CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Elsevier - Publisher Connector

ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Functional Analysis 219 (2005) 306–339
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Duality and normal parts of operator modules

Bojan Magajna

Department of Mathematics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia

Received 26 June 2003; received in revised form 21 November 2003; accepted 15 July 2004
Communicated by D. Sarason

Abstract

For an operator bimodule X over von Neumann algebras $A \subseteq \mathrm{B}(\mathscr{H})$ and $B \subseteq \mathrm{B}(\mathscr{K})$, the space of all completely bounded A, B-bimodule maps from X into $\mathrm{B}(\mathscr{K}, \mathscr{H})$, is the bimodule dual of X. Basic duality theory is developed with a particular attention to the Haagerup tensor product over von Neumann algebras. To X a normal operator bimodule X_{n} is associated so that completely bounded A, B-bimodule maps from X into normal operator bimodules factorize uniquely through X_{n}. A construction of X_{n} in terms of biduals of X, A and B is presented. Various operator bimodule structures are considered on a Banach bimodule admitting a normal such structure.
© 2004 Elsevier Inc. All rights reserved.

MSC: primary 46L07; 46H25; secondary 47L25

Keywords: Operator bimodule; von Neumann algebra; Relative tensor products

1. Introduction

One of the aims of this article is to show that the classical duality theory of Banach spaces and the more recent duality of operator spaces [10,23,38,39] effectively extends to the situation where a Banach space is replaced by a normal operator bimodule X over von Neumann algebras A and B. The role of the dual is played by the A', B'-bimodule $X^\natural$ consisting of all completely bounded A, B-bimodule maps from X into $\mathrm{B}(\mathcal{K}, \mathcal{H})$, where $\mathcal{H}$ and $\mathcal{K}$ are proper Hilbert modules over A and B, respectively. Among the basic tools (or motivations) for such an extension are the operator-valued

E-mail address: bojan.magajna@fmf.uni-lj.si.

0022-1236/$ - see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2004.07.016

Hahn–Banach and bipolar theorems [2,24,44]. Some general aspects of duality of operator bimodules were considered also in [37,40,3]. Here, we study mainly bimodules over von Neumann algebras and emphasize the normality considerations. We shall explain briefly an application to W*-correspondences (Section 3).

In Section 2, we collect definitions of various (known) classes of bimodules, introduce abbreviations for their names and summarize some preliminary results.

In Section 3, we develop our basic technique and prove some typical duality theorems. In the formulation of results we are guided by classical functional analysis, but since the range of 'functionals' here is $\mathrm{B}(\mathcal{K}, \mathcal{H})$ instead of $\mathbb{C}$, the proofs of main results require methods completely different from the classical ones. Our starting point will be a duality result for the Haagerup tensor product of normal operator bimodules (Theorem 3.2), which extends the duality for the usual Haagerup tensor product of operator spaces obtained by Blecher and Smith [12]. This will enable us to relate the bimodule duals to the usual operator space duals. Many classical results can be generalized at least to strong bimodules. In Section 3, we also consider very briefly relations between the properties of a given bimodule map T and its bimodule adjoint $T^\natural$.

Because of the central role of the extended module Haagerup tensor product, we relate in Section 4 its bimodule dual to the normal version of the Haagerup tensor product studied by Effros and Ruan [22]. We also describe the module versions of the extended and the normal Haagerup tensor products of two von Neumann algebras over a common von Neumann subalgebra as concrete spaces of operators and thus generalize some results of Blecher–Smith [12] and Effros–Kishimoto [17].

For a general operator A, B-bimodule X we shall show that the closure X_{n} of the image of X in its bimodule bidual $X^{\natural\natural}$ is a normal operator A, B-bimodule having the following universal property: for each completely bounded A, B-bimodule map ϕ from X to a normal operator A, B-bimodule Y there exists a unique A, B-bimodule map $\tilde{\phi}$ from X_{n} into Y such that $\phi = \tilde{\phi}\iota$, where ι is the canonical map from X into X_{n}. The bimodule X_{n} is described in Section 5 in an alternative way and is called the normal part of X, although in general it is not contained in X. We also consider how $X^{\natural\natural}$ sits completely isometrically in the operator space bidual $X^{\sharp\sharp}$ of X (Theorem 5.8), where $X^{\sharp\sharp}$ is equipped with the canonical normal operator bimodule structure over the universal von Neumann envelopes of A and B.

In Section 6, the discussion is specialized to central bimodules over an abelian von Neumann algebra C. A C-bimodule X is called central if $xc = cx$ for all $c \in C$ and $x \in X$. First we observe that a central operator C-bimodule X is normal if and only if for each $x \in \mathrm{M}_n(X)$ the function $\Delta \ni t \mapsto \|x(t)\|$ on the spectrum Δ of C is continuous, where $x(t)$ is the coset of x in $\mathrm{M}_n(X)/[(\ker \mathrm{t})\mathrm{M}_n(X)]$. Then we characterize concretely the normal part of a central C-bimodule X in terms of its decomposition along Δ (Theorem 6.9). We also prove that for a strong central C-bimodule X and a subbimodule Y in X the quotient X/Y is normal if and only if Y is strong.

If a Banach A, B-bimodule X over von Neumann algebras A and B admits a norm structure of a *normal* operator A, B-bimodule then the maximal such structure $\mathrm{MAXN}_A(X)_B$ turns out to be different from the maximal operator A, B-bimodule $\mathrm{MAX}_A(X)_B$ on X (Section 7). Indeed, $\mathrm{MAXN}_A(X)_B$ is just the normal part of $\mathrm{MAX}_A(X)_B$. Even more surprisingly, if X admits a structure of a normal *dual*

operator A, B-bimodule, the maximal such structure, $\mathrm{MAXND}_A(X)_B$, is different from $\mathrm{MAXN}_A(X)_B$. This provides new examples of operator spaces which are duals as Banach spaces, say $V^\sharp$, but without any operator space predual on V. Earlier such examples are in [28,19].

2. Basic classes of bimodules, notation and other preliminaries

Throughout the paper A, B and C will be C*-algebras with unit 1, in fact von Neumann algebras most of the time. By a Banach A, B-bimodule we mean a Banach space X which is an A, B-bimodule such that $1x = x = x1$ and $\|axb\| \leqslant \|a\|\|x\|\|b\|$ for all $a \in A$, $b \in B$ and $x \in X$. The class of all such bimodules is denoted by ${}_A\mathrm{BM}_B$, and the space of all bounded A, B-bimodule maps from X to Y by $\mathrm{B}_A(X, Y)_B$.

A Hilbert A-module is just a Hilbert space $\mathcal{H}$ together with a $*$-representation π of A on $\mathcal{H}$. We shall always assume that $\pi(1) = 1_{\mathcal{H}}$. If A is a von Neumann algebra and π is normal then $\mathcal{H}$ is called *normal*. If π is injective, $\mathcal{H}$ is called *faithful*. If π is cyclic then $\mathcal{H}$ is called *cyclic*. If each finite subset of $\mathcal{H}$ is contained in a closed cyclic submodule $[A\xi]$ then $\mathcal{H}$ is *locally cyclic*. The importance of such modules originates from a well known result of [43] recalled in Theorem 7.2. Over a von Neumann algebra A there exists a canonical Hilbert module $\mathcal{H}$, called the standard form. We shall only need a property of this module abstracted in the following:

Definition 2.1. *A proper module* over a von Neumann algebra A is a faithful normal Hilbert A-module $\mathcal{H}$ such that all normal states on A and on A' (the commutant of A in $\mathrm{B}(\mathcal{H})$) are vector states (that is, of the form $x \mapsto \langle x\xi, \xi\rangle$, $\xi \in \mathcal{H}$).

Note that a proper A-module $\mathcal{H}$ contains (up to a unitary equivalence) all normal cyclic Hilbert A-modules (since all normal states on A come from vectors in $\mathcal{H}$) and is locally cyclic by Smith [43, 2.3]. Since for a separable $\mathcal{H}$ locally cyclic vectors are cyclic by Haagerup and Winslow [26, 2.7], it follows from [27, 7.2.9] that a proper separable A-module is essentially just the standard form.

For operator spaces X and Y, $\mathrm{CB}(X, Y)$ denotes the set of completely bounded linear maps from X to Y. Occasionally we shall use the notation OS for the class of operator spaces. If $X, Y \in \mathrm{OS} \cap {}_A\mathrm{BM}_B$, let $\mathrm{CB}_A(X, Y)_B = \mathrm{B}_A(X, Y)_B \cap \mathrm{CB}(X, Y)$. We are now going to recall the definitions of various classes of operator modules. We will follow the usual terminology, but since some classes of modules have very long names (such as 'normal dual operator A, B-bimodules') and appear repeatedly, it will be convenient to introduce notation for them.

Definition 2.2. (i) The class ${}_A\mathrm{OM}_B$ of *operator A, B-bimodules* consists of all bimodules $X \in {}_A\mathrm{BM}_B \cap \mathrm{OS}$ which can be completely isometrically and homomorphically represented in a $\mathrm{B}(\mathcal{H})$. In other words, for some Hilbert module $\mathcal{H}$ over A and B the space $\mathrm{CB}_A(X, \mathrm{B}(\mathcal{H}))_B$ contains a complete isometry.

(ii) If in (i) A and B are von Neumann algebras and $\mathcal{H}$ can be chosen to be normal over A and B, then X is a *normal operator A, B-bimodule* ($X \in {}_A\mathrm{NOM}_B$).

As we shall observe below, the class of normal operator modules contains normal Hilbert modules and will play an important role here.

Operator bimodules are characterized by the CES theorem [13], which was later generalized and sharpened [7, Section 5; 10, Section 4.6; 38, Chapter 16].

Theorem 2.3 (*Blecher and Le Merdy [10], Paulsen [38]*). *A bimodule* $X \in \mathrm{OS} \cap {}_A\mathrm{BM}_B$ *is an operator bimodule if and only if* $\mathrm{M}_n(X)$ *is a Banach* $\mathrm{M}_n(A), \mathrm{M}_n(B)$*-bimodule for each* $n = 1, 2 \ldots$.

Given $X \in \mathrm{OS}$, *there exist* C^**-algebras* $A_l(X)$ *and* $A_r(X)$ *such that X is an operator* $A_l(X), A_r(X)$*-bimodule and every left (right) operator A-module structure on X is given by a* $*$*-homomorphism from A into* $A_l(X)$ *(into* $A_r(X)$*). In particular, if X is a left operator A module and a right operator B-module, then X is automatically an operator* A, B*-bimodule.*

Normal operator bimodules are characterized by the following result, the first part of which is not hard to deduce also from Theorem 5.8 below (and its proof).

Theorem 2.4 (*Magajna [32, 3.3; 34, 6.1]*). *A bimodule* $X \in {}_A\mathrm{OM}_B$ *is normal if and only if for each* $n \in \mathbb{N}$ *and* $x \in \mathrm{M}_n(X)$ *the mappings* $\mathrm{M}_n(A) \ni a \mapsto \|ax\|$ *and* $\mathrm{M}_n(B) \ni b \mapsto \|xb\|$ *are weak* lower semicontinuous. If A and B are* σ*-finite, this is the case if and only if for all* $x \in \mathrm{M}_n(X)$ *and sequences of projections* (e_j) *and* (f_j) *increasing to* 1 *in* $\mathrm{M}_n(A)$ *and* $\mathrm{M}_n(B)$ *(resp.) we have* $\lim_j \|e_j x\| = \|x\| = \lim_j \|x f_j\|$.

We recall that a von Neumann algebra A is σ-finite if each orthogonal family of nonzero projections in A is countable.

Definition 2.5. (i) A *dual Banach* A, B*-bimodule* is a dual Banach space $X = V^\sharp \in {}_A\mathrm{BM}_B$ such that the maps $X \ni x \mapsto ax$ and $X \ni x \mapsto xb$ are weak* continuous for all $a \in A$ and $b \in B$. Then the preadjoints of these maps define a B, A-bimodule structure on V. Conversely, for every $V \in {}_B\mathrm{BM}_A$, $X = V^\sharp$ becomes a *dual Banach* A, B*-bimodule* by

$$\langle axb, v\rangle = \langle x, bva\rangle \ (x \in X, \ \ v \in V).$$

The category of such bimodules is denoted by ${}_A\mathrm{DBM}_B$ and the space of all weak* continuous (hence bounded) A, B-bimodule maps from X to Y by $\mathrm{N}_A(X, Y)_B$.

(ii) If A and B are von Neumann algebras, a bimodule $X = V^\sharp \in {}_A\mathrm{DBM}_B$ is a *normal dual Banach bimodule* ($X \in {}_A\mathrm{NDBM}_B$) if the maps $A \ni a \mapsto \langle ax, v\rangle$ and $B \ni b \mapsto \langle xb, v\rangle$ are weak* continuous for all $x \in X$ and $v \in V$.

Definition 2.6. An operator bimodule X is a *dual operator* A, B*-bimodule* ($X \in {}_A\mathrm{DOM}_B$) if X is the operator space dual of some $V \in {}_B\mathrm{BM}_A \cap \mathrm{OS}$ equipped with the A, B-bimodule action as in Definition 2.5. For such bimodules X, Y let $\mathrm{NCB}_A(X, Y)_B = \mathrm{N}_A(X, Y)_B \cap \mathrm{CB}(X, Y)$.

Remark 2.7. A Hilbert A-module $\mathcal{H}$ is regarded as an operator A-module by considering $\mathcal{H}$ as the column operator space [10] or [39]. Then $\mathcal{H}$ is dual to the conjugate Hilbert space $\mathcal{H}^*$ with the row operator space structure and the right module action

$$\xi^* a = (a^* \xi)^*, \quad (\xi \in \mathcal{H},\ a \in A), \tag{2.1}$$

where ξ^* denotes ξ regarded as an element of $\mathcal{H}^*$. *In this paper $\mathcal{H}$ will always mean a column Hilbert space and $\mathcal{H}^*$ the corresponding operator space dual.*

Definition 2.8. For von Neumann algