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Multiplicative Ideal Theory of Noncommutative Krull Pairs. II: Factorization of One-Sided Ideals

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5. STRONG KRULL PAIRS: EXAMPLES

We use the terminology and the results of Part I throughout. In particular, (E, R) denotes a Krull pair, that is, E is a subring of R, maximal in its Asano class such that the E is d-Noetherian (the Noether condition for divisorial thick two-sided ideals of E holds).

To establish a factorization of *all* one-sided thick ideals of E we impose one more condition:

5.I. A Krull pair (E, R) is called *strong* if every maximal element F in the Asano class α of E is d-Noetherian and R is a left-Asano quotient ring of E.

This condition entails that every thick divisorial left ideal of every maximal order in α is symbolically invertible, that is, a member of the Brandt groupoid $B_d(\alpha_d) = K$ (see Theorem 3.6 and Corollary 3.8). In Section 6, the groupoid K is investigated from a general point of view. This is used in Section 7 to obtain a factorization theory for one-sided ideals of a strong Krull pair. But since we now have completed the essential definitions, it is appropriate first to point out examples of Krull pairs and strong Krull pairs showing that important classes of rings are of the type considered in this paper.

(a) *The Commutative Case.*

We begin with a more general result.

5.1. PROPOSITION. *Let (E, R) be a Krull pair such that the center Z of E is a domain, and the center of R contains the quotient field K of Z. Then Z is a Krull domain.*

Proof. For a prime P of E let $P^{(n)}$ be the symbolic product of n factors P. By setting $v_P(x) = n$ if $x \in P^{(n)}$, $x \notin P^{(n+1)}$, for $x \in Z$, $x \neq 0$, we define a discrete valuation of Z. ($v_P(xy) = v_P(x) + v(y)$ is seen using the fact that every two-

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sided ideal Ex has the ideal factorization $Ex = P^{(v_P(x))} * A$, where $A \not\subseteq P$, and the commutativity of $G_d(E)$ is proved in Theorem 4.5. v_P may be trivial on Z; in fact, all v_P are trivial on Z (hence K) if and only if $Z = K$. Clearly, Z is the intersection of the valuation rings in K which belong to the v_P, since $E \cap K = Z$. $v_P(x) \neq 0$ for only finitely many P is clear since Ex is a symbolic product of only finitely many primes of E (see Theorem 4.5). These properties characterize Krull domains (see Bourbaki [2]).

5.2. COROLLARY. *Let Z be a (commutative) domain, K its quotient field. Then (Z, K) is a Krull pair if and only if Z is a Krull domain.*

Proof. Specializing 5.1 ($K = R$), we obtain the "only if" contention. Now suppose that Z is a Krull domain. It is easy to see that our concepts of divisorial ideals and modules are equivalent to the usual ones in the commutative case (cf. Bourbaki, in particular [2, Sect. 1, Proposition 1,c]). If β is the Asano class of Z, M_β is the set of "Z-lattices in K." The maximality of Z in β follows from the fact that Krull rings are completely integrally closed (cf. Fossum [5, Chap. 1, Sect. 3]). Trivially, every subring of a commutative ring is bounded, and every nonzero ideal of a domain thick.

We want to point out that even in the commutative case our approach is a bit more general than, say, Bourbaki's, for we did not exclude rings containing divisors of zero. In this case, (E, R) being a Krull pair with R commutative, the usual Krull ring ideal theory holds for thick ideals by Theorem 4.5.

(b) *Asano's Noncommutative Arithmetics*

If E is a bounded maximal Asano order, and R is its Asano quotient ring, then our M_β is the set of fractional left or right ideals of orders Asano-equivalent to E. In fact, Asano [1] and Jacobson [8] show implicitly that these fractional ideals form a module system. The ascending chain condition for two-sided ideals now implies that (E, R) is a Krull pair. By the maximality of prime ideals (one of Asano's axioms) we may apply Proposition 4.7 to obtain Theorem 4.5 with ordinary ideal products, i.e., the fundamental theorems of Asano's on two-sided ideals.

Factorization of one-sided ideals is approached by Asano as follows: By imposing a restricted descending chain condition, the Artin–Wedderburn structure theory is available for factor rings modulo two-sided primes. This yields that these factor rings are principal ideal rings. The factorization theory of principal ideal rings is then used to find the desired ideal theory of E. These methods cannot be applied for a general Krull pair (E, R), because there is no structure theory of the factor ring even available if one imposes descending chain conditions on divisorial one-sided ideals containing a prime of E. However, for a strong Krull pair, Asano's techniques can be replaced to some extent by a closer inspection of the Brandt groupoid and its ordering (Sections 6 and 7).

(c) *Finite Central Extensions of Commutative Krull Domains*

The "orders over Krull domains" as defined in Fossum [6] also yield a wide class of interesting Krull pairs. Fossum considers a simple central algebra R of finite dimension over the quotient field K of a commutative Krull domain Z. (We do not bother with the easy modifications if R is only semisimple, or K is a subfield of the center.) If now H is an order in Fossum's sense (a "Z-lattice" of full rank which is a ring of Z-integral elements), we only reformulate statements of the paper [6] in saying that the Asano class β of H consists of all Z-orders in R and that M_β is identical with the set of Z-lattices (of full rank) in R (cf. 5.4 below). The maximal elements of β are just the "maximal orders." Fossum shows, by a classical discriminant argument, that each order H is a subring of some maximal order E.

5.3. If E is a maximal Z-order (in the sense of [6]), Z a Krull domain, and R the full quotient ring of E, then (E, R) is a strong Krull pair.

Proof. Since E is an arbitrary maximal order, and the boundedness is trivial in central quotient rings, we need only show that E is d-Noetherian. For a hight one prime ideal p of Z write, as usual, $Z_p = \{y^{-1}x;\ x, y \in Z, y \notin p\}$, and for $A \in M_\beta$ let $A_p = Z_p \cdot A$. Using the fact that $E = \bigcap E_p$ (p runs through all hight one prime ideals of Z), and that all these E_p are maximal Z_p-orders, it is not difficult to see that for a divisorial left ideal A of E (in our sense) there holds $A = \bigcap A_p$. From this we obtain, by the arguments proving [5, Theorem 3.6], that E is d-Noetherian, even d-left-Noetherian.

(d) *A Small Infinite Matrix Ring over Krull Domains*

5.4. Let R be a central quotient ring of the ring E and Z be the center of E. Then E is bounded in R. A is in M_β (β the Asano class of E) if and only if there are $z_1, z_2 \in R^* \cap Z$ such that $z_1 E \subseteq A \subseteq z_2^{-1}E$. (If $c \in R^*$, then $c^{-1} = z^{-1}t$ for some $z \in Z \cap R^*$ and $t \in E$. Hence $c^{-1}E = z^{-1}tE \subseteq z^{-1}E^2 = z^{-1}E$, $Ez = zE \subseteq cE$. This argument shows that E is bounded, and may be used to find z_1 and z_2 such that $z_1 E \subseteq uE$, $vE \subseteq z_2^{-1}E$, where u, v are elements in R^* such that $uE \subseteq A \subseteq vE$, existing by 2.3, if $A \in M_\beta$.)

