k, \lambda_0^l|}
\]

Notice that we needed \( t, t' \in W \) to ensure that \( |t \wedge t'| \geq N \).
Lemma 3.
\[
\frac{\log 2}{\log(1/\gamma)} + \frac{\log 2}{\log(1/\lambda_0)} > 1 \Rightarrow S := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{2^{-k-l}}{\max\{\gamma^k, \lambda_0^l\}} < \infty.
\]
The lemma and (7) imply (4), so Theorem 1 will be established. Lemma 3 is an easy exercise but we provide the proof for completeness.

Proof of Lemma 3. Let \( r(k) = k \log \gamma / \log \lambda_0 \), so that \( \gamma^k = \lambda_0^{r(k)} \). We have
\[
S = \sum_{k=0}^{\infty} 2^{-k} \left( \sum_{l< r(k)} \frac{2^{-l}}{\lambda_0^l} + \sum_{l \geq r(k)} \frac{2^{-l}}{\gamma^k} \right)
= \sum_{k=0}^{\infty} 2^{-k} \left( \frac{2\lambda_0^{-r(k)}}{2\lambda_0 - 1} + \frac{2^{-r(k)+1}}{\gamma^k} \right)
\leq \text{const} \sum_{k=0}^{\infty} 2^{-k} (\log \gamma / \log \lambda_0 - k (\log \gamma / \log 2)).
\]
This series converges since \( (\log 2 / \log(1/\gamma)) + (\log 2 / \log(1/\lambda_0)) > 1 \) implies
\[
1 + \frac{\log(1/\gamma)}{\log(1/\lambda_0)} > \frac{\log(1/\gamma)}{\log 2}.
\]
The lemma is proved. □

3. PROOF OF PROPOSITION 2
Let \( k = |s \wedge s'| \) and \( l = |t \wedge t'| \). Denote
\[
\phi(\gamma) = (1 - \gamma) \sum_{i=k}^{\infty} (s_i - s'_i) \gamma^i,
\]
and
\[
\psi(\lambda) = (1 - \lambda) \sum_{i=l}^{\infty} (t_i - t'_i) \lambda^i.
\]
Observe that
\[
\phi(\gamma) + \psi(\lambda) = \Pi_{\gamma}(s) - \Pi_{\gamma}(s') + \Pi_{\lambda}(t) - \Pi_{\lambda}(t').
\]

Lemma 4. If \( l \geq N \) then
\[
|\psi'(\lambda)| > \delta l \lambda^{l-1}, \quad \text{for} \ \lambda \in I,
\]
where \( N \) and \( \delta \) satisfy (2).

Proof of Lemma 4. We have \( \psi(\lambda) = (1 - \lambda) \sum_{i=l}^{\infty} g_i \lambda^i \) where \( g_i = t_i - t'_i \in \{-1, 0, 1\} \). Then
\[|\psi'(\lambda)| = \left| \sum_{i=1}^{\infty} g_i(i\lambda^{i-1} - (i + 1)\lambda^i) \right| \geq |g_l(l\lambda^{l-1} - (l + 1)\lambda^l)| - \sum_{i=l+1}^{\infty} |g_i(i\lambda^{i-1} - (i + 1)\lambda^i)|.\]

Observe that \(i\lambda^{i-1} - (i + 1)\lambda^i > 0\) for \(\lambda < \frac{1}{2}\) and \(i \geq 1\), so, using that \(|g_l| = 1\) and \(|g_i| \leq 1\), we obtain for \(\lambda \in [\lambda_0, \lambda_1]\):

\[|\psi'(\lambda)| \geq l\lambda^{l-1} - (l + 1)\lambda^l - \sum_{i=l+1}^{\infty} (i\lambda^{i-1} - (i + 1)\lambda^i) = l\lambda^{l-1} - 2(l + 1)\lambda^l = \lambda^{l-1}(l - 2(l + 1)\lambda) \geq l\lambda^{l-1}(1 - 2(1 + 1/l)\lambda).\]

Since \(l \geq N\), condition (2) implies (8). \(\square\)

We will also need two simple estimates:

\[(9) \quad |\phi(\gamma)| \geq (1 - \gamma) \left( \gamma^k - \sum_{i=k+1}^{\infty} \gamma^i \right) = \gamma^k(1 - 2\gamma),\]

and

\[(10) \quad |\psi(\lambda)| \leq (1 - \lambda) \sum_{i=1}^{\infty} \lambda^i = \lambda^l.\]

Let \(f(\lambda) = \phi(\gamma) + \psi(\lambda)\). Now we can start estimating the integral (6) which takes the form

\[I_{s.t.s',t'}(p_0) = \int I \frac{\sin[p_0 f(\lambda)]}{f(\lambda)} d\lambda,\]

where \(I = [\lambda_0, \lambda_1]\). We shall split the interval of integration into (in general) three parts: first separate the part where \(|\phi(\gamma)|\) dominates \(|\psi(\lambda)|\), and then split the remaining part according to the size of \(|\phi(\gamma) + \psi(\lambda)|\). Set

\[I_1 = \{ \lambda \in I : \lambda^l < \gamma^k(\frac{1}{2} - \gamma) \}, \quad I_2 = I \setminus I_1,\]

and write

\[I_{s.t.s',t'}(p_0) = \int_{I_1} + \int_{I_2} = \mathcal{I}_1 + \mathcal{I}_2.\]

