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On Subordinacy and Analysis of the Spectrum of One-Dimensional Schrödinger Operators

D. J. GILBERT AND D. B. PEARSON

Department of Applied Mathematics,
University of Hull, HU6 7RX, England.

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By considering the behaviour as $N \to \infty$ of the ratio of $L_2[0, N]$ norms of solutions of $-d^2u/dr^2 + V(r)\,u = xu$, $0 \leqslant r < \infty$, $x \in \mathbb{R}$, a characterisation of the absolutely continuous and singular spectra of one-dimensional Schrödinger operators is deduced. The analysis is applicable to all operators for which $L = -d^2/dr^2 + V(r)$ is regular at 0 and in the limit point case at infinity, with $V(r)$ locally integrable. © 1987 Academic Press, Inc.

1. INTRODUCTION

The theory of the subordinacy presented in this paper enables a detailed and rigorous characterisation of the spectrum of one-dimensional Schrödinger operators on $[0, \infty)$ to be established. It is shown that when 0 is a regular endpoint, the absolutely continuous, singular continuous, and pure point spectra may be identified in terms of the relative asymptotic behaviour of certain linearly independent solutions of the corresponding Schrödinger equation

$$Lu = xu$$

$$L = -\frac{d^2}{dr^2} + V(r) \qquad 0 \leqslant r < \infty \tag{1.1}$$

at each real point x. This provides a new and systematic method of spectral analysis for Schrödinger operators which is advantageous in several respects. In the first place, the theory is independent of detailed properties of the potential $V(r)$; only very general requirements, as, for example, that $V(r)$ be locally integrable and that L be in the limit point case at infinity, need to be met. Moreover, the information which is needed to apply the

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theory to particular cases should be comparatively easy to obtain. It is only necessary to determine for each **real** x whether a solution of $Lu = xu$ exists which is "smaller" than all other linearly independent solutions at infinity, and if so, whether this solution satisfies the boundary condition at 0. The usefulness, in practice, of a method which characterises the spectrum in terms of the behaviour of solutions of the Schrödinger equation is already well known to physicists, who customarily identify the spectrum with the set of real x for which the solution of $Lu = xu$ satisfying the boundary condition at 0 is bounded [1, Chap. 10, Section 16; 2, pp. 71, 82]. This characterisation of the spectrum is not, in general, correct (see, e.g., [3, p. 648]); however, the general belief on which it is based, viz. that a correlation exists between the behaviour of solutions and the spectrum, is corroborated by the results of this paper. Finally, it seems likely that considerable generalisation of the theory is possible, and extension to a wide class of Schrödinger operators where 0 may be a singular endpoint will be considered in a subsequent paper.

After summarising some relevant results from existing theory in Section 2, we explain what is meant by a minimal support of a measure and describe a decomposition of the spectrum in terms of minimal supports of the absolutely continuous, singular continuous, and pure point parts of the spectral measure μ and a generalised derivative $d\mu/d\kappa$. We then indicate how an analogous decomposition of the spectrum, featuring the boundary behaviour of an analytic function $m(z)$, may be derived using a known relationship between $m(z)$ and the spectral function $\rho(\lambda)$ (cf. [4, Section 2]). This material is treated more fully in [5, Chap. II].

Analysis of the spectrum using the boundary behaviour of analytic functions was extensively studied by Titchmarsh, and successfully applied to a number of standard cases [6, Chap. V]. In theory this approach provides the solution to the problem; from prior knowledge of the potential and the boundary condition at 0, the function $m(z)$ can be obtained from properties of solutions of $Lu = zu$ for z in $\mathbb{C} \setminus \mathbb{R}$. In practice, the method is frequently inoperable owing to the difficulty of obtaining sufficiently detailed information about the solutions to derive $m(z)$ explicitly.

We shall use the systematic correlation between the boundary behaviour of $m(z)$ and the spectrum to obtain an equally systematic correlation between the asymptotic behaviour of solutions of the Schrödinger equation at real energies and the spectrum and, in so doing, obviate the need to determine $m(z)$ explicitly. Only relatively crude information about the asymptotic behaviour of solutions of $Lu = xu$ for x in $\mathbb{R}$ will then be required to obtain a complete analysis of the spectrum.

In Section 3 we introduce the idea of subordinacy and derive some straightforward continuity properties of the $L_2[0, N]$ norms of solutions of $Lu = zu$ as z varies in $\mathbb{C}$. In Sections 4 and 5 we deduce necessary and sufficient conditions for the existence of subordinate solutions of $Lu = xu$ in terms of the boundary behaviour of $m(z)$ as z approaches the real axis at x normally. These necessary and sufficient conditions enable a complete description of the spectrum in terms of the existence or otherwise of subordinate solutions to be established in Section 6. To illustrate the force of these ideas, we conclude by showing that some well-known results in spectral theory and analysis are immediate consequences of our theory; we hope to apply the theory to some hitherto unsolved problems in a subsequent publication.

2. Analysis of the Spectrum Using Boundary Properties of Analytic Functions

We are concerned with the spectral analysis of self-adjoint operators arising from the differential expression (1.1), where $V(r)$ is integrable on every finite subinterval of $[0, \infty)$. L is then a differential expression of the Sturm–Liouville type with a regular endpoint at 0. According to the well known "Weyl Alternative" [7], L must satisfy one or the other of the following conditions:

(a) for each z in $\mathbb{C}$, every solution of $Lu = zu$ is in $L_2\,[0, \infty)$; in this case L is said to be in the **limit circle case** at infinity.

(b) for each z in $\mathbb{C}$, no more than one linearly independent solution of $Lu = zu$ is in $L_2[0, \infty)$; in this case L is said to be in the **limit point case** at infinity and for each z in $\mathbb{C}\setminus\mathbb{R}$, precisely one solution is in $L_2[0, \infty)$.

We shall only consider the second alternative, since in almost all cases of physical and mathematical interest L is of the limit point type.

Let an operator $H(\alpha)$ be defined by

$$Hu = Lu \qquad \text{for} \quad u \in \mathscr{D}$$

where $u \in \mathscr{D}$ if

(i) u, $Lu \in L_2\,[0, \infty)$

(ii) u, u' are absolutely continuous in every closed subinterval of $[0, \infty)$.

(iii) $\cos \alpha u(0) + \sin \alpha u'(0) = 0$ for some fixed α in $[0, \pi)$.

Then if L is regular at 0 and in the limit point case at infinity, $H(\alpha)$ is self-adjoint for each α in $[0, \pi)$ [8, Chap. X, Section 3].

Associated with each such operator $H(\alpha)$, there exists a monotonically increasing spectral function $\rho_\alpha(\lambda)$ which is unique up to an additive constant [9, Chap. 9, Section 3]. The spectrum of $H(\alpha)$ is the complement of the set of points in a neighbourhood of which $\rho_\alpha(\lambda)$ is constant [10, Chap. 4], and the unique decomposition of $\rho_\alpha(\lambda)$ into absolutely continuous, singular continuous, and pure point parts [11, Theorem 19.61] enables the absolutely continuous, singular continuous, and point spectra to be defined similarly. It may be shown that these definitions are in agreement with the more usual descriptions in terms of resolvent operators [5, p. 56].

Intimately related to the spectral function $\rho_\alpha(\lambda)$ is a complex valued function $m(z, \alpha)$ which is defined and analytic in $\mathbb{C}\backslash\mathbb{R}$ and has a positive imaginary part in the upper half plane. $m(z, \alpha)$ may be defined for all z in $\mathbb{C}\backslash\mathbb{R}$ by the condition

$$\int_0^\infty |u_m(r, z, \alpha)|^2 \, dr < \infty, \tag{2.1}$$

where $u_m(r, z, \alpha) = u_2(r, z, \alpha) + m(z, \alpha)\, u_1(r, z, \alpha)$, and $u_1(r, z, \alpha)$ and $u_2(r, z, \alpha)$ are those solutions of $Lu = zu$ which satisfy

$$\begin{aligned} u_1(0, z, \alpha) &= -\sin\alpha \qquad & u_2(0, z, \alpha) &= \cos\alpha \\ u_1'(0, z, \alpha) &= \cos\alpha \qquad & u_2'(0, z, \alpha) &= \sin\alpha. \end{aligned} \tag{2.2}$$

If $\|\cdot\|$ denotes the $L_2[0, \infty)$ norm, $u_m(r, z, \alpha)$ also satisfies the relationship

$$\|u_m(r, z, \alpha)\|^2 = \frac{\operatorname{Im} m(z, \alpha)}{\operatorname{Im} z} \tag{2.3}$$

for all z in $\mathbb{C}\backslash\mathbb{R}$ [9, Chap. 9, Section 2]. If $\alpha_1, \alpha_2 \in [0, \pi)$ are two distinct boundary conditions, then the corresponding functions $m(z, \alpha_1)$ and $m(z, \alpha_2)$ are related by

$$m(z, \alpha_2) = \frac{1 + \cot\gamma \; m(z, \alpha_1)}{\cot\gamma - m(z, \alpha_1)}, \tag{2.4}$$

where $\gamma = (\alpha_1 - \alpha_2)$ [9, Chap. 9, Problem 8]. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Formulae connecting $\rho_\alpha(\lambda)$ and $m(z, \alpha)$ are

$$\rho_\alpha(v) - \rho_\alpha(\mu) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_\mu^v \operatorname{Im} m(x + iy, \alpha) \, dx \tag{2.5}$$

for all $\mu, v \in \mathbb{R}$ which are points of continuity of $\rho_\alpha(\lambda)$ [9, Chap. 9, Theorem 3.1], and, inversely,

$$m(z, \alpha) = \int_{-\infty}^\infty \frac{1}{(\lambda - z)} \, d\rho_\alpha(\lambda) + \cot\alpha \tag{2.6}$$

which holds for all $\alpha \in (0, \pi)$; if $\alpha = 0$, a slightly different formulation of (2.6) is required [12, Section 2.3].