Now let Z be a Krull domain and K be its quotient field. In the ring S of, say, row-finite matrices over K choose a set of matrix units e_{ij} ($1 \leqslant i, j < \infty$), $e_{ij}e_{kz} = \delta_{jk} \cdot e_{iz}$. The unit of S is $1 = \sum_{i=1}^{\infty} e_{ii}$. We consider the subring R of S, consisting of the elements

$$r = x \cdot 1 + \sum_{1 \leqslant i,j \leqslant N} z_{ij}e_{ij}\,,\ x, z_{ij} \in K,\ N \in \mathbb{N};$$

(N may be arbitrary large.) We write this in a shorter form,

$$r = x + \sum x_{ij}e_{ij}\,,$$

tacitly assuming almost all x_{ij} to be 0. We denote R by $K'_{\infty\times\infty}$ and consider the subring $E = Z'_{\infty\times\infty} = \{rx + \sum x_{ij} \cdot e_{ij} \in R$ and $x, x_{ij} \in Z\}$.

Clearly, R is a central quotient ring of E. Hence E is bounded by 5.4. Let β be the Asano class of E, and suppose that there is an order F in β such that $E \subset F$. By Theorem 4.5 we find a $z \in Z \cap R$ such that $E \subset F \subseteq z^{-1}E$. If now $r = x + \sum x_{ij}e_{ij} = \sum y_{ij}e_{ij}$, then for every entry $y = y_{ij}$ of the matrix r we find

$$ye_{11} = e_{1i}re_{j1} \in EFE \subseteq F^3 = F.$$

This implies that all the natural powers, $(ye_{11})^n = y^ne_{11}$, are in F, hence $y^n \in z^{-1}Z$. Since Krull rings are completely integrally closed, this is only possible for $y \in Z$. We have seen $r \in E$, hence E is maximal in β.

Next we want to show that E is d-Noetherian. For this purpose it is necessary to inspect two-sided ideals of E more closely. If $r = x + \sum x_{ij}e_{ij} = \sum y_{ij}e_{ij}$ we define $\varphi(r) = x$, and $D(r) = \sum y_{ij}Z$. $D(r)$ is an ideal of Z, and φ a ring epimorphism $E \to Z$. The kernel U of φ consists precisely of all finite matrices (that is, those having almost all entries 0). U is not thick (no finite matrix is in R^*). For any thick ideal A of E we may define $D(A) = \sum_{r\in A} D(r)$, and $\varphi(A) = \sum_{r\in A} \varphi(r) \cdot Z$. $D(A)$ and $\varphi(A)$ are ideals of Z.

5.5. If A is a two-sided ideal of E then $A = D(A) \cdot U + \varphi(A) \cdot E$. A is thick, iff $\varphi(A) \neq 0$. There holds $\varphi(A) \subseteq D(A)$. A is divisorial iff $D(A) = \varphi(A)$, and $D(A)$ is a divisorial ideal of Z.

Proof. First we show that $D(A) \cdot U \subseteq A$. Let $x \in D(A)$. Then there exist m elements $r^{(l)} = \sum y_{ij}^{(l)}e_{ij} \in A$, such that $\sum_{l=1}^m \sum_{i,j} y_{ij}^{(l)}Z \ni x$. x is a finite sum of elements in the ideals $y_{ij}^{(l)}Z$. For any triple (i, j, l) and $z \in Z$, we have $z \cdot e_{1i}r^{(l)}e_{j1} = z \cdot y_{ij}^{(l)}e_{11} \in A$, hence $x \cdot e_{11} \in A$, and $x \cdot e_{ij} = e_{i1}xe_{11}e_{1j} \in A$. This proves $D(A) \cdot U \subseteq A$.

If $x \in \varphi(A)$ then there exist $r_1, ..., r_m \in A$, such that $x = \sum_{i=1}^m \varphi(r_i) = \varphi(r)$, where $r = \sum_{i=1}^m r_i \in A$. Therefore we have $r = x + \sum_{1\leqslant i,j\leqslant N} x_{ij}e_{ij}$ for some natural number N. Note that all $x \cdot e_{ii} \in A$, since $x = x_{N+1,N+1} \in D(A)$. Similarly $(x + x_{ii})e_{ii}$ and $x_{ij}e_{ij} \in A$, if $1 \leqslant i, j \leqslant N$ and $i \neq j$. This implies $\sum_{1\leqslant i,j\leqslant N} x_{ij}e_{ij} \in A$, and finally $x \in A$. Hence $\varphi(A) \subseteq A$, and $\varphi(A) \cdot E \subseteq A$.

If conversely $r \in A$, then $r - \varphi(r) \in A \cap U$, since $\varphi(r) \in A$. But $A \cap U \subseteq D(A) \cdot U$ and $\varphi(r) \in \varphi(A) \cdot E$ imply $r = (r - \varphi(r)) + \varphi(r) \in D(A) \cdot U + \varphi(A) \cdot E$. In view of the formula $A = D(A) \cdot U + \varphi(A) \cdot E$ the remark on thickness is trivial. $\varphi(A) \subseteq D(A)$ follows from the definitions. Note $\varphi(A)^{-1} \supseteq D(A)^{-1}$.

Next we show $A^{-1} = D(A)^{-1} \cdot E$, if A is thick. ($D(A)^{-1}$, $\varphi(A)^{-1}$ are meant to be inverses in K.) Since E is a maximal order we have $A^{-1} = E/A$. Clearly

$$\begin{aligned} D(A)^{-1} \cdot E \cdot A &= D(A)^{-1} \cdot D(A) \cdot U + D(A)^{-1} \cdot \varphi(A) \cdot E \\ &\subseteq Z \cdot U + \varphi(A)^{-1} \cdot \varphi(A) \cdot E \\ &\subseteq Z \cdot U + Z \cdot E \subseteq E. \end{aligned}$$

It follows that $D(A)^{-1} \cdot E \subseteq A^{-1}$. Conversely, if $r \in A^{-1}$, that is $r \cdot A \subseteq E$; we obtain $r \cdot D(A) \cdot U \subseteq E$. But U, hence $r \cdot D(A) \cdot U$, contains only finite matrices, that is $r \cdot D(A) \cdot U \subseteq U$, hence $U \cdot r \cdot U \cdot D(A) \subseteq U^2 = U$. This yields that for all coefficients y_{ij} of $r = \sum y_{ij}e_{ij}$ there holds $y_{ij}D(A) \subseteq Z$, hence $y_{ij} \in Z/D(A) = D(A)^{-1}$, proving $r \in D(A)^{-1} \cdot E$ and $D(A)^{-1} \cdot E = A^{-1}$.

A similar reasoning shows $\bar{A} = (D(A)^{-1} \cdot E)^{-1} = \overline{D(A)} \cdot E$. A is thick and divisorial, iff $D(A) \cdot U + \varphi(A) \cdot E = \overline{D(A)} \cdot E$. This is only possible if $\varphi(A) = \overline{D(A)}$, which, conversely, implies $\overline{D(A)} = D(A) = \varphi(A)$, and $A = \overline{D(A)} \cdot E$. It is now evident that $D \mapsto DE$ defines a monotone bijective map of the set of divisorial ideals in Z onto the set of divisorial two-sided ideals of $Z'_{\infty\times\infty}$. Hence $Z'_{\infty\times\infty}$ is d-Noetherian. In view of Proposition 5.1 we have proved

5.6. Proposition. *Let Z be a commutative domain with quotient field K. Then the infinite matrix rings $(Z'_{\infty\times\infty}, K'_{\infty\times\infty})$, as defined in this section, form a Krull pair if and only if Z is a Krull ring.*

It should be noted that $Z'_{\infty,\infty}$ is not, as a ring, finitely generated over its center Z. The structure of natural numbers as an index set of matrix units did not enter into the discussion. A similar construction for an arbitrary index set would allow for noncountable generating sets. But $Z'_{\infty,\infty}$ is locally algebraic over Z.

It is an entertaining exercise to compare the results of Section 4, applied to the Krull pair of Proposition 5.6, with a more direct and comprehensive discussion using the formula $A = D(A) \cdot U + \varphi(A) \cdot E$. It can be shown that for $Z'_{\infty,\infty}$ the ascending chain condition for divisorial left ideals does not apply, even if Z is the ring of rational integers.

6. Normal Module Systems

6.1. A module system K is called ***normal*** if (i) $K = K_n$, i.e., every order of K is maximal; (ii) $K = K^*$, i.e., every element of K is invertible; (iii) there is an order E in K such that the bimodule group $G_K(E)$, defined in 1.6, is Abelian and freely generated by the maximal ideals in E.

By Theorems 3.6 and 4.5, and by 4.11 we know that if (E, R) is a Krull pair, and R a left Ore quotient ring of E then the system K of thick divisorial modules in R for which A/A and $A\backslash A$ are d-Noetherian and maximal in the Asano class of E, form a normal module systems. The multiplication, then, is the symbolic one. In this section anything is done in the abstract system K. Therefore, we write the product of A, B in K as AB, $A \cdot B$, or sometimes $A \circ B$, if it is proper. In applying the results to a Krull pair, the asterisks have to be inserted. Note that what is here called, say, a left ideal of E, is to be taken with respect to the system K, and means in the application to the Krull system (E, R) a "thick divisorial left ideal of E."

In the rest of this paper, K denotes a normal module system. K carries two kinds of structure: the original module system structure (it is a special kind of partially ordered semigroup), and the Brandt groupoid structure (that is a proper multiplication which is not everywhere defined). In our applications the Brandt groupoid was derived from the module system. Here we do a partial converse for a normal system. We want to describe the basic operations of the module system in terms of the Brandt groupoid operations, which are $(A, B) \to (A \circ B, A^{-1})$.

We first describe the partial ordering, which is done by singling out the "positive" elements, in this case called "integer."

6.II. $A \in K$ is called *integer* iff $A^2 \subseteq A$. (This is equivalent to A being contained in its left (or right) order $E = A/A$, or to A being an ideal in E.)

6.1. If $AB = A \circ B$ and A, B are integers, then $A \circ B$ is an integer. (For $A \circ B \subseteq (A\backslash A) \circ B = (B/B)B = B \subseteq B\backslash B$, and $B\backslash B$ is the right order of AB. This may be wrong if the product is not proper.)

6.III. If E and F are orders of K then $D_{EF} = (FE)^{-1}$ is called *distance* (from E to F). We reserve the letter D for distances. $FE \supseteq E$ implies $D_{EF} \subseteq E$, hence D_{EF} is an integer and a left E; right F-ideal. Note $D_{EE} = E$.

6.2. Let X, $Y \in K$. Then $X \subseteq Y$ if and only if there are integers $A, B \in K$ such that $A \circ Y \circ B = X$. If, in addition, $X/X = Y/Y$ then we may take $A = Y/Y$, hence $Y \circ B = X$.

Proof. It is convenient to indicate the left and right orders by subscripts.

6.IV. For any $A \in K$ write $A = A_{GH}$ if and only if $A/A = G$ and $A\backslash A = H$. Let $X = X_{EF}$, $Y = Y_{GH}$, $X \subseteq Y$. Then $A = D_{EG}$, $B = Y^{-1}X$ will do, for $B = B_{HF}$ implies $AYB = D_{EG}GX = (GE)^{-1}.(GE) \cdot X = EX = X$ and $B = Y^{-1}X \subseteq Y^{-1}Y = H$ shows B to be an integer.

Conversely, if $A \circ Y \circ B = X$, with A, B integers, then $A = A_{GE} \subseteq E$, $B = B_{FH} \subseteq F$, thus $Y = AXB \supseteq EXF = X$.

6.3. COROLLARY. *If $X \subseteq Y$, and Y is an integer, then X is an integer.*

6.4. COROLLARY. *$A \in K$ is an integer if and only if A is contained in some order.*

Next we describe multiplication and residuation in terms of Brandt groupoid operations.

6.5. If $A, B \in K$, and $A = A_{EF}$, $B = B_{GH}$, then

$$AB = A \circ D_{GF}^{-1} \circ B, \qquad A/B = A \circ D_{FH} \circ B^{-1}.$$

Proof. The first equation is trivial. Since $B^{-1}B = H$, we see that $AD_{FH}B^{-1}B = A \circ D_{FH} \subseteq A$, showing $A \circ D_{FH} \circ B^{-1} \subseteq A/B$. On the other hand, we conclude from $(A/B)B \subseteq A$ that $A = AF \supseteq (A/B)\,BHF = (A/B)\,BD_{FH}^{-1}$, finally $AD_{FH}B^{-1} \supseteq (A/B)\,BD_{FH}^{-1}D_{FH}B^{-1} = (A/B)\,BHB^{-1} \supseteq (A/B)$.

6.6. Corollary. *If E, F are orders then $E/F = D_{EF}$.*

Hence if we "know" the set of distances, the module system operations can be expressed by those of the Brandt groupoid.

6.7. If $A = A_{EF}$, and G, H are orders, then (G, H) contains a unique smallest X containing A, and a unique largest Y contained in A, namely, $X = GAH = D_{EG}^{-1} \circ A \circ D_{HF}^{-1}$, and $Y = G\backslash A/H = D_{GE} \circ A \circ D_{FH}$.

Proof. For the equations see 6.5. That X has the required properties is clear since G is a left unity for (G, H). $Y \supseteq A$ follows from 6.2 and the remark following 6.III. Clearly $Y \in (G, H)$. Now, if $Z = Z_{GH} \subseteq A$, then $D_{GE}^{-1}ZD_{FH}^{-1} = EGZHF = EZH \subseteq EAH = A$, hence $Z = D_{GE}D_{GE}^{-1}ZD_{HF}^{-1}D_{FH} \subseteq D_{GE}AD_{FH} = Y$.

6.8. Corollary. *If $A = A_{EF}$, then A has a unique integral proper minimal left factor in $G(E)$, containing A, namely AE, and then $A = (AE) \circ D_{EF}$. A is an integer if and only if AE is an integer.*

Proof. $A = AE \circ D_{EF}$ and the first assertion only specializes 6.7. If AE is an integer so is A by Corollary 6.3. If A is an integer then $A \subseteq E$, thus $AE \subseteq E$, and AE is an integer too.

Next we characterize distances.

6.9. Proposition. *Let $A \in K$, and define E, F by $A = A_{EF}$. Then the following statements on A are equivalent:*

(i) *A is a distance (i.e., $A = D_{EF}$);*

(ii) *A is the unique largest integer in (E, F);*

(iii) *$A \subseteq E$, and there is no integer $C \in G(E)$, such that $A \subseteq C \subset E$;*

(iv) *A is an integer, and if for an integer $B = B_{GG}$ there holds $A \subseteq B$, then B is an order (i.e., $B = G$);*

(v) *$A(A/A) = A/A$;*

(vi) *$A^2 = A$.*

Proof. By Corollary 6.8, $AE = E$ and $A = D_{EF}$ are equivalent, showing that (i), (ii), and (v) are equivalent. $AE = E$ can be written $AAA^{-1} = AA^{-1}$ which is equivalent to $A^2 = AAA^{-1}A = AA^{-1}A = A$, that is (vi). (iv) entails (iii), and (iii) entails (i). Last, we show that (iii) entails (iv). If B is as supposed in (iv) then by 6.2 there exist integers X, Y such that $A = X \circ B \circ Y =$

$X \circ B \circ X^{-1} \circ (X \circ Y) \subseteq XBX^{-1}$. But XBX^{-1} is an integer element of $G(E)$. By (iii) we get $XBX^{-1} = E$, or $B = X^{-1}EX = X^{-1}X = BB^{-1} = G$.

6.10. COROLLARY. *If $A \subseteq C$, C is an integer, and A a distance, then C is a distance.*

6.11. PROPOSITION. *If E, F are orders in K and $B \in (E, F)$ then $A \to B^{-1} \circ A \circ B$ is an isomorphism of lattice ordered groups $G(E) \to G(F)$, which does not depend on $B \in (E, F)$.*

Only the independence from B needs a proof.

Using commutativity, $A = A \circ (D_{EF}B^{-1})(D_{EF}B^{-1})^{-1} = (D_{EF}B^{-1})^{-1}AD_{EF}B^{-1} = B \circ D \circ A_{EF}^{-1} \circ D_{EF} \circ B^{-1}$, hence $B^{-1}AB = D_{EF}^{-1}AD_{EF}$, as required.

6.V. We form the abstract lattice-ordered group $\mathbb{G}$ of those classes $\{B^{-1}AB, B/B = E\} = :\Delta$. Then there is, for each order F, one and only one $\Delta_F \in \Delta \cap G(F)$, and $\Delta \to \Delta_F$ is an isomorphism of lattice-ordered groups.

It is convenient to let $\mathbb{G}$ act on K from the right and left, defining, if $X = X_{GH}$, $\Delta X = \Delta_G \circ X$, $X\Delta = X \circ \Delta_H$.

6.12. In the action just defined $\mathbb{G}$ centralizes K, that is $\Delta X = X\Delta$ for each $X \in K$.

Proof.

$$\Delta X \Delta^{-1} = \Delta_G X \Delta_H^{-1} = D_{GH}\Delta_H(D_{GH}^{-1} X \Delta_H^{-1})$$

$$= D_{GH}(D_{GH}^{-1} X \Delta_H^{-1}) \Delta_H = X.$$

6.VI. We define two functions $\Phi, \Psi: K \to \mathbb{G}$ by setting $A = \Psi A \cdot D_{EF} = \Phi A \cdot D_{FE}^{-1}$ for every $A \in (E, F)$. We call Φ *lower bound* and Ψ *upper bound*.

By 6.7 and Corollary 6.8 $(\Phi A)_E$ is the unique largest E-bimodule contained in A, $(\Phi A)_F$ the unique largest F-bimodule contained in F, $(\Psi A)_E$ the unique smallest E-bimodule containing A, and $(\Psi A)_F$ the unique smallest F-bimodule containing $A = A_{EF}$. Note $\Phi(A^{-1}) = (\Psi A)^{-1}$. A is *an* integer iff ΨA is *an* integer, and A is a distance iff $\Psi A = 1$ (see Corollary 6.8 and Proposition 6.9).

6.13. If $A \subseteq B$, then $\Phi A \subseteq \Phi B$ and $\Psi A \subseteq \Psi B$.

Proof. This is clear if A and B have the same right (or left) order. If not, then there exist integers X, Y such that $A = X \circ B \circ Y \subseteq BY \subseteq B$. A and BY have the same right order, BY and B the same left order.

6.14. Let $A, B \in K$, $A \in (E, F)$, $B \in (G, H)$. Then there holds

(i) $(\Phi A \cdot \Phi B)_E = ABD_{HG}D_{GF}D_{FE} = \Phi(AB) \cdot D_{HE}^{-1}D_{HG}D_{GF}D_{FE}$,

(ii) $(\Psi A \cdot \Psi B)_E = ABD_{GH}^{-1}D_{GF}D_{EF}^{-1} = \Psi(AB) \cdot D_{EH}D_{GH}^{-1}D_{GF}D_{EF}^{-1}$.

Proof. By 6.V and 6.11 we find

$$(\Phi A)_E \cdot (\Phi(B))_E = AD_{FE}(D_{GF}D_{FE})^{-1}\, BD_{HG}(D_{GF}D_{FE})$$
$$= AF(FG)\, BD_{HG}D_{GF}D_{FE} = ABD_{HG}D_{GF}D_{FE}.$$

(ii) is proved similarly.

6.15. Corollary. *If $A, B \in K$, then $\Phi A \cdot \Phi B \subseteq \Phi(AB)$. If, in addition, $AB = A \circ B$, then $\Psi A \cdot \Psi B \supseteq \Psi(AB)$.*

Proof. Since $D_{HG}D_{GF}D_{FE}$ is a proper product of integer modules, it is an integer, and 6.14 shows $(\Phi A \cdot \Phi B)_E \subseteq AB$. $(\Phi(AB))_E$ is the largest E-bimodule contained in AB, hence $(\Phi A \cdot \Phi B)_E \subseteq (\Phi AB)_E$. From this we derive $\Psi A \cdot \Psi B \supseteq \Psi(AB)$ using the formulas $(\Phi X^{-1})^{-1} = \Psi(X)$, and $B^{-1} \circ A^{-1} = (A \circ B)^{-1}$.

6.16. Corollary. *If $A, B \in K$, $A \in (E, F)$, $B \in (G, H)$, then*

(i) *$\Phi A \cdot \Phi B = \Phi(AB)$ is equivalent to $FG \subseteq EH$.*

(ii) *$\Psi A \cdot \Psi B = \Psi(AB)$ is equivalent to $D_{EH}D_{GF} = D_{EF}$, or $HD_{GF}E = HE$.*

If, in addition, $AB = A \circ B$, then $\Phi A \cdot \Phi B = \Phi(AB)$, and $\Psi A \cdot \Psi B = \Psi(AB)$ is equivalent to $F \subseteq EH$, and $F \subseteq HE$. ($D_{HE}^{-1}D_{GH}D_{GF}D_{FE}$ is equivalent to $EH = EFGH$, or $FG \subseteq EH$.) These properties of upper and lower bounds are used in Section 7. The next proposition is announced for its own sake.

6.17. Proposition. *Let E, F be orders of the normal module system K. Then $A \to A\backslash A$ maps the set of left E-ideals (in K) which contain the distance D_{EF} bijectively onto the set of orders contained in FE.*

Proof. $D_{EF} \subseteq A \subseteq E$, $E = A/A$ implies $A = D_{EG}$, $D_{EF} = D_{EG} \circ D_{GF}$, where $G = A\backslash A$. Hence $1 = \Psi D_{EF} = 1 \cdot 1 = \Psi D_{EG} \cdot \Psi D_{GF}$, and by Corollary 6.16 we see $G \subseteq FE$. Conversely, if $G \subseteq FE$, we conclude $GE \subseteq FE$, $D_{EG} \supseteq D_{EF}$. Thus $G \to D_{EG}$ is the inverse map of $A \to A\backslash A$.

7. Factorization of One-Sided Ideals

Let (E, R) be a strong Krull pair and K be the Brandt groupoid of thick divisorial modules with left-order maximal in the Asano-class of E. First we want to look at lattice properties of K. If A, $B \in K$, the set theoretic intersection $A \cap B$ may be not in K. But we have the following

7.1. If $A, B \in K$ have the same left (or right) order F then $A \cap B$ and $\overline{A + B} \in K$.

Proof. By 4.6 there are (two-sided) thick divisorial ideals X, Y, U, V of F such that $X \subseteq A$, $Y \subseteq B$, $U \subseteq A^{-1}$, $V \subseteq B^{-1}$, hence $X * Y \subseteq X \cap Y \subseteq A \cap B \subseteq A + B \subseteq \overline{A + B} \subseteq U^{-1} * V^{-1}$; 2.3 yields $A \cap B$, $A + B$, $\overline{A + B} \in M_\beta$. Because of $A \cap B \subseteq \overline{A \cap B} \subseteq \bar{A} \cap \bar{B} = A \cap B$, the module $A \cap B$ is divisorial.

7.2. PROPOSITION. *Let (E, R) be a strong Krull pair. For any maximal element F of the Asano class α of E the set ${}_FK$ of left F-modules in $K = B_d(\alpha_d)$ forms a lattice if meet and join are defined by $A \cap B$ to be the set intersection, and $A \cup B = \overline{A + B}$. If $A, B \in {}_FK$ and $C \in K$, then $(A \cup B) * C = (A * C) \cup (B * C)$ and $C * (A \cup B) = (C * A) \cup (C * B)$.*

Proof. By 7.1 it is easily seen that ${}_FK$ is a lattice-ordered semigroup under symbolic multiplication. For additive subgroups of R we have $(A + B)C = AC + BC$ hence, for $A, B \in {}_FK$ and $C \in K$, $\overline{AC} + \overline{BC} \subseteq \overline{(A + B)C} = (A + B) * C \subseteq (A \cup B) * C$ holds, which implies $(A * C) \cup (B * C) \subseteq (A \cup B) * C$.

Using 2.14, we find, on the other hand, $(A * C) \cup (B * C) \supseteq \overline{AC + BC} = \overline{(A + B)C} \supseteq \overline{(\overline{A + B})C} = (A \cup B)C$, which implies $(A * C) \cup (B * C) \supseteq (A \cup B) * C$. We have shown $(A \cup B) * C = (A * C) \cup (B * C)$. If C is on the left side we must not apply 2.14 because we do not know whether $A + B$ has a maximal right order. But then, for any $X \in {}_FK$ we may write $C * X = C \circ D^{-1} \circ X$, where D is the distance $D = D_{FG}$, $G = C\backslash C$ (see 6.5), and conclude as follows:

$$\begin{aligned} C * (A \cup B) &\supseteq (C * A) \cup (C * B) = (C * D^{-1} * A) \cup (C * D^{-1} * B) \\ &= (C * C^{-1}) * ((C * D^{-1} * A) \cup (C * D^{-1} * B)) \\ &\supseteq C * ((C^{-1} * C * D^{-1} * A) \cup (C^{-1} * C * D^{-1} * B)) \\ &= C * ((D^{-1} * A) \cup (D^{-1} * B)) \supseteq C * (A \cup B). \end{aligned}$$

7.I. We call a normal module system K a *normal lattice system* if it has the properties which are proved in Proposition 7.2 for the system $K = B_d(\alpha_d)$.

In order to spare asterisks we operate (in the rest of this section) in a normal lattice system K and write multiplication in K without on asterisk. All results are, after inserting the asterisks, propositions on thick divisorial ideals (and modules) of the system $K = B_d(\alpha_d)$ of a strong Krull pair.

We investigate now how the prime factorization of the universal bimodule group $\mathbb{G}$ of K can be used to obtain factorizations for the elements of K.

7.II. (i) If $A_i \in K$ ($i = 1,\ldots, r$) the product $A_1 \cdot \cdots \cdot A_r$ is called *smooth* if $\Phi(A_1 \cdots A_r) = \Phi A_1 \cdots \Phi A_r$ and $\Psi(A_1 \cdots A_r) = \Psi A_1 \cdots \Psi A_r$.

(ii) An element $Q \in K$ is called *smoothly atomic* if Q is an integer, not an order, and for any smooth proper product $Q = A \circ B$ with integers A, B there holds $A = Q$ or $B = Q$. (If $A = Q$, this implies $\Phi B = 1$, hence B must be an order; in fact, $B = Q^{-1}Q$.)

It turns out that every integer A can be written as a proper smooth product of smoothly atomic factors and the smoothly atomic elements can be characterized (Theorem 7.6 below). Proposition 7.7 says to what extent this factorization is unique. We make some preparatory observations.

It should be noted that ABC is a smooth product if AB and BC are so. Inserting two-sided factors in proper products does not affect smoothness, as expressed by

7.3. Let $A \circ B$ be a smooth product, $A = A_{EF}$, and $X \in G(F)$. Then $A \circ X \circ B$ is a smooth product.

Proof. Let $\Phi X = \Delta$. Then, since X is two-sided, we have $\Delta = \Psi X$, and $A \circ X \circ B = (A\Delta)B = \Delta(AB)$. It follows from Corollary 6.16 that for any $U \in K$, $\Delta \in \mathbb{G}$ there holds $\Phi(\Delta U) = \Delta\Phi(U)$, and $\Psi(\Delta U) = \Delta\Psi(U)$. From $\Phi(AB) = \Phi(A)\,\Phi(B)$ we conclude, therefore, that $\Phi(AXB) = \Phi(\Delta AB) = \Delta\Phi(AB) = \Delta \cdot \Phi A \cdot \Phi B = \Phi A \cdot \Delta \cdot \Phi B = \Phi A \cdot \Phi X \cdot \Phi B$, and similarly $\Psi(AXB) = \Psi A \cdot \Psi X \cdot \Psi B$.

7.4. Let $A = A_{EF} \in K$, $\Phi A = \Gamma\Delta$, Γ, Δ integer elements of the universal bimodule group $\mathbb{G}$ of the normal lattice system K. Then $A = \Gamma_E \Delta_F$ (which is not smooth for $E \neq F$).

Proof. Use 6.5, 6.12, and 6.V in order to obtain $\Gamma_E\Delta_F = \Gamma_E D_{FE}^{-1}\Delta_F = \Gamma D_{FE}^{-1}\Delta = \Gamma\Delta D_{FE}^{-1} = \Phi A \cdot D_{FE}^{-1} = A$.

7.5. Let A, Γ, Δ be as in 7.4, and A be a distance. Then $A = (A \cup \Delta_E)(A \cup \Gamma_F)$ which is a smooth (but not necessarily proper) product.

Proof. Note $\Delta_E A \subseteq A$, $A\Gamma_F \subseteq A$, $\Delta_E\Gamma_F = A = A^2$ (see 6.2, 7.4). Using Proposition 7.2, we find $(A \cup \Delta_E)(A \cup \Gamma_F) = A^2 \cup \Delta_E A \cup A\Gamma_F \cup \Delta_E\Gamma_F = A$, since $A \subseteq A \cup \Delta_E \subseteq E$, $A \subseteq A \cup \Gamma_F \subseteq F$, $A \cup \Delta_E$, and $A \cup \Gamma_F$ are distances (6.10). As for smoothness, we show $\Phi(A \cup \Delta_E) = \Delta$ and $\Phi(A \cup \Gamma_F) = \Gamma$. By 6.13 we see $\Phi(A \cup \Delta_E) \supseteq \Delta$, and $\Phi(A \cup \Gamma_F) \supseteq \Gamma$. Using 6.15 we obtain $\Delta\Gamma = \Phi A = \Phi((A \cup \Delta_E)(A \cup \Gamma_F)) \supseteq \Phi(A \cup \Delta_E)\,\Phi(A \cup \Gamma_F) \supseteq \Delta\Gamma$. This is only possible if $\Phi(A \cup \Delta_E) = \Delta$, and $\Phi(A \cup \Gamma_F) = \Gamma$.

Now we can prove the following fundamental facts:

7.6. Theorem. *Let K be a normal lattice system. Then* (i) *holds*: $Q \in K$ *is smoothly atomic if and only if Q is an integer and ΦQ is a prime.* (*In other words,* *a left ideal Q of the order E is smoothly atomic if and only if it contains a two-sided prime ideal of E, and is different from E. We say "Q belongs to ΦQ.")*

(ii) *If $A \in K$ is an integer, and $\Phi A = \Pi_1 \cdot \cdots \cdot \Pi_r$, where the Π_i ($i = 1,..., r$) are (not necessarily different) primes of $\mathfrak{G}$, then there exist smoothly atomic elements $Q_1,..., Q_r$ of K, such that $A = Q_1 \cdot \cdots \cdot Q_r$, and $\Phi(Q_i) = \Pi_i$ ($i = 1,..., r$). (The Q_i depend on the sequel in which the Π_i are taken. But since A determines ΦA uniquely, the different primes of $\mathfrak{G}$ to which the atomic factors belong and their multiplicities are the same for any such factorization.)*

Proof. (i) If Q is integer, ΦQ is a prime, and $Q = A \circ B$ is a smooth product of integers $A, B \in K$, then $\Phi Q = \Phi(AB) = \Phi A \Phi B$ implies, say, $\Phi A = 1$, $\Phi B = \Phi Q$. Since A is an integer, $\Phi A = 1$ entails that A is an order, namely, the left order of $A \circ B = Q$. Hence, $B = Q$, and Q is shown to be smoothly atomic.

If $A = D_{EF} \in K$ is a distance, not an order, and ΦA is not prime, then for some prime Π and integer $\Gamma \neq 1$ we have $\Phi A = \Pi\Gamma$. From 7.5 we obtain $A = QX$ where $Q = A \cup \Pi_E$, $X = A \cup \Gamma_F$, $\Phi Q = \Pi$, $\Phi X = \Gamma$. Observing $A \subseteq Q \subseteq E$, we find (by 6.2) an integer $Y \in K$ such that $A = Q \circ Y$, namely, $Y = Q^{-1} \circ A$. Then $Y = Q^{-1}QX \supseteq X$, hence $\Phi Y \supseteq \Phi X = \Gamma$, and $\Pi\Gamma = \Phi(QY) \supseteq \Phi Q \cdot \Phi Y \supseteq \Pi\Gamma$ (see 6.15). This is only possible, if $\Phi Y = \Phi X = \Gamma$, hence $\Phi(Q \circ Y) = \Phi(Q) \cdot \Phi(Y)$. Since A is a distance we have $\Psi A = 1$. From $A \subseteq Q \subseteq E$, $A \subseteq Y \subseteq F$ we obtain, using 6.15, that $\Psi Q = 1$, $\Psi Y = 1$, hence the product $A = Q \circ Y$ is smooth. This shows that A is not smoothly atomic. If $A = \Psi A \cdot D_{EF}$ is an integer, not a distance and not an order, then $\Psi A \subset 1$. $A = (\Psi A)_E \circ D_{EF}$ is smooth by 7.3. Hence, in this case, A is smoothly atomic, iff $E = F$, and $(\Psi A)_E = (\Phi A)_E = A$ is prime: (i) is proved.

As to (ii), we note first that we may assume A to be a distance: if $A = \Psi A \cdot D_{EF}$ then $A = (\Psi A)_E \circ D_{EF}$ is a smooth product. If (ii) is known for D_{EF}, that is, $D_{EF} = Q_1 \circ \cdots \circ Q_r$, Q_i smoothly atomic, and $\Psi A = \Pi \cdot \Gamma$, Π prime, Γ integer, then for any i ($i = 1,..., r - 1$) we have $A = (\Gamma \cdot Q_1 \cdot \cdots \cdot Q_i) \cdot (\Pi Q_{i+1} \cdot \cdots \cdot Q_r)$ by 6.12, and if G is the right order of Q_i we may write $A = \Gamma \cdot (Q_1 \circ \cdots \circ Q_i \circ \Pi_G \circ Q_{i+1} \circ \cdots \circ Q_r)$. Π_G is smoothly atomic and belongs to Π, hence the factor right of Γ is written as a smooth product of smoothly atomic factors (see 7.3). Assuming we can start with an arbitrary sequel of Π_i such that $\Phi(Q_i) = \Pi_i$, we may insert the prime factors of ΨA at any position we want, and are done.

In order to prove that (ii) is correct for $A = D_{EF}$ being a distance, we observe that we have already found a smooth factorization $A = Q \circ Y$, where $\Phi Q = \Pi$ was an arbitrary prime factor of ΦA. $\Phi Y = \Pi^{-1}\Phi(A)$ has one prime factor less than ΦA, and Y is a distance. Hence, by induction, Y has a factorization $Y = Q_2 \circ \cdots \circ Q_r$ as required in (ii), and $A = Q \circ Q_2 \circ \cdots \circ Q_r$ is the desired factorization of A.

The factorization of Theorem 7.6(ii) is generally not unique. For two factors we describe which factors are possible.

7.7. PROPOSITION. *Let* $A = A_{EF} \in K$ *be an integer,* $\Phi A = \Gamma\Delta$, $\Gamma, \Delta \subsetneq 1$. *Then* (i) *and* (ii) *are equivalent statements for an* $U \in K$:

(i) $A \cup \Gamma_E \subseteq U \subseteq A(A \cup \Delta_F)^{-1}$, *and* $UU^{-1} = E$.

(ii) *There exists a* $V \in K$ *such that* $A = U \circ V$, U, V *are integers and* $\Phi U = \Gamma$, $\Phi V = \Delta$.

Proof. Assume (ii). From $\Phi U = \Gamma$ we conclude $\Gamma_E \subseteq U$, and since $A = U \circ V \subseteq U$ this implies $A \cup \Gamma_E \subseteq U$. Similarly $A \cup \Delta_F \subseteq V$, hence $U = AV^{-1} \subseteq A(A \cup \Delta_F)^{-1}$. $AA^{-1} = UU^{-1}$ is clear.

Assume (i). Define V by $V = U^{-1}A$.

$A \subseteq U$ implies V to be integer. We have $V = U^{-1}A \supseteq (A(A \cup \Delta_F)^{-1})^{-1}A = A \cup \Delta_F$. Hence by 6.13, $\Phi V \supseteq \Delta$, $\Phi U \supset \Gamma$, and 6.15 yields $\Delta\Gamma = \Phi A = \Phi(UV) \supseteq \Phi U \cdot \Phi V = \Delta\Gamma$. This is only possible if $\Phi U = \Delta$, and $\Phi V = \Gamma$.

By 7.7 cases of unique smooth factorizations are pointed out.

7.8. COROLLARY. *If* $A = A_{EF} \in K$ *is an integer,* $\Phi A \subsetneq \Pi$, *and* $A \cup \Pi_E$ *is a maximal left ideal of* E, *then* $Q = A \cup \Pi_E$ *is the unique smoothly atomic proper left factor of* A *belonging to* Π.

Proof. Applying 7.7 with $\Gamma = \Pi$, $\Delta = \Pi^{-1}\Phi A$, we have for any atomic smooth proper left factor U that $A \cup \Pi_E \subseteq U \subseteq A(A \cup \Delta_F)^{-1}$ holds. $A(A \cup \Delta_F)^{-1}$ is a left ideal of E, and different from E, for $E = A(A \cup \Delta_F)^{-1}$ would imply $A = E(A \cup \Delta_F) \supseteq A \cup \Delta_F$, and $\Phi A \supseteq \Phi(A \cup \Delta_F) \supseteq \Delta$, which is impossible. The maximality of $A \cup \Pi_E$ entails $A \cup \Pi_E = U = A(A \cup \Delta_F)^{-1}$.

7.9. PROPOSITION. *Let* $A_{EF} = A = Q_1 \circ \cdots \circ Q_r$ *such that* (i) *all* Q_i *are maximal left ideals in their left order,* (ii) $\Phi Q_i = \Pi$ $(i = 1,..., r)$, *and* (iii) $\Phi A = \Pi^r$. *Then, if* $A = Q_1' \circ \cdots \circ Q_s'$, *with all* Q_i' $(i = 1,..., s)$ *smoothly atomic, it follows that* $r = s$, *and* $Q_1 = Q_1'$, $Q_2 = Q_2'$,..., $Q_r = Q_r'$. *If further* $A \subseteq B \subset F$, *and* B *is a right ideal of* F, *then, for some* i, *there holds* $B = Q_i \circ Q_{i+1} \circ \cdots \circ Q_s$.

Proof. First we show $A \cup \Pi_E = Q_1$. From 7.7 we know $A \cup \Pi_E \subseteq Q_1 \subseteq A(A \cup \Pi_F^{r-1})^{-1}$. If $A' = Q_2 \cdot \cdots \cdot Q_r$, then $\Phi A' \supseteq \Phi Q_2 \cdot \cdots \cdot \Phi Q_r = \Pi^{r-1}$, and $\Pi^r = \Phi(A) \supseteq \Pi \cdot \Phi(A')$, hence $\Phi A' = \Pi^{r-1}$. Since Proposition 7.9 is trivial for $r = 1$, we may assume $r > 1$ and by induction on r, that Proposition 7.9 is proved for A' instead of A. Clearly $A' = Q_1^{-1}A \subseteq (A \cup \Pi_E)^{-1}A \subseteq A^{-1}A = F$, hence by Proposition 7.9 (applied to A' and $B' = (A \cup \Pi_E)^{-1}A$), we find some $i \geqslant 2$, such that $B' = (A \cup \Pi_E)^{-1}A = Q_i \circ \cdots \circ Q_r$. Now $\Pi^r = \Phi A = \Phi((A \cup \Pi_E)B') \supseteq \Pi \cdot \Pi^{r-i+1}$, hence $r \leqslant r - i + 2$, $i \leqslant 2$, that is, $i = 2$, and $B' = Q_2 \circ \cdots \circ Q_r$. From $A = Q_1 \circ B' = (A \cup \Pi_E) \circ B'$ we conclude $Q_1 = A \cup \Pi_E$, as desired. In particular, $A \cup \Pi_E$ is a maximal left ideal of E, and 7.8 shows that $Q_1 = A \cup \Pi_E = Q_1'$, and $B' = Q_2' \circ \cdots \circ Q_s'$. Applying the induction hypothesis once more, we obtain $s = r$, and $Q_2 = Q_2',\ldots, Q_r = Q_r'$. Finally, if $A \subseteq B \subset F$, B a right ideal of F, the contention is trivial if $A = B$ $(i = 1)$. If $A \subset B$, there exists an integer $Y \in K$ such that $A = Y \circ B$, and Y is not an order. Then $Y \cup \Pi_E$ is not an order because, by 7.7, we have $\Phi(Y \cup \Pi_E) = \Pi$. (Note $\Phi Y \subsetneq \Pi$, since Y is not an order, and $\Phi(\Pi_E) \subsetneq \Pi$ imply $\Phi(Y \cup \Pi_E) \subsetneq \Pi$.) We conclude $A \cup \Pi_E \subseteq Y \cup \Pi_E \subset E$, and $Q_1 = A \cup \Pi_E = Y \cup \Pi_E \supseteq Y$. Hence there exists an integer $Z \in K$ such that $Y = Q_1 \circ Z$, and therefore $Q_1 \circ A' = Q_1 \circ Z \circ B$. This implies $A' = Z \circ B \subseteq B$, hence by the induction hypothesis, $B = Q_i \circ \cdots \circ Q_r$ $(i \geqslant 2)$, and Proposition 7.9 is proved. Note that if A is as in Proposition 7.9, there is a unique fine chain of left F-ideals from A to F.

*We want to apply Proposition 7.9 to the following: Let Z be a Dedekind ring, K be its quotient field, and R be a simple central K-algebra of K-dimension 4. Let E be a maximal Z-order (see 5c). For a prime ideal p of Z we suppose that p does not ramify, and Z mod p is finite. Then $pE = P$ is a prime ideal of E, and the factor ring modulo P is a full ring at 2×2 matrices over a field. If Π is the prime of $\mathfrak{G}$ such that $\Pi_E = pE = P$ then for any maximal order F there holds: Every smoothly atomic left ideal Q of F, such that $\Phi Q = \Pi$, is either pF or a maximal left ideal of F (because in the ring of 2×2 matrices over a field all proper left ideals are maximal). Now consider a left ideal A of E such that $\Phi A = \Pi^r$ for some $r \geqslant 1$. (This is equivalent to the norm of A being a power of p.) From $A = \Psi A \cdot D_{EF}$ and $\Pi^r \subseteq \Psi A \subseteq 1$, it follows that A is a distance if and only if $P = pE$ is not a proper left factor of A (for this implies $\Psi A = 1$). Now factorize A smoothly according to Theorem 7.6: $A = Q_1 \circ \cdots \circ Q_r$. A is a distance iff none of the Q_i is two-sided, hence iff all Q_i are maximal left ideals of their left orders.