By (9) and (10),

\[|f(\lambda)| = |\phi(\gamma) + \psi(\lambda)| \geq |\phi(\gamma)| - |\psi(\lambda)| \geq \gamma^k(\frac{1}{2} - \gamma), \quad \text{for } \lambda \in I_1,\]

so \(|\mathcal{I}_1| \leq |I_1| \gamma^{-k}(\frac{1}{2} - \gamma)^{-1}\). However, if \(\lambda_0 > \gamma^k(\frac{1}{2} - \gamma)\), then \(I_1\) is empty, so \(|\mathcal{I}_1| = 0\). Therefore,

\[(11) \quad |\mathcal{I}_1| \leq \max\{\gamma^k(\frac{1}{2} - \gamma), \lambda_0^k\}.\]

By Lemma 4, the function \(f(\lambda) = \phi(\gamma) + \psi(\lambda)\) is monotone on \(I\). Recall that \(l \geq N \geq 2\), let
\[ \mathcal{F} = \{ \lambda \in I_2 : |f(\lambda)| \leq \lambda^l(l-1)^{-1} \}, \]
and set
\[ a = \inf \mathcal{F}, \quad b = \sup \mathcal{F}. \]

Further, let
\[ I_{21} = [a, b], \quad I_{22} = I_2 \setminus I_{21}, \]
and split the domain of integration accordingly
\[ \mathcal{I}_2 = \int_{I_{21}} + \int_{I_{22}} = \mathcal{I}_{21} + \mathcal{I}_{22}. \]

Since \(|f(\lambda)| > \lambda^l(l-1)^{-1}\) for \(\lambda \in I_{22},\)
\[ |\mathcal{I}_{22}| < \int_{I_{22}} (l-1)\lambda^{-l}d\lambda < \tau^{1-l} < \tau^{-l}, \]
where \(\tau = \inf I_{22}.\) But \(I_{22} \subset I_2 \subset [\lambda_0, \lambda_1],\) hence \(\tau^l \geq \max\{\gamma^k(\frac{1}{2} - \gamma), \lambda_0 \}.\) Thus,
\[ |\mathcal{I}_{22}| < \frac{1}{\max\{\gamma^k(\frac{1}{2} - \gamma), \lambda_0 \}}. \]

It remains to estimate \(\mathcal{I}_{21}.\) We are going to apply the following result from [6]: if \(f(x)\) is \(C^2\)-smooth and monotone on \([a, b]\) then
\[ \left| \int_a^b \frac{\sin[p_0 f(\lambda)]}{f(\lambda)} d\lambda \right| \leq \text{const} \left[ \frac{1}{\inf_{[a,b]} |f''(\lambda)|} + \frac{\sup_{[a,b]} |f''''(\lambda)||f(b) - f(a)|}{(\inf_{[a,b]} |f''(\lambda)|)^3} \right]. \]

We have \([a, b] = I_{21}.\) By Lemma 4,
\[ |f'(\lambda)| = |\psi'(\lambda)| > \delta l\lambda^{l-1} \geq \delta l a^{l-1}, \quad \text{for } \lambda \in [a, b]. \]

By an obvious estimate,
\[ |f''(\lambda)| = |\psi''(\lambda)| < c_1 l^2 \lambda^l < c_1 l^2 b^l, \quad \text{for } \lambda \in [a, b], \]
where the constant \(c_1\) depends on \(J\) only. By the definition of \([a, b] = I_{21},\)
\[ |f(b) - f(a)| \leq b^l/(l-1) + a^l/(l-1) < 2b^l/(l-1). \]

Applying (13) we obtain
\[ |\mathcal{I}_{21}| \leq \text{const} \left[ \frac{1}{\delta l a^{l-1}} + \frac{c_1 l^2 b^l(2b^l/(l-1))}{\delta^3 l^3 a^{3l-3}} \right]. \]

From (14) and (15) it follows that
\[ \frac{2b^l}{l-1} > |f(b) - f(a)| = \int_a^b |f'(\lambda)| d\lambda \geq \delta \int_a^b l\lambda^{l-1} d\lambda = \delta(b^l - a^l). \]

Therefore,
\[ b^l \left(1 - \frac{2}{\delta(l-1)}\right) < a^l. \]

For \(l \geq 1 + 4/\delta\) this implies \(b^l < 2a^l.\) Since \(a \geq \lambda_0,\) it follows that for all \(l,\)
with a constant $c_2$ depending on $\lambda_0$. Combining this with (16), we obtain $|I_{21}| \leq c_3 a^{-l}$ for some $c_3 = c_3(l)$. Since $a \in I_2$, we have $a^{-l} \geq \max\{\gamma^k(\frac{1}{2} - \gamma), \lambda_0^l\}$, so

$$|I_{21}| \leq \frac{1}{c_3} \max\{\gamma^k(\frac{1}{2} - \gamma), \lambda_0^l\}.$$ 

Together with (11) and (12) this concludes the proof of Proposition 2. \square

**Added in proof:** Theorem 1 has been generalized by Y. Peres and the author to the case of sums $K + C_\lambda$ where $K$ is any compact subset of $\mathbb{R}$ and $C_\lambda$ is a family of homogeneous Cantor sets satisfying the open set condition.

**REFERENCES**