Where there is no ambiguity, we shall henceforth omit the α-dependence of $\rho(\lambda)$, $m(z)$, $u_1(r, z)$, $u_2(r, z)$, and $u_m(r, z)$. Let μ denote the Borel–Stieltjes measure generated by $\rho(\lambda)$, let κ denote Lebesgue measure and define

$$\frac{d\mu}{d\kappa}(x) = \lim_{\kappa(I_x) \to 0} \left\{ \frac{\mu(I_x)}{\kappa(I_x)} : I_x \text{ is an interval of } \mathbb{R} \text{ containing } x \right\}$$

whenever this limit exists. We shall briefly summarise some relevant properties of $m(z)$ and $(d\mu/d\kappa)(x)$. $m(z)$ is said to have normal limit at the point $x \in \mathbb{R}$ if $m(z)$ converges to a finite limit or to infinity as z approaches x from above along the normal to the real axis at x. The following result may be deduced from some theorems of Plessner [13, Sätze I, IV] using properties of conformal mappings.

LEMMA 1. *$m(z)$ has a finite normal limit Lebesgue almost everywhere on $\mathbb{R}$.*

The next lemma is essentially contained in [14, Chap. IV, Theorem 9.1].

LEMMA 2. *Let S denote*

$$\{x \in \mathbb{R} : (d\mu/d\kappa)(x) \text{ does not exist finitely or infinitely}\}$$

Then $\mu(S) = \kappa(S) = 0$.

Let $m_+(x)$ denote $\lim_{y \downarrow 0} m(x+iy)$ for each x in $\mathbb{R}$ for which this normal limit exists, and let $\operatorname{Im} m_+(x)$ be defined similarly. Using integration by parts on the imaginary part of (2.6), it may be shown that

LEMMA 3. *If $(d\mu/d\kappa)(x)$ exists finitely or infinitely, then* $\operatorname{Im} m_+(x)$ *also exists and* $(d\mu/d\kappa)(x) = (1/\pi) \operatorname{Im} m_+(x)$.

Lemma 3 is a variant of the well-known Fatou theorem (see, e.g., [15, Chap. IV; 16, Section 3].

Our analysis of the spectrum will be in terms of minimal supports of the spectral measure μ and of its absolutely continuous, singular continuous, and pure point parts which we denote by $\mu_{\text{a.c.}}$, $\mu_{\text{s.c.}}$, and $\mu_{\text{p.}}$, respectively. Minimal supports of a Borel–Stieltjes measure ι on $\mathbb{R}$ are defined as follows:

DEFINITION 1. A subset S of $\mathbb{R}$ is said to be a minimal support of a measure ι on $\mathbb{R}$ if the following conditions are satisfied:

(i) $\iota(\mathbb{R} \setminus S) = 0$

(ii) If S_0 is a subset of S with $\iota(S_0) = 0$, then $\kappa(S_0) = 0$.

A minimal support of a measure ι gives an indication of where the measure is concentrated and is uniquely determined up to sets of κ- and ι-measure zero. Indeed, it may be shown that the set of all minimal supports of a given measure ι is an equivalence class under $\sim$, where if $S \triangle S'$ denotes the symmetric difference $(S \backslash S') \cup (S' \backslash S)$, the relation $\sim$ is defined by

$$S \sim S' \Leftrightarrow \kappa(S \triangle S') = \iota(S \triangle S') = 0 \tag{2.8}$$

for $S, S' \subseteq \mathbb{R}$ [5, Lemma 2.20].

The following classification of the spectrum in terms of minimal supports may be deduced using de la Vallée-Poussin's decomposition theorem [14, Chap. IV, Theorem 9.6], the Lebesgue–Radon–Nikodym theorem [17, Theorem 6.9], and Lemma 2:

LEMMA 4. *Minimal supports $\mathcal{M}$, $\mathcal{M}_{\text{a.c.}}$, $\mathcal{M}_{\text{s.}}$, $\mathcal{M}_{\text{s.c.}}$, and $\mathcal{M}_{\text{p.}}$ of μ, $\mu_{\text{a.c.}}$, $\mu_{\text{s.}}$, $\mu_{\text{s.c.}}$, and $\mu_{\text{p.}}$, are as follows, where $E = \{x \in \mathbb{R}: (d\mu/d\kappa)(x)$ exists$\}$:*

(i) $\mathcal{M} = \{x \in E: 0 < (d\mu/d\kappa)(x) \leqslant \infty\}$

(ii) $\mathcal{M}_{\text{a.c.}} = \{x \in E: 0 < (d\mu/d\kappa)(x) < \infty\}$

(iii) $\mathcal{M}_{\text{s.}} = \{x \in E: (d\mu/d\kappa)(x) = \infty\}$

(iv) $\mathcal{M}_{\text{s.c.}} = \{x \in E: (d\mu/d\kappa)(x) = \infty, \mu(x) = 0\}$

(v) $\mathcal{M}_{\text{p.}} = \{x \in E: (d\mu/d\kappa)(x) = \infty, \mu(x) > 0\}$.

Let $\mathcal{M}'$ denote $\{x \in E': 0 < \operatorname{Im} m_+(x) \leqslant \infty\}$, where $E' = \{x \in \mathbb{R}: \operatorname{Im} m_+(x)$ exists$\}$. Then $\mathcal{M} \subseteq \mathcal{M}'$ by Lemma 3. Moreover, if

$$U = \{x \in \mathbb{R}: \operatorname{Im} m_+(x) \text{ exists finitely or infinitely, but } (d\mu/d\kappa)(x) \text{ does not}\}$$

then $\mathcal{M}' \backslash \mathcal{M} \subseteq U \subseteq S$, where S is defined as in Lemma 2. Since $\kappa(S) = \mu(S) = 0$, $\mathcal{M}' \sim \mathcal{M}$ so that $\mathcal{M}'$ is a minimal support of μ. Analogous results for $\mu_{\text{a.c.}}$, $\mu_{\text{s.}}$, etc. follow in the same way, and we have

PROPOSITION 1. *Minimal supports $\mathcal{M}'$, $\mathcal{M}'_{\text{a.c.}}$, $\mathcal{M}'_{\text{s.}}$, $\mathcal{M}'_{\text{s.c.}}$ and $\mathcal{M}'_{\text{p.}}$ of μ, $\mu_{\text{a.c.}}$, $\mu_{\text{s.}}$, $\mu_{\text{s.c.}}$, and $\mu_{\text{p.}}$ are as follows, where $E' = \{x \in \mathbb{R}: \operatorname{Im} m_+(x)$ exists$\}$:*

(i) $\mathcal{M}' = \{x \in E': 0 < \operatorname{Im} m_+(x) \leqslant \infty\}$

(ii) $\mathcal{M}'_{\text{a.c.}} = \{x \in E': 0 < \operatorname{Im} m_+(x) < \infty\}$

(iii) $\mathcal{M}'_{\text{s.}} = \{x \in E': \operatorname{Im} m_+(x) = \infty\}$

(iv) $\mathcal{M}'_{\text{s.c.}} = \{x \in E': \operatorname{Im} m_+(x) = \infty, \mu(x) = 0\}$

(v) $\mathcal{M}'_{\text{p.}} = \{x \in E': \operatorname{Im} m_+(x) = \infty, \mu(x) > 0\}$.

Conventionally, spectral analysis is concerned with identification of the spectrum rather than with minimal supports of the spectral measure. The relationship between the set of minimal supports and the spectrum is to some extent clarified by the following:

LEMMA 5. *Let $\sigma(H)$, μ denote the spectrum and the spectral measure respectively of a Schrödinger operator H. Then there exists a minimal support $\mathcal{M}_\mu$ of μ such that* $\mathrm{cl}(\mathcal{M}_\mu)=\sigma(H)$.

Proof. Let $\mathcal{M}(H)$ be a minimal support of μ and define $\mathcal{M}_\mu=\mathcal{M}(H)\cap\sigma(H)$. Since $\mathcal{M}_\mu\subseteq\mathcal{M}(H)$ and $\mu(\mathcal{M}(H)\backslash\mathcal{M}_\mu)=0$ (see, e.g., [14, Chap. IV, Lemma 9.4(i)]), $\mathcal{M}_\mu$ is also a minimal support of μ by Definition 1. Moreover, $\mathcal{M}_\mu\subseteq\sigma(H)$ and $\sigma(H)$ is a closed set, so $\mathrm{cl}(\mathcal{M}_\mu)\subseteq\sigma(H)$.

Now consider $\mathscr{C}(\mathrm{cl}(\mathcal{M}_\mu))=\mathbb{R}\backslash\mathrm{cl}(\mathcal{M}_\mu)$. Clearly $\mathscr{C}(\mathrm{cl}(\mathcal{M}_\mu))$ is open, and, since $\mathcal{M}_\mu$ is a minimal support of μ, $\mu(\mathscr{C}\,\mathrm{cl}(\mathcal{M}_\mu))\leqslant\mu(\mathbb{R}\backslash\mathcal{M}_\mu)=0$ by Definition 1. It follows that $\rho(\lambda)$ is constant on $\mathscr{C}(\mathrm{cl}(\mathcal{M}_\mu))$, so that $\mathrm{cl}(\mathcal{M}_\mu)\supseteq\sigma(H)$ by the definition of the spectrum. This completes the proof of the lemma.

Analogous results hold with respect to the absolutely continuous and singular spectra of H, which are closed sets. However, it should be noted that, in general, the sets $\mathcal{M}_{\text{a.c.}}$, $\mathcal{M}'_{\text{a.c.}}$, and $\mathcal{M}_{\text{s.}}$, $\mathcal{M}'_{\text{s.}}$, of Lemma 4 and Proposition 1 are not contained in the respective spectra [5, Exs. 2.10, 2.11].

Let $\mu^{(\alpha)}_{\text{a.c.}}$, $\mu^{(\alpha)}_{\text{s.}}$ denote the absolutely continuous and singular parts of the Borel–Stieltjes measure generated by $\rho_\alpha(\lambda)$, and let $E_{\text{a.c.}}(\alpha)$, $E_{\text{s.}}(\alpha)$ denote the equivalence classes of minimal supports of $\mu^{(\alpha)}_{\text{a.c.}}$ and $\mu^{(\alpha)}_{\text{s.}}$, respectively. The striking contrast between the behaviour of $E_{\text{a.c.}}(\alpha)$ and $E_{\text{s.}}(\alpha)$ as α varies is demonstrated in the following lemma.

LEMMA 6. (i) $E_{\text{a.c.}}(\alpha_1)=E_{\text{a.c.}}(\alpha_2)$ *for all boundary conditions* α_1 *and* α_2.

(ii) *If* $\mu^{(\alpha_1)}_{\text{s.}}(\mathbb{R})$, $\mu^{(\alpha_2)}_{\text{s.}}(\mathbb{R})>0$ *then* $E_{\text{s.}}(\alpha_1)\neq E_{\text{s.}}(\alpha_2)$ *for any* $\alpha_2\neq\alpha_1$ $(\mathrm{mod}\ \pi)$; *moreover, for each pair of distinct boundary conditions* $\{\alpha_1,\alpha_2\}$ *there exist* $\mathcal{M}(\alpha_1)\in E_{\text{s.}}(\alpha_1)$ *and* $\mathcal{M}(\alpha_2)\in E_{\text{s.}}(\alpha_2)$ *such that* $\mathcal{M}(\alpha_1)\cap\mathcal{M}(\alpha_2)=\varnothing$.

Proof. From (2.4)

$$\operatorname{Im} m(z,\alpha_2)=\frac{\operatorname{Im} m(z,\alpha_1)(1+\cot^2\gamma)}{(\cot\gamma-\operatorname{Re} m(z,\alpha_1))^2+(\operatorname{Im} m(z,\alpha_1))^2},\tag{2.9}$$

where $\gamma=(\alpha_1-\alpha_2)$. Using the notation of Proposition 1 we have

(i) $\kappa(\mathcal{M}'_{\text{a.c.}}(\alpha_1)\triangle\mathcal{M}'_{\text{a.c.}}(\alpha_2))=0$ by (2.9) and Lemma 1.

Since $\mu_{\text{a.c.}}^{(\alpha_1)}$ and $\mu_{\text{a.c.}}^{(\alpha_2)}$ are absolutely continuous with respect to Lebesgue measure, it follows that $E_{\text{a.c.}}(\alpha_1) = E_{\text{a.c.}}(\alpha_2)$.

(ii) $$\mathcal{M}'_{\text{s.}}(\alpha_1) \cap \mathcal{M}'_{\text{s.}}(\alpha_2) = \varnothing \tag{2.10}$$

by (2.9) where $\mathcal{M}'_{\text{s.}}(\alpha_1) \in E_{\text{s.}}(\alpha_1)$, $\mathcal{M}'_{\text{s.}}(\alpha_2) \in E_{\text{s.}}(\alpha_2)$. It follows from (2.10) and Definition 1 that

$$\mu_{\text{s.}}^{(\alpha_2)}(\mathcal{M}'_{\text{s.}}(\alpha_1)) = \mu_{\text{s.}}^{(\alpha_1)}(\mathcal{M}'_{\text{s.}}(\alpha_2)) = 0$$

so if $\mu_{\text{s.}}^{(\alpha_1)}(\mathbb{R})$, $\mu_{\text{s.}}^{(\alpha_2)}(\mathbb{R}) > 0$, then $\mathcal{M}'_{\text{s.}}(\alpha_1) \notin E_{\text{s.}}(\alpha_2)$ and $\mathcal{M}'_{\text{s.}}(\alpha_2) \notin E_{\text{s.}}(\alpha_1)$. This proves all our assertions and completes the lemma.

We deduce from Lemma 6 that, whereas the absolutely continuous parts of the respective spectral measures are equivalent under two distinct boundary conditions, the singular parts are orthogonal; this fact has already been noted by Aronszajn [4, Theorem 1]. In Sections 3 and 4 we shall relate the boundary behaviour of Im $m(z)$ as z approaches $x \in \mathbb{R}$ normally to the nature of solutions of $Lu = xu$. The crucial distinction will be between those x for which Im $m_+(x, \alpha)$ exists finitely and is nonzero for all α, and those x for which there is a boundary condition α_1 such that Im $m_+(x, \alpha_1)$ exists and is zero; with this in mind, we deduce the following corollary to Lemma 6. Since $E_{\text{a.c.}}(\alpha)$ is independent of α, it will be referred to simply as $E_{\text{a.c.}}$.

COROLLARY 1. *The set* $S = \{x \in \mathbb{R}$: *there is no boundary condition* α *for which* Im $m_+(x, \alpha)$ *exists and equals zero*$\}$ *is in* $E_{\text{a.c.}}$.

Proof. By Lemma 6, it suffices to prove that S is a minimal support of $\mu_{\text{a.c.}}(\alpha_1)$ for some fixed boundary condition α_1; this will be achieved if we show that $S \sim \mathcal{M}'_{\text{a.c.}}(\alpha_1)$, where $\sim$ is the equivalence relation of (2.8) with $\iota = \mu_{\text{a.c.}}^{(\alpha_1)}$.

Now for Lebesgue almost all x in $\mathcal{M}'_{\text{a.c.}}(\alpha_1)$, Im $m_+(x, \alpha_2)$ exists finitely and is strictly greater than zero for every $\alpha_2 \neq \alpha_1 \pmod{\pi}$ by Lemma 1 and (2.9). Hence

$$\kappa(\mathcal{M}'_{\text{a.c.}}(\alpha_1) \backslash S) = 0. \tag{2.11}$$

Moreover, by Lemma 1 and the definition of S, $0 < \text{Im } m_+(x, \alpha_1) < \infty$ Lebesgue almost everywhere on S. That is,

$$\kappa(S \backslash \mathcal{M}'_{\text{a.c.}}(\alpha_1)) = 0. \tag{2.12}$$

Since $\mu_{\text{a.c.}}^{(\alpha_1)}$ is absolutely continuous with respect to κ, (2.11) and (2.12) together imply that $S \sim \mathcal{M}'_{\text{a.c.}}(\alpha_1)$, which proves the corollary.

3. The Concept of a Subordinate Solution

A consideration of the relative size of solutions of $Lu = xu$ at infinity is more problematic when solutions are oscillatory. The definition of subordinacy which is introduced in this section avoids the drawbacks of a pointwise comparison of solutions and enables the boundary behaviour of $m(z)$ at each x in $\mathbb{R}$ to be related to the relative asymptotic size of certain linearly independent solutions of $Lu = xu$.

A solution $u(r, x)$ of $Lu = xu$ is said to be oscillatory on $[0, \infty)$ if for every $R \in \mathbb{R}^+$ there exists $R_0 > R$ such that $u(r, x)$ vanishes at $r = R_0$; solutions which are not oscillatory are said to be nonoscillatory. By Sturm's separation theorem, whenever one solution of $Lu = xu$ is oscillatory, every solution of $Lu = xu$ is oscillatory [18, Chap. XI, Corollary 3.1]; as appropriate, therefore, we may refer to $Lu = xu$ as being oscillatory or nonoscillatory at x. A simple consequence of Sturm's comparison theorem [18, Chap. XI, Theorem 3.1] is that $p \in \mathbb{R}$ always exists such that $Lu = xu$ is nonoscillatory for $x < p$ and oscillatory for $x > p$, where the possibility that $p = \pm\infty$ is not excluded; p is known as the parabolic point [3, p. 637]. A further consequence of Sturm's comparison theorem is that $Lu = xu$ is always nonoscillatory for $x < \liminf V(r)$, and oscillatory for $x > \limsup V(r)$. Hence p must lie in the interval $[\liminf V(r), \limsup V(r)]$, its precise location depending on properties of $V(r)$.

The relationship of solutions to the spectrum has for some time been more clearly understood where solutions of $Lu = xu$ are nonoscillatory. If $x < p$, then there always exists an $L_2[0, \infty)$ solution [3, (v)]; if, in addition $V(r)$ is semi-bounded, the $L_2[0, \infty)$ solution converges to zero as $r \to \infty$ [19, Chap. V, Theorem 2; 20, Section 2]. Since $Lu = xu$ is oscillatory at x if and only if the restriction of the spectrum to $(-\infty, x)$ is an infinite set [19, Chap. I, Theorem 31], the spectrum on $(-\infty, p)$ consists of isolated eigenvalues with, possibly, an accumulation point at p. The eigenvalues corresponding to distinct boundary conditions interlace [21], with every point in $(-\infty, p)$ being an eigenvalue for some boundary condition α [22]. An interesting result due to Hartman and Wintner is that if $Lu = xu$ is nonoscillatory, then there always exists a principal solution $u(r, x)$ which is unique up to scalar multiples and for which

$$\lim_{r \to \infty} \frac{u(r, x)}{v(r, x)} = 0 \tag{3.1}$$

for each linearly independent solution $v(r, x)$ [18, Chap. XI, Theorem 6.4]; note that $u(r, x)$ need not be bounded as $r \to \infty$ unless $V(r)$ is semi-bounded [3, (v)]. The restriction of the spectrum to $(-\infty, p)$ may therefore be identified as

$$\{x \in (-\infty, p)\text{: there exists an } L_2[0, \infty) \text{ solution of } Lu = xu \text{ satisfying the boundary condition at } 0\} \tag{3.2}$$

or equivalently as

$$\{x \in (-\infty, p)\text{: there exists a principal solution of } Lu = xu \text{ satisfying the boundary condition at } 0\} \tag{3.3}$$

and its nature is isolated pure point.

No such simple generalisations have been available for that part of the spectrum where solutions of the corresponding Schrödinger equation are oscillatory. Possible types of spectrum on (p, ∞) include absolutely continuous, singular continuous, dense point, isolated point, and nowhere dense essential spectra; these can occur in almost every conceivable combination, at least on finite subintervals. That such spectra can arise from Schrödinger operators follows from the inverse method of Gel'fand and Levitan [23, see especially p. 257]; for some interesting cases see [4, Section 5]. Some connections between the behaviour of solutions and the spectrum on $[p, \infty)$ have been noted; for example, a sufficient, but not necessary condition for x to lie in the essential spectrum is the absence of an $L_2[0, \infty)$ solution of $Lu = xu$ [24, Korollar 3.2, Anhang zu Teil 3]. However, if such relationships as are already known are insufficient to identify the essential spectrum on $[p, \infty)$, still less do they discriminate between its constituent parts.

The definition of a subordinate solution given below extends the idea of a principal solution in such a way that it can have meaning where solutions are oscillatory and even where no solutions are in $L_2[0, \infty)$. This will enable us to deduce a complete classification of the spectrum on $\mathbb{R}$ (or, more precisely, of minimal supports of μ, $\mu_{\text{a.c.}}$, $\mu_{\text{s.}}$, $\mu_{\text{s.c.}}$, and $\mu_{\text{p.}}$ on $\mathbb{R}$) in terms of the existence or otherwise of certain types of solutions of $Lu = xu$ at each point x.

Let $\|f(r)\|_N$ denote $(\int_0^N |f(r)|^2\, dr)^{1/2}$.

DEFINITION 2. If L is regular at 0 and limit point at infinity, then a solution $u_s(r, z)$ of $Lu = zu$ is said to be subordinate at infinity if for every linearly independent solution $u(r, z)$

$$\lim_{N \to \infty} \frac{\|u_s(r, z)\|_N}{\|u(r, z)\|_N} = 0. \tag{3.4}$$

It is trivial to observe that for each fixed z in $\mathbb{C}$ there can be no more than one linearly independent solution of $Lu = zu$ which is subordinate at infinity. Moreover, if for fixed z in $\mathbb{C}$ there exist solutions $u_s(r, z)$ and $u(r, z)$ which satisfy (3.4), then

$$\lim_{N\to\infty} \frac{\|u_s(r, z)\|_N}{\|u_t(r, z)\|_N} = 0$$

for every solution $u_t(r, z)$ which is not a scalar multiple of $u_s(r, z)$.

We remark that, when $x \in \mathbb{R}$ and $Lu = xu$ is nonoscillatory, Definition 2 and (3.1) identify the same sets of solutions. The reason the $L_2[0, \infty)$ norm has been chosen in the definition is that the formula (2.3) will be used to establish a relationship between the boundary behaviour of $m(z)$ as $z \downarrow x$ normally and the existence or otherwise of a subordinate solution of $Lu = xu$. Since the continuity behaviour of $L_2[0, N]$ norms of solutions is of central importance in this connection, we begin by establishing a few elementary properties.

Let $\phi(r, z_1)$ and $\psi(r, z_1)$ be linearly independent solutions of $Lu = z_1 u$ and suppose the solution $\theta(r, z_1)$ of $Lu = z_1 u$ satisfies $\theta(0, z_1) = \xi$, $\theta'(0, z_1) = \eta$. Then, by the variation of constants formula [9, Chap. 3, Theorem 6.4], the solution $\theta(r, z_2)$ of $Lu = z_2 u$ satisfying $\theta(0, z_2) = \xi$ and $\theta'(0, z_2) = \eta$ is given by

$$\theta(r, z_2) = \theta(r, z_1) + \int_0^r \frac{[\phi(t, z_1)\,\psi(r, z_1) - \phi(r, z_1)\,\psi(t, z_1)]}{W(\phi(t, z_1), \psi(t, z_1))} \times (z_2 - z_1)\,\theta(t, z_2)\,dt. \tag{3.5}$$

The Wronskian $W(\phi(r, z_1), \psi(r, z_1))$ is independent of r [18, Chap. XI, Section 2(v)], and is readily evaluated at $r = 0$.

For each fixed z in $\mathbb{C}$, define unique solutions $u_1(r, z)$, $u_2(r, z)$, and $u_{(k)}(r, z)$ of $Lu = zu$ to satisfy

$$u_1(0, z) = -\sin\alpha \qquad u_1'(0, z) = \cos\alpha$$

$$u_2(0, z) = \cos\alpha \qquad u_2'(0, z) = \sin\alpha$$

$$u_{(k)}(r, z) = u_2(r, z) + k u_1(r, z) \qquad k \in \mathbb{C}.$$

For each fixed z in $\mathbb{C}\backslash\mathbb{R}$, define

$$u_m(r, z) = u_2(r, z) + m(z)\,u_1(r, z),$$

where $m(z) = m(z, \alpha)$ (cf. (2.1)). For those x in $\mathbb{R}$ for which $m_+(x)$ exists finitely define

$$u_{m_+}(r, x) = u_2(r, x) + m_+(x)\,u_1(r, x).$$

If $m_+(x)$ exists finitely and is real, define

$$m(x) = m_+(x)$$
$$u_m(r, x) = u_2(r, x) + m(x)\, u_1(r, x).$$

In the following lemma, we examine the behaviour of $\| u_1(r, z)\|_N$, $\| u_{(k)}(r, z)\|_N$, and $\| u_m(r, z)\|_N$ when both $x \in \mathbb{R}$ and $N < \infty$ are fixed. Since $z = x + iy$, these norms are functions of y, defined on $(-\infty, \infty)$ in the case of $\| u_1(r, z)\|_N$ and $\| u_{(k)}(r, z)\|_N$, and, in general, on $\mathbb{R}\backslash\{0\}$ in the case of $\| u_m(r, z)\|_N$.

LEMMA 7. *Let $x \in \mathbb{R}$ and $N \in \mathbb{R}^+$ be fixed. If*

$$\gamma_1 = 2\,|\,y_2 - y_1|\, \| u_1(r, z_1)\|_N\, \| u_m(r, z_1)\|_N,$$
$$\gamma_{(k)} = \frac{4y\, \| u_{(k)}(r, z)\|_N\, \| u_m(r, z)\|_N}{|k - m_+(x)|},$$

where $z = x + iy$, $z_1 = x + iy_1$, and $z_2 = x + iy_2$, then

(i) *if $y_1 > 0$,*

$$|\, \| u_1(r, z_2)\|_N - \| u_1(r, z_1)\|_N | \leqslant \gamma_1\, \| u_1(r, z_2)\|_N.$$

If $m_+(x)$ exists finitely and γ_1 is suitably adjusted, then this inequality also holds for $y_1 = 0$.

(ii) If $y_1, y_2 > 0$,

$$|\, \| u_m(r, z_2)\|_N - \| u_m(r, z_1)\|_N |$$
$$\leqslant \gamma_1\, \| u_m(r, z_2)\|_N + |m(z_2) - m(z_1)|\, \| u_1(r, z_1)\|_N.$$

If $m_+(x)$ exists finitely and γ_1 is suitably adjusted, then this inequality also holds if y_1 or $y_2 = 0$.

(iii) If $m_+(x)$ exists finitely and $k \in \mathbb{C}$ is such that $k \neq m_+(x)$,

$$|\, \| u_{(k)}(r, x)\|_N - \| u_{(k)}(r, z)\|_N | \leqslant \gamma_{(k)}\, \| u_{(k)}(r, x)\|_N$$

whenever $y > 0$ is sufficiently small.

(iv) With the hypothesis of (iii), if $y > 0$ is sufficiently small

$$|\, \| u_m(r, z)\|_N - \| u_{m_+}(r, x)\|_N |$$
$$\leqslant \gamma_{(\kappa)}\, \| u_{m_+}(r, x)\|_N$$
$$+ \frac{2\,|m(z) - m(x)|}{|m_+(x) - k|}\, [\| u_{(k)}(r, z)\|_N + \| u_{m_+}(r, z)\|_N].$$

Proof. (i) Setting $\phi(r, z_1) = \theta(r, z_1) = u_1(r, z_1)$, $\theta(r, z_2) = u_1(r, z_2)$, $\psi(r, z_1) = u_m(r, z_1)$ in (3.5), we have for $y_1 > 0$,

$$u_1(r, z_2) = u_1(r, z_1) + \int_0^r [u_m(t, z_1)\, u_1(r, z_1) - u_m(r, z_1)\, u_1(t, z_1)]\, i(y_2 - y_1)\, u_1(t, z_2)\, dt,$$

where we note that $u_1(r, z_1)$ and $u_m(r, z_1)$ are linearly independent and $W(u_m(r, z_1), u_1(r, z_1)) = 1$. Hence for $r \leqslant N$,

$$|u_1(r, z_2) - u_1(r, z_1)| \leqslant |y_2 - y_1|\, |u_1(r, z_1)| \int_0^N |u_m(t, z_1)\, u_1(t, z_2)|\, dt + |y_2 - y_1|\, |u_m(r, z_1)| \int_0^N |u_1(t, z_1)\, u_1(t, z_2)|\, dt$$

Taking $L_2[0, N]$ norms of both sides and using the Minkowski and Cauchy–Schwarz inequalities, we deduce that

$$\| u_1(r, z_2) - u_1(r, z_1) \|_N \leqslant \gamma_1 \, \| u_1(r, z_2) \|_N$$

and a further application of Minkowski's inequality gives the result. If $m_+(x)$ exists finitely, the argument applies equally to $z_1 = x$.

(ii) Define $u_{m_2}(r, z_1) = u_2(r, z_1) + m(z_2)\, u_1(r, z_1)$. By Minkowski's inequality

$$\begin{aligned} &| \, \| u_m(r, z_2) \|_N - \| u_m(r, z_1) \|_N \, | \\ &\quad \leqslant \| u_m(r, z_2) - u_m(r, z_1) \|_N \\ &\quad \leqslant \| u_m(r, z_2) - u_{m_2}(r, z_1) \|_N + \| u_{m_2}(r, z_1) - u_m(r, z_1) \|_N \\ &\quad \leqslant \| u_m(r, z_2) - u_{m_2}(r, z_1) \|_N + | m(z_2) - m(z_1) | \, \| u_1(r, z_1) \|_N. \end{aligned} \tag{3.6}$$

If $y_2 > 0$, $u_m(r, z_2)$ and $u_{m_2}(r, z_1)$ are defined and satisfy the same boundary conditions at $r = 0$. Hence, as in the proof of (i),

$$u_m(r, z_2) = u_{m_2}(r, z_1) + \int_0^r [u_m(t, z_1)\, u_1(r, z_1) - u_m(r, z_1)\, u_1(t, z_1)] \times i(y_2 - y_1)\, u_m(t, z_2)\, dt$$

whenever $y_1 > 0$. This yields

$$\| u_m(r, z_2) - u_{m_2}(r, z_1) \|_N \leqslant \gamma_1 \, \| u_m(r, z_2) \|_N$$

which together with (3.6) gives the result. If $m_+(x)$ exists finitely the argument applies equally to $z_1 = x$ or $z_2 = x$.

(iii) From the definitions of $u_{(k)}(r, z)$ and $u_m(r, z)$,

$$W(u_{(k)}(r, z), u_m(r, z)) = m(z) - k.$$

Now $k \neq m_+(x)$, and $m(z)$, regarded as a function of y, is continuous; hence for sufficiently small y

$$|k - m(z)| > \frac{|k - m_+(x)|}{2} > 0 \tag{3.7}$$

is satisfied, so that $u_{(k)}(r, z)$ and $u_m(r, z)$ are linearly independent solutions of $Lu = xu$ [18, Chap. XI, Section 2(vii)].

If we now set $z_1 = z$, $z_2 = x$, $\theta(r, z_1) = \phi(r, z_1) = u_{(k)}(r, z)$, $\theta(r, z_2) = u_{(k)}(r, x)$, and $\psi(r, z_1) = u_m(r, z)$ in (3.5), we obtain for $y > 0$,

$$u_{(k)}(r, x) = u_{(k)}(r, z) + \int_0^r \frac{[u_{(k)}(t, z)\, u_m(r, z) - u_{(k)}(r, z)\, u_m(t, z)](-iy)\, u_{(k)}(t, x)}{(m(z) - k)}\, dt.$$

Taking $L_2[0, N]$ norms as before, this together with (3.7) yields the result.

(iv) Define $u_{m_x}(r, z) = u_2(r, z) + m_+(x)\, u_1(r, z)$. By Minkowski's inequality

$$\big|\, \|u_m(r, z)\|_N - \|u_{m_+}(r, x)\|_N \big| \leqslant \|u_{m_x}(r, z) - u_{m_+}(r, x)\|_N + |m(z) - m_+(x)|\, \|u_1(r, z)\|_N \tag{3.8}$$

(cf. (3.6)). For sufficiently small y

$$u_1(r, z) = \frac{u_m(r, z) - u_{(k)}(r, z)}{m(z) - k}$$

so that by (3.7) and Minkowski's inequality

$$\|u_1(r, z)\|_N \leqslant \frac{2}{|k - m_+(x)|} (\|u_m(r, z)\|_N + \|u_{(k)}(r, z)\|_N). \tag{3.9}$$

Setting $z_1 = z$, $z_2 = x$, $\theta(r, z_1) = u_{m_x}(r, z)$, $\theta(r, z_2) = u_{m_+}(r, x)$, $\phi(r, z_1) = u_{(k)}(r, z)$, and $\psi(r, z_1) = u_m(r, z)$ in (3.5), we have for $y > 0$

$$u_{m_+}(r, x) = u_{m_x}(r, z) + \int_0^r \frac{[u_{(k)}(t, z)\, u_m(r, z) - u_{(k)}(r, z)\, u_m(t, z)](-iy)\, u_{m_+}(t, x)}{(m(z) - k)}\, dt.$$

Taking $L_2[0, N]$ norms as before, this together with (3.7), (3.8), and (3.9) gives the result.

By setting $\theta(r, z_1) = \phi(r, z_1) = u_1(r, z_1)$, $\theta(r, z_2) = u_1(r, z_2)$, and $\psi(r, z_1) = u_2(r, z_1)$ in (3.5) we obtain for all $y_1, y_2 \in \mathbb{R}$

$$\big| \|u_1(r, z_2)\|_N - \|u_1(r, z_1)\|_N \big| \leqslant \gamma_2 \|u_1(r, z_2)\|_N,$$

where $\gamma_2 = 2|y_2 - y_1| \, \|u_1(r, z_1)\|_N \, \|u_2(r, z)\|_N$. Hence if $N < \infty$ and $x \in \mathbb{R}$ are fixed, $\|u_1(r, z)\|_N$ is a continuous function of y on all of $\mathbb{R}$ irrespective of whether $m(x)$ is defined; similarly $\|u_{(k)}(r, z)\|_N$ is a continuous function of y on $\mathbb{R}$. We have therefore:

LEMMA 8. *If $N < \infty$ and $x \in \mathbb{R}$ are fixed, then $\|u_1(r, z)\|_N$ and $\|u_{(k)}(r, z)\|_N$ are continuous functions of y on $\mathbb{R}$.*

Lemmas 7 and 8 will be used in the following sections where we determine necessary and sufficient conditions for the existence of a subordinate solution of $Lu = xu$ at the real point x.

4. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF SUBORDINATE SOLUTIONS

We now show that if $m_+(x)$ exists finitely and is real, or if $|m(z)| \to \infty$ as $y \downarrow 0$, then the subordinacy of the $L_2[0, \infty)$ solution $u_m(r, z)$ of $Lu = zu$ for $y > 0$ is reflected in the subordinacy of corresponding solutions of $Lu = xu$.

For fixed $x \in \mathbb{R}$, let $\varepsilon, y' > 0$ be related by

$$\frac{\varepsilon^2}{4} = \sup_{0 < y \leqslant y'} |m(z) - m_+(x)| + y' \tag{4.1}$$

and define $N(y')$ by

$$|y'|^{1/2} \|u_1(r, z')\|_{N(y')} = 1, \tag{4.2}$$

where $z = x + iy$ and $z' = x + iy'$. Note that y' is a continuous monotonic function of ε, and that $y' \downarrow 0$ as $\varepsilon \downarrow 0$; observe also that for each $y' > 0$, $N(y')$ exists satisfying (4.2) since $u_1(r, z')$ is not in $L_2[0, \infty)$, and that $N(y')$ is a function of ε.

We investigate the relationship between $N(y')$ and ε when $m(z)$ converges to a finite real limit and ε is small in the following lemma.

LEMMA 9. *Let $x \in \mathbb{R}$ be fixed and suppose that $m(z)$ converges to a finite real limit $m(x)$ as $y \downarrow 0$. If ε, y', and $N(y')$ are related as in* (4.1) *and* (4.2), *then*

(i) $\|u_m(r,x)\|_{N(y')}/\|u_1(r,x)\|_{N(y')} < \varepsilon$ *whenever* ε *is sufficiently small.*

Moreover,

(ii) $N(y')$ *depends continuously on* ε.

(iii) $N(y') \to \infty$ *as* $\varepsilon \downarrow 0$.

Proof. (i) Suppose $\varepsilon < 1$. Using the hypothesis and (4.1), we have

$$|m(z') - m(x)| < \varepsilon/4 \tag{4.3}$$

$$|\operatorname{Im} m(z')|^{1/2} < \varepsilon/2. \tag{4.4}$$

From (2.3), (4.2), and (4.4)

$$\frac{\|u_m(r,z')\|_{N(y')}}{\|u_1(r,z')\|_{N(y')}} < \frac{\varepsilon}{2} \tag{4.5}$$

and

$$\gamma_1 = 2y' \ \|u_1(r,z')\|_{N(y')} \ \|u_m(r,z')\|_{N(y')} < \varepsilon. \tag{4.6}$$

Let z' and x be identified with z_1 and z_2 respectively in parts (i) and (ii) of Lemma 7. Then by (4.6),

$$\|u_1(r,x)\|_{N(y')} > (1+\varepsilon)^{-1} \ \|u_1(r,z')\|_{N(y')}$$

and

$$\|u_m(r,x)\|_{N(y')} < (1-\varepsilon)^{-1} \left[\|u_m(r,z')\|_{N(y')} + |m(z') - m(x)| \ \|u_1(r,z')\|_{N(y')} \right],$$

respectively. Hence

$$\frac{\|u_m(r,x)\|_{N(y')}}{\|u_1(r,x)\|_{N(y')}} < \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \left[\frac{\|u_m(r,z')\|_{N(y')}}{\|u_1(r,z')\|_{N(y')}} + |m(z') - m(x)| \right]$$

from which the result now follows by (4.3) and (4.5).

(ii) We first show that for each $y'' > 0$, $N(y')$ is a continuous function of y' at y''. Let $\delta > 0$ be given and $y'' > 0$ be fixed. Since $u_1(r, z'')$ cannot vanish on a nontrivial r-interval,

$$|y''|^{1/2} \ \|u_1(r,z'')\|_{N(y'')-\delta} < 1$$

by (4.2). This, together with the continuity of $\|u_1(r,z')\|_N$ in y' which was proved in Lemma 8, implies that

$$|y'|^{1/2} \ \|u_1(r,z')\|_{N(y'')-\delta} < 1 \tag{4.7}$$

for y' sufficiently close to y''. Similarly

$$|y'|^{1/2} \|u_1(r, z')\|_{N(y'')+\delta} > 1 \tag{4.8}$$

for y' sufficiently close to y''. Since $\|u_1(r, z')\|_N$ is strictly increasing with N for each fixed z', it follows from (4.2), (4.7), and (4.8) that

$$N(y'') - \delta < N(y') < N(y'') + \delta$$

for y' sufficiently close to y''; hence $N(y')$ is continuous in y' at y''.

The continuous dependence of $N(y')$ on ε now follows since y' is a continuous function of ε by (4.1).

(iii) Since $y' \downarrow 0$ as $\varepsilon \downarrow 0$ by (4.1), it suffices to prove that $N(y') \to \infty$ as $y' \downarrow 0$. Suppose that $N(y') \not\to \infty$ as $y' \downarrow 0$. Then $M \in \mathbb{R}^+$ and a sequence $\{y'_k\}$ exist such that $y'_k \downarrow 0$ as $k \to \infty$ and $N(y'_k) < M$ for each k. It then follows from (4.2) that for each y'_k

$$\|u_1(r, x + iy'_k)\|_M \geqslant \|u_1(r, x + iy'_k)\|_{N(y'_k)} = |y'_k|^{-1/2}$$

so that $\|u_1(r, x + iy'_k)\|_M \to \infty$ as $y'_k \downarrow 0$. This is impossible since $\|u_1(r, z)\|_M$ is a continuous function of y on $\mathbb{R}$ by Lemma 8.

We have therefore shown by contradiction that $N(y') \to \infty$ as $\varepsilon \downarrow 0$, and the proof is complete.

Since $N(y')$ is primarily of interest as a function of ε, we shall relabel $N(y')$ as $n(\varepsilon)$. As a result of Lemma 9, it is now straightforward to obtain sufficient conditions for the existence of a subordinate solution of $Lu = xu$ at the real point x.

COROLLARY 2. *Let $x \in \mathbb{R}$ be fixed. Then*

(i) *if $m(z)$ converges to a finite real limit as $y \downarrow 0$, $u_m(r, x)$ is a subordinate solution of $Lu = xu$;*

(ii) *if $|m(z)| \to \infty$ as $y \downarrow 0$, $u_1(r, x)$ is a subordinate solution of $Lu = xu$.*

Proof. (i) By Lemma 9 there exists an interval $(0, a]$ such that $n(\varepsilon)$ is a continuous function of ε on $(0, a]$ satisfying

$$\frac{\|u_m(r, x)\|_{n(\varepsilon)}}{\|u_1(r, x)\|_{n(\varepsilon)}} < \varepsilon. \tag{4.9}$$

Let η such that $0 < \eta < a$ be given, and let K_η denote the least upper bound of $n(\varepsilon)$ on $[\eta, a]$. By Lemma 9(iii), $n(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$; it therefore follows from the continuity of $n(\varepsilon)$ that $n(\varepsilon)$ takes every value in $[K_\eta, \infty)$ as ε ranges over $(0, a]$. Moreover, if ε is in $(0, a]$, then whenever $n(\varepsilon) > K_\eta$,

ε is in $(0, \eta)$ by the definition of K_η. We may therefore deduce from (4.9) that for ε in $(0, a]$

$$\frac{\| u_m(r, x)\|_{n(\varepsilon)}}{\| u_1(r, x)\|_{n(\varepsilon)}} < \eta$$

whenever $n(\varepsilon) > K_\eta$. Since $n(\varepsilon)$ takes every value in $[K_\eta, \infty)$ when ε is in $(0, a]$, it follows that

$$\lim_{N \to \infty} \frac{\| u_m(r, x)\|_N}{\| u_1(r, x)\|_N} = 0.$$

Therefore $u_m(r, x)$ is a subordinate solution of $Lu = xu$.

(ii) The hypothesis means that $|m(z, \alpha_1)| \to \infty$ as $y \downarrow 0$ for some given real boundary condition α_1. If $\alpha_2 \in [0, \pi)$ is a distinct boundary condition, it follows from (2.4) that $m_+(x, \alpha_2) = -\cot(\alpha_1 - \alpha_2)$ which is finite and real. Hence by (i)

$$u_m(r, x, \alpha_2) = u_2(r, x, \alpha_2) + m(x, \alpha_2)\, u_1(r, x, \alpha_2)$$

is a subordinate solution of $Lu = xu$. Since

$$\cos \alpha_1 u_m(0, x, \alpha_2) + \sin \alpha_1 u_m'(0, x, \alpha_2) = 0,$$

$u_m(r, x, \alpha_2)$ is a scalar multiple of $u_1(r, x) = u_1(r, x, \alpha_1)$. This completes the proof of the corollary.

Remark 1. Suppose that a sequence $\{y_n\}$ exists such that $y_n \downarrow 0$ and $m(x + iy_n) \to g \in \mathbb{R}$ as $n \to \infty$. Then by modifying the arguments of Lemma 9 and Corollary 2(i), it may be shown that a sequence $\{N_n\}$ exists for which $N_n \to \infty$ as $n \to \infty$ and

$$\lim_{n \to \infty} \frac{\| u_{(g)}(r, x)\|_{N_n}}{\| u_1(r, x)\|_{N_n}} = 0, \tag{4.10}$$

where $u_{(g)}(r, x) = u_2(r, x) + g u_1(r, x)$. Note that when (4.10) is satisfied, we have also

$$\lim_{n \to \infty} \frac{\| u_{(g)}(r, x)\|_{N_n}}{\| u(r, x)\|_{N_n}} = 0 \tag{4.11}$$

for every solution $u(r, x)$ which is independent of $u_{(g)}(r, x)$.

Significant implications for the spectrum are now almost immediate. Since the set

$$S = \{x \in \mathbb{R} : |m(z)| \to \infty, \operatorname{Im} m(z) \not\to \infty \text{ as } y \downarrow 0\}$$

has Lebesgue measure zero by Lemma 1, and μ-measure zero by Proposition 1, we have

$$\begin{aligned} \mathcal{M}''_{\text{s.}} &= \{x \in \mathbb{R} : |m(z)| \to \infty \text{ as } y \downarrow 0\} \sim \mathcal{M}'_{\text{s.}} \\ \mathcal{M}''_{\text{s.c.}} &= \{x \in \mathbb{R} : |m(z)| \to \infty \text{ as } y \downarrow 0, \mu(x) = 0\} \sim \mathcal{M}'_{\text{s.c.}} \qquad (4.12) \\ \mathcal{M}''_{\text{p.}} &= \{x \in \mathbb{R} : |m(z)| \to \infty \text{ as } y \downarrow 0, \mu(x) > 0\} = \mathcal{M}'_{\text{p.}}, \end{aligned}$$

where $\sim$ is the equivalence relation defined in (2.8) with $\iota = \mu$. By Corollary 2(ii), if $x \in \mathcal{M}''_{\text{s.}}$, then a subordinate solution of $Lu = xu$ exists which satisfies the boundary condition at 0. However, if such a solution is in $L_2[0, \infty)$, x must be an eigenvalue and hence an element of $\mathcal{M}''_{\text{p.}}$. Therefore, a nonsquare integrable subordinate solution of $Lu = xu$ always exists satisfying the boundary condition at 0 when $x \in \mathcal{M}''_{\text{s.c.}}$. Subordinate solutions which are not in $L_2[0, \infty)$ are thus of crucial importance when there is singular continuous spectrum; indeed, a remarkable correspondence will emerge once a converse to Corollary 2 is proved in the following section.

5. Necessary Conditions for the Existence of Subordinate Solutions

Can a subordinate solution of $Lu = xu$ exist if $m(z)$ converges to a finite limit at x which is not real or if $m(z)$ fails to converge finitely or infinitely as $z \to x$? The answer to both questions turns out to be no, so that a subordinate solution of $Lu = xu$ can only exist if one or other of the sufficient conditions of Corollary 2 is satisfied. To substantiate this claim, we need some further estimates of solutions, and in the following lemma we apply the methods of Lemma 9, with revised definitions of y' and $N(y')$, to the case where $m_+(x)$ exists finitely but is not real.

Lemma 10. *Let $x \in \mathbb{R}$ and $k \in \mathbb{C}$ be fixed and suppose that $m(z)$ converges to a finite limit $m_+(x)$ as $y \downarrow 0$. Suppose also that $k \neq m_+(x)$ and* Im $m_+(x) = l_x > 0$. *Then for sufficiently large N*

(i) there exists $K \in \mathbb{R}^+$ which is independent of x such that

$$\frac{\|u_{m_+}(r, x)\|_N}{\|u_1(r, x)\|_N} \leqslant K l_x$$

(ii) there exists $M \in \mathbb{R}^+$ which depends on k and x such that

$$\frac{\|u_{m_+}(r, x)\|_N}{\|u_{(k)}(r, x)\|_N} \leqslant M l_x$$

where $u_{(k)}(r, x) = u_2(r, x) + k u_1(r, x)$.

Proof. (i) Let $\varepsilon, y' > 0$ satisfy

$$\varepsilon < l_x \tag{5.1}$$

$$\frac{\varepsilon^2}{2^{10} l_x} = \sup_{0 < y \leqslant y'} |m(z) - m_+(x)| + y' \tag{5.2}$$

$$|y'|^{1/2} \, \|u_1(r, z')\|_{N(y')} = l_x^{-1/2}/2^3. \tag{5.3}$$

From (5.1) and (5.2)

$$|m(z') - m_+(x)| < \varepsilon/2^{10} \tag{5.4}$$

and from (5.2) and the hypothesis,

$$|\operatorname{Im} m(z') - l_x|^{1/2} < \varepsilon l_x^{-1/2}/2^5.$$

Hence

$$|\operatorname{Im} m(z')|^{1/2} < l_x^{1/2} + \varepsilon l_x^{-1/2}/2^5$$

which together with (2.3), (5.1), and (5.3) implies

$$\frac{\|u_m(t, z')\|_{N(y')}}{\|u_1(r, z')\|_{N(y')}} < 8 l_x + \frac{\varepsilon}{4} < 9 l_x,$$

$$\gamma_1 = 2y' \, \|u_1(r, z')\|_{N(y')} \, \|u_m(r, z')\|_{N(y')} < 1/3;$$

z' and x are now identified with z_1 and z_2 respectively in Lemma 7(i), (ii), and the remainder of the proof follows a pattern similar to that of Lemma 9 and Corollary 2(i).

(ii) Let $\varepsilon, y' > 0$ satisfy

$$\varepsilon < \min\{1, l_x, l_x |m_+(x) - k|^{-1/2}/2^4\} \tag{5.5}$$

$$\varepsilon^4 \, |m_+(x) - k| \, l_x^{-1} = \sup_{0 < y \leqslant y'} |m(z) - m_+(x)| + y' \tag{5.6}$$

and suppose also that y' is sufficiently small to satisfy the estimate of Lemma 7(iii). Define $N(y')$ by

$$|y'|^{1/2} \, \|u_{(k)}(r, z')\|_{N(y')} = (|m_+(x) - k| \, l_x^{-1/2})/2^4. \tag{5.7}$$

From (5.5) and (5.6)

$$|m(z') - m_+(x)| < \varepsilon^2 l_x^{-1} \, |m_+(x) - k| \tag{5.8}$$

and, using the hypothesis,

$$|\operatorname{Im} m(z')|^{1/2} < l_x^{1/2} + \varepsilon^2 l_x^{-1/2} |m_+(x) - k|^{1/2}. \tag{5.9}$$

It now follows from (2.3), (5.5), (5.7), and (5.9) that

$$\frac{\|u_m(r, z')\|_{N(y')}}{\|u_{(k)}(r, z')\|_{N(y')}} < \frac{(2^4+1)\, l_x}{|m_+(x) - k|}, \tag{5.10}$$

$$\gamma_{(k)} = \frac{4y' \, \|u_{(k)}(r, z')\|_{N(y')} \, \|u_m(r, z')\|_{N(y')}}{|m_+(x) - k|} < \frac{1}{3}. \tag{5.11}$$

Identifying z' with z in Lemma 7(iii), (iv) yields

$$\frac{\|u_{m_+}(r, x)\|_{N(y')}}{\|u_{(k)}(r, x)\|_{N(y')}} \leqslant \frac{2(1+\gamma_{(k)})}{(1-\gamma_{(k)})} \frac{|m(z') - m_+(x)|}{|m_+(x) - k|} + \frac{(1+\gamma_{(k)})}{(1-\gamma_{(k)})}\left[\frac{2\,|m(z') - m_+(x)|}{|m_+(x) - k|} + 1\right] \frac{\|u_m(r, z')\|_{N(y')}}{\|u_{(k)}(r, z')\|_{N(y')}}.$$

Applying (5.8), (5.10), and (5.11) to this inequality, the remainder of the proof is completed as in Lemma 9 and Corollary 2(i).

Lemma 10 remains valid in a modified sense if there exists a sequence $\{Y_n\}$ in $\mathbb{R}^+$ such that $Y_n \to 0$ and $m(x + iY_n)$ converges to a finite limit g as $n \to \infty$. In this case sequences $\{M_p\}$ and $\{N_q\}$ exist such that M_p, $N_q \to \infty$ and

$$\frac{\|u_{(g)}(r, x)\|_{M_p}}{\|u_1(r, x)\|_{M_p}} = O(\operatorname{Im} g), \qquad \frac{\|u_{(g)}(r, x)\|_{N_q}}{\|u_{(k)}(r, x)\|_{N_q}} = O(\operatorname{Im} g) \tag{5.12}$$

as $p, q \to \infty$, where $u_g(r, x) = u_2(r, x) + g u_1(r, x)$, $k \neq g$. Since this is easily verified by making suitable minor adjustments to the arguments of Lemma 10, we omit a detailed proof.

Remark 2. Suppose that a subordinate solution of $Lu = xu$ is known to exist for some $x \in \mathbb{R}$. Then the set of scalar multiples of

$$u(r, x) = a u_1(r, x) + b u_2(r, x),$$

where $a \in \mathbb{R}$, $b \in \mathbb{C}$, and $u(r, x)$ is subordinate, is the set of all subordinate solutions at x. Now $\|\overline{u(r, x)}\|_N = \|u(r, x)\|_N$ for all $N \in \mathbb{R}^+$, so $\overline{u(r, x)}$ is a subordinate solution and hence a scalar multiple of $u(r, x)$. It follows that if $a \neq 0$, then b is real, so that a subordinate solution at a real point x is always a scalar multiple of a real solution.

The following corollary to Lemma 10 establishes necessary conditions for the existence of subordinate solutions, and provides a converse to Corollary 2.

COROLLARY 3. *If a subordinate solution of $Lu = xu$ exists at the real point x then, as $y \downarrow 0$, either $m(z)$ converges to a finite real limit or $|m(z)| \to \infty$.*

Proof. Suppose first that a subordinate solution of $Lu = xu$ exists and that $m(z)$ converges to a finite limit at x. Then, using the notation and conclusions of Lemma 10,

$$\frac{\|u_{m_+}(r, x)\|_N}{\|u_1(r, x)\|_N} = O(l_x), \qquad \frac{\|u_{m_+}(r, x)\|_N}{\|u_{(k)}(r, x)\|_N} = O(l_x) \tag{5.13}$$

as $N \to \infty$, where $k \in \mathbb{C}$ is such that $k \neq m_+(x)$. If $u_{m_+}(r, x)$ is not subordinate, there exists a solution $u(r, x)$ which is independent of $u_{m_+}(r, x)$ such that

$$\lim_{N \to \infty} \frac{\|u(r, x)\|_N}{\|u_{m_+}(r, x)\|_N} = 0 \tag{5.14}$$

by Definition 2. However, since $u(r, x)$ is a scalar multiple of some element of the solution set $\{u_1(r, x)\} \cup \{u_{(k)}(r, x) : k \in \mathbb{C},\ k \neq m_+(x)\}$, (5.14) is incompatible with (5.13). The supposition that $u_{m_+}(r, x)$ is not subordinate must therefore be false, so that by Remark 2, $u_{m_+}(r, x)$ is a scalar multiple of a real solution. It follows that $m(z)$ converges to a finite real limit at x.

Now suppose that $m(z)$ does not converge finitely or infinitely as $y \downarrow 0$. Then there exist sequences $\{y_s\}$, $\{Y_t\}$ in $\mathbb{R}^+$ and g, h in $\mathbb{C} \cup \{\infty\}$ with $g \neq h$ such that y_s, $Y_t \to 0$ and $m(x + iy_s) \to g$, $m(x + iY_t) \to h$ as s, $t \to \infty$. We shall prove by contradiction that no subordinate solution of $Lu = xu$ can exist.

Consider first the case $|g|$, $|h| < \infty$. Define $u_{(g)}(r, x) = u_2(r, x) + g u_1(r, x)$, $u_{(h)}(r, x) = u_2(r, x) + h u_1(r, x)$ and suppose that a subordinate solution of $Lu = xu$ exists. If $u_{(g)}(r, x)$ is not subordinate, then there exists a solution $u(r, x)$ which is independent of $u_{(g)}(r, x)$ such that

$$\lim_{N \to \infty} \frac{\|u(r, x)\|_N}{\|u_{(g)}(r, x)\|_N} = 0 \tag{5.15}$$

by Definition 2. Arguing as above, we see that (5.15) conflicts with (5.12) (or (4.11) if $\operatorname{Im} g = 0$); hence $u_{(g)}(r, x)$ must be subordinate, and, similarly $u_{(h)}(r, x)$ must be subordinate. However, $u_{(g)}(r, x)$ and $u_{(h)}(r, x)$ are linearly independent solutions of $Lu = xu$, so they cannot both be subordinate. This contradiction shows that no subordinate solution can exist if $|g|$, $|h| < \infty$.

If $|g| = \infty$, then by (2.4) a boundary condition α_2 exists such that the sequence $\{m(x+iy_s, \alpha_2)\}$ converges to a finite real limit $-\cot(\alpha_1 - \alpha_2)$, where we have taken $m(z) = m(z, \alpha_1)$. We may then infer from Remark 1 that a sequence $\{M_p\}$ exists such that $M_p \to \infty$ as $p \to \infty$ and

$$\lim_{p \to \infty} \frac{\| u_m(r, x, \alpha_2) \|_{M_p}}{\| u_1(r, x, \alpha_2) \|_{M_p}} = 0$$

so that the existence of a subordinate solution implies $u_m(r, x, \alpha_2)$ is subordinate. Now $u_m(r, x, \alpha_2)$ is a scalar multiple of $u_1(r, x) = u_1(r, x, \alpha_1)$ (see proof of Corollary 2(ii)), so if a subordinate solution exists and $|g| = \infty$ and $|h| < \infty$, both $u_1(r, x)$ and $u_{(h)}(r, x)$ must be subordinate. This is however impossible, so no subordinate solution can exist in the case $|g| = \infty$, $|h| < \infty$.

The conclusions of the corollary now follow, since all other possibilities have been eliminated.

6. Subordinacy and the Spectrum

Corollaries 2 and 3 form a complete set of necessary and sufficient conditions for the existence of a subordinate solution of $Lu = xu$ at the real point x in terms of the behaviour of the function $m(z)$ as z approaches x along the normal to the real axis at x. Combining these two results, we obtain the following existence theorem:

THEOREM 1. *A subordinate solution of $Lu = xu$ exists at the real point x if and only if as $y \downarrow 0$ either $m(z)$ converges to a finite real limit, in which case $u_m(r, x)$ is subordinate, or $|m(z)| \to \infty$, in which case $u_1(r, x)$ is subordinate.*

Further discussion of this theorem is contained in [5, 25]. If $\lim_{y \downarrow 0} |m(z, \alpha_1)| = \infty$ for some boundary condition α_1, then it follows from (2.4) that for any distinct boundary condition α_2, $m_+(x, \alpha_2) = -\cot(\alpha_1 - \alpha_2)$ which is finite and real. This means that Theorem 1 may be expressed in the following alternative form:

THEOREM 2. *A subordinate solution of $Lu = xu$ exists at the real point x if and only if there exists a boundary condition α such that $m(z, \alpha)$ converges to a finite real limit $m(x, \alpha)$ as $y \downarrow 0$, in which case*

$$u_m(r, x, \alpha) = u_2(r, x, \alpha) + m(x, \alpha)\, u_1(r, x, \alpha)$$

is subordinate.

Noting that $u_1(r, x)$ is a solution of $Lu = xu$ which satisfies the boundary condition at 0, we shall now derive minimal supports of the decomposed parts of the spectral measure μ in terms of the subordinacy of solutions.

THEOREM 3. *Minimal supports* $\mathcal{M}'''$, $\mathcal{M}'''_{\text{a.c.}}$, $\mathcal{M}'''_{\text{s.}}$, $\mathcal{M}'''_{\text{s.c.}}$, *and* $\mathcal{M}'''_{\text{p.}}$ *of* μ, $\mu_{\text{a.c.}}$, $\mu_{\text{s.}}$, $\mu_{\text{s.c.}}$, *and* $\mu_{\text{p.}}$ *are as follows*:

(i) $\mathcal{M}''' = \mathbb{R} \setminus \{x \in \mathbb{R}$: *a subordinate solution of* $Lu = xu$ *exists but does not satisfy the boundary condition at* $0\}$

(ii) $\mathcal{M}'''_{\text{a.c.}} = \{x \in \mathbb{R}$: *no subordinate solution of* $Lu = xu$ *exists*$\}$

(iii) $\mathcal{M}'''_{\text{s.}} = \{x \in \mathbb{R}$: *a subordinate solution of* $Lu = xu$ *exists which satisfies the boundary condition at* $0\}$

(iv) $\mathcal{M}'''_{\text{s.c.}} = \{x \in \mathbb{R}$: *a subordinate solution of* $Lu = xu$ *exists which satisfies the boundary condition at* 0 *but is not in* $L_2[0, \infty)\}$

(v) $\mathcal{M}'''_{\text{p.}} = \{x \in \mathbb{R}$: *a subordinate solution of* $Lu = xu$ *exists which satisfies the boundary condition at* 0 *and is in* $L_2[0, \infty)\}$

Proof. We need only prove (ii) and (iii) and since (v) is well known, (iii) and (v) imply (iv), and (ii) and (iii) imply (i).

(ii) This is immediate from Corollary 1 and Theorem 2.

(iii) It is sufficient to prove that $\mathcal{M}'''_{\text{s.}} \sim \mathcal{M}'_{\text{s.}}$, or, equivalently, by (4.12) that $\mathcal{M}'''_{\text{s.}} \sim \mathcal{M}''_{\text{s.}}$, where $\sim$ is the equivalence relation defined in (2.8) with $\iota = \mu$. By Corollary 2(ii), $\mathcal{M}''_{\text{s.}} \subseteq \mathcal{M}'''_{\text{s.}}$; moreover, whenever $x \in \mathcal{M}'''_{\text{s.}}$, $u_1(r, x)$ is a subordinate solution of $Lu = xu$ which implies by Theorem 1 that $x \in \mathcal{M}''_{\text{s.}}$. Hence $\mathcal{M}'''_{\text{s.}} = \mathcal{M}''_{\text{s.}}$, and the result is proved.

Theorem 3 shows that complexity of spectral behaviour is reflected in a very precise way in properties of solutions of the corresponding Schrödinger equations. The result is a natural extension of (3.2) and (3.3) which characterise the restriction of the spectrum to $(-\infty, p)$, where p is the parabolic point; note that whereas subordinate solutions, principal solutions, and $L_2[0, \infty)$ solutions of $Lu = xu$ are indistinguishable when $x < p$, subordinate solutions need not be square integrable when $x \geqslant p$, and principal solutions have no meaning when $Lu = xu$ is oscillatory at x. We see that in the context of those Schrödinger equations for which L is regular at 0 and in the limit point case at infinity, subordinate solutions may be regarded as an extension of $L_2[0, \infty)$ solutions; indeed, subordinate solutions bear precisely the same relation to the minimal supports of the singular part of the spectral measure as do square integrable solutions to the minimal supports of the pure point part.

The most obvious application of Theorem 3 is to the detailed spectral analysis of particular Schrödinger operators or, more precisely, to analysis of that part of the spectrum for which solutions of $Lu = xu$ are oscillatory, since the theorem yields no new information about the spectrum on

$(-\infty, p]$. In relatively straightforward cases, for example, where the nature of the spectrum is known already and only its location needs to be determined, it may still be simplest to make use of well-established results. The fact that the essential spectrum is the union of the closure of the set of all x for which $Lu = xu$ has no $L_2[0, \infty)$ solution and a nowhere dense set [24, Section 3], together with the well-known characterisation of the point spectrum (as listed in Theorem 3), will frequently be sufficient to completely determine the spectrum. It is where little is known in advance or where the essential spectrum is made up of several types of spectra, that the discrimination afforded by Theorem 3 may be most useful.

Direct application of Theorem 3 will not always be straightforward, however, even where there is sufficient information available to determine the minimal supports exactly. In theory there is no difficulty in identifying the absolutely continuous spectrum; note that this type of spectrum is not void if and only if $\kappa(\mathcal{M}'''_{\text{a.c.}}) > 0$. The presence or absence of a singular continuous spectrum may be more difficult to establish, particularly if embedding in the absolutely continuous spectrum is a possibility. Although a necessary condition for the singular continuous spectrum to be nonempty is that $\mathcal{M}'''_{\text{s.c.}}$ be an uncountable set, this condition is not sufficient as may be deduced using the inverse method of Gel'fand and Levitan [23] together with Lemma 3, Corollary 2, and [26, Section 11.82, Lemma 1]. However, we may assert that if $\kappa(\mathcal{M}'''_{\text{a.c.}} \cap I) = \mu(\mathcal{M}'''_{\text{p.}} \cap I) = 0$ and $\mathcal{M}'''_{\text{s.c.}} \cap I$ is an uncountable set for some real interval I, then there is a nontrivial singular continuous spectrum on I. Suppose not, then the singular continuous spectrum is empty on I; then $\mu \ll \kappa$ on I and $d\mu/d\kappa = 0$ Lebesgue almost everywhere on I, which together imply that ρ is a constant function on I [26, Section 11.71] and hence that I is in the resolvent set. It follows that $m(z)$ may be analytically continued across I [27, Section 5, Theorem (i)], and so $m_+(x)$ exists finitely and is real at each x in I. Therefore by Theorem 1, $\mathcal{M}'''_{\text{s.c.}} \cap I = 0$, which proves the assertion by contradiction.

As a result of Theorem 3, the proofs of a number of results in spectral theory and analysis are now very simply accomplished. We describe some examples.

Example 1. It was recognised by Weyl that the essential spectrum is preserved under a change of boundary condition, but unclear to him whether certain constituent subsets (namely the set of "Häufungspunkten des Punktspektrums" and the "Streckenspektrum") were individually invariant under one-dimensional perturbations of this type [7, Section 17]. This problem was later investigated from a slightly different angle by Aronszajn, whose analysis was substantially that of Lemma 6 [4, Section 3, Theorem 1). It may be seen that the invariance of minimal supports of the absolutely continuous measure and the contrasting orthogonality of minimal supports of the singular measure under a change of boundary condition which were established in Lemma 6 are not immediately obvious corollaries to Proposition 1. However, the necessity of this behaviour is apparent at once from Theorem 3; for, clearly, the presence or absence of a subordinate solution of $Lu = xu$ at a given point x is independent of the boundary condition at $r = 0$, whereas it is impossible for more than one distinct boundary condition to be satisfied by a subordinate solution at x.

EXAMPLE 2. If $V(r)$ is in $L_1[0, \infty)$, then L is regular at 0 and in the limit point case at infinity; moreover, the spectrum of each Schrödinger operator arising from L is absolutely continuous on $(0, \infty)$ and discrete on $(-\infty, 0)$. This analysis of the spectrum was demonstrated by Titchmarsh using boundary properties of $m(z)$ [6, Section 5.3].

By suitably rearranging the expression which is obtained for a solution $u(r, x)$ of $Lu = xu$ using the variation of constants formula (3.5), it may be shown that, up to a multiplicative constant, an arbitrary solution of $Lu = xu$ has the asymptotic form

$$u(r, x) = \sin(\sqrt{x}\, r + \xi(x)) + o(1)$$

as $r \to \infty$, whenever $x > 0$; the function $\xi(x)$ depends on the boundary conditions which are satisfied by $u(r, x)$ at 0 (cf. [6, loc. cit]). It is therefore evident that for fixed $x > 0$, the linearly independent solutions of $Lu = xu$ differ from one another "at infinity" only by a phase shift, and we may deduce, using Definition 2, that for each $x > 0$ no subordinate solutions of $Lu = xu$ exists. The conclusion that the spectrum is absolutely continuous on $(0, \infty)$ is then immediate by Theorem 3. If $x < 0$, the solutions $u_1(r, x)$ (see (2.2)) of $Lu = xu$ have the asymptotic form $(\exp sr)(\eta(s) + o(1))$ as $r \to \infty$, where $s = \sqrt{-x}$ and $\eta(s)$ is an analytic function of s for $s > 0$ (cf. [6, loc. cit]). Since the zeros of $\eta(s)$ must be isolated, $Lu = xu$ is non-oscillatory for $x < 0$; hence the parabolic point is 0, and the spectrum is discrete on $(-\infty, 0)$.

Comparing Titchmarsh's proof with the method outlined above, it is seen that a considerable economy of effort has been achieved by the use of Theorem 3.

In conclusion, we remark that the results of this paper apply equally to any Sturm–Liouville operator for which the corresponding differential expression is regular at 0 and in the limit point case at infinity. We have emphasised the application to Schrödinger operators because the particular decomposition of the spectrum we have considered is especially relevant to quantum mechanics.

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