By Proposition 7.9, *the tuple* $(Q_1,\ldots, Q_r)$ is *uniquely determined.* Together with Theorem 7.11 and the fact that the factorization of ideals A with ΦA a power of a ramified prime is trivial (namely commutative), this provides a very full insight in the factorization of left ideals of E.* If the K-dimension of R is >4, the uniqueness statement of Proposition 7.9 applies only to part of the set of distances, because, as examples show, in Proposition 7.7(i) we need not have equality. But note the following:

7.10. Proposition. *Suppose in Proposition 7.7 there holds, in addition, $\Gamma \cup \Delta = 1$ (that is, Γ, Δ are relatively prime). Then there holds $A \cup \Gamma_E = A(A \cup \Delta_F)^{-1}$, $A = (A \cup \Gamma_E) \circ (A \cup \Delta_F)$, and, if $A = U \circ V$, U, $V \in K$ integers, and $\Phi U \cup \Delta = 1$, $\Phi V \cup \Gamma = 1$, then $U = A \cup \Gamma_E$, $V = A \cup \Delta_F$.*

Proof. From the assumptions we get $A \subseteq U$, and $\Gamma\Delta = \Phi A \subseteq \Phi U$. This implies, since ΦU and Δ are relatively prime, that $\Phi U \supseteq \Gamma$, and similarly $\Phi V \supseteq \Delta$. Now we find $\Gamma\Delta = \Phi A \supseteq \Phi U \cdot \Phi V \supseteq \Gamma\Delta$. This entails $\Phi U = \Gamma$, $\Phi V = \Delta$. Hence U, V are as required in Proposition 7.7(ii), and therefore

$A \cup \Gamma_E \subsetneq U$. There exists an integer $Z \in K$ such that $A \cup \Gamma_E = U \circ Z \subsetneq Z$, which implies $\Gamma \subseteq \Phi(A \cup \Gamma_E) \subseteq \Phi Z$. Comparing $A = U \circ Z \circ (A \cup \Gamma_E)^{-1} \circ A$ with $A = U \circ V$ we conclude $V = Z \circ (A \cup \Gamma_E)^{-1} A \subsetneq Z$, since $(A \cup \Gamma_E)^{-1} A$ is an integer. $V \subsetneq Z$ yields $\Phi Z \supsetneq \Phi V = \Gamma$, $\Phi Z \supsetneq \Gamma \cup \Delta = 1$. Since Z is an integer, it must be an order, and $(A \cup \Gamma_E) = U \circ Z = U$. We can argue similarly for any U in Proposition 7.7(ii) with Γ, Δ as in Proposition 7.10, in particular for $U = A(A \cup \Delta_F)^{-1}$, and we see $U = A \cup \Gamma_E = A(A \cup \Delta_F)^{-1}$. This shows that the right order of $A \cup \Gamma_E$ is the left order of $A \cup \Delta_F$, and from $A = U \circ V = U \circ (A \cup \Delta_F)$ we get $V = A \cup \Delta_F$. Proposition 7.10 allows us to establish some sort of unique multiplicative "primary" decomposition.

7.III. $X \in K$ is called *primary* (a primary module) it for some prime $\Pi \in \mathbb{G}$ and rational integers m, n there holds $\Pi^n \subsetneq X \subsetneq \Pi^m$, that is, ΦX and ΨX are in the subgroup of $\mathbb{G}$ generated by the prime Π (we refer to X as Π-primary).

This concept is not to be confused with primary ideals of rings in the usual sense but the concepts coincide in the case of ideals of Dedekind rings.

7.11. THEOREM (Multiplicative primary decomposition). *Let K be a normal lattice system, $A \subsetneq K$, A not an order. Then there are uniquely determined different primes $\Pi_1, \dots, \Pi_t$ of $\mathbb{G}$, such that for Π_i-primary elements A_i ($i = 1, \dots, t$), all not orders, there holds $A = A_1 \circ \cdots \circ A_t$. The A_i are determined uniquely too (but depend on the sequel of the Π_i ($i = 1, \dots, t$) which is arbitrary). $A_1 \circ \cdots \circ A_t$ is a smooth product. A is an integer iff all A_i are integers, and a distance iff all A_i are distances.*

Proof. Recall $A = \Psi A \cdot D_{EF}$, hence $\Phi A = \Psi A \cdot \Phi D_{EF}$, if $A \in (E, F)$. Then we decompose ΦD_{EF} into a product of prime powers:

$$\Phi D_{EF} = \Pi_1^{n_1} \cdot \cdots \cdot \Pi_s^{n_s},$$

all the Π_i different, and all $n_i > 0$ (the product may be empty if $E = F$). We use Theorem 7.11 inductively, beginning with $\Gamma = \Pi_1^{n_1}$, in order to obtain a smooth product $D_{EF} = U_1 \circ \cdots \circ U_s$, with $\Phi U_i = \Pi_i^{n_i}$, $\Psi U_i = 1$ ($i = 1, \dots, s$). We write $\Psi A = \Pi_1^{m_1} \cdots \Pi_s^{m_s} \cdot \Pi_{s+1}^{m_{s+1}} \cdots \Pi_t^{m_t}$, with $m_{s+1}, \dots, m_t \neq 0$, and using $A_i = \Pi_i^{m_i} X_i$ ($i = 1, \dots, t$), $A_j = \Pi_j^{m_j}$ ($j = s + 1, \dots, t$) we obtain the smooth product $A = A_1 \circ \cdots \circ A_t$, where none of the A_i ($i = 1, \dots, t$) is an order. The sequel of the $\Pi_1, \dots, \Pi_s$ (for possible decompositions of D_{EF}) is arbitrary, and so is the sequel of the $\Pi_1, \dots, \Pi_t$ for possible decompositions of A.

Let $A = A_1 \circ \cdots \circ A_t = A_1' \circ \cdots \circ A_r'$, where A_1 is not an order, both A_1, A_1' are Π_1-primary, $A_2, \dots, A_t$, $A_2', \dots, A_r'$ are primary with respect to primes different from Π_1. There exists a positive exponent r, and an integer element $\Gamma \in \mathfrak{G}$ such that $\Pi_1 \not\supseteq \Gamma$, and $\Pi_1{}^r A_1$, $\Pi_1{}^r A_1'$, $\Gamma A_2 \cdot \cdots \cdot A_t$, $\Gamma A_2' \cdot \cdots \cdot A_r'$ all are integers.

$(\Pi_1{}^r A_1)(\Gamma A_2 \cdot \cdots \cdot A_t) = (\Pi_1{}^r A_1')(\Gamma A_2' \cdot \cdots \cdot A_m')$ implies, by Proposition 7.10, that $\Pi_1{}^r A_1 = \Pi_1{}^r A_1'$, $A_1 = A_1'$ and $A_2 \cdot \cdots \cdot A_t = A_2' \cdot \cdots \cdot A_m'$. The uniqueness follows by induction on t. Now the ΨA_i are powers (possibly with negative or zero exponent) of different primes, hence in $\Psi A_1 \cdot \cdots \cdot \Psi A_t$ nothing can be cancelled. Therefore, $\Psi A \subseteq 1$ (resp., $\Psi A = 1$) is equivalent to $\Psi A_i \subseteq 1$ (resp., $\Psi A_i = 1$) for all $i = 1,\ldots, t$. These values at Ψ characterize integral modules (resp., distances).

Note added in proof. There is little overlapping of [10] with this paper, since Brung's "Noncommutative Krull Rings" are quite different from our "Krull Pairs." For example, the maximal order of the Hurwitz quaternion algebra is not a Krull ring in Brung's sense.

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[1] A complete bibliography appears with Part I of this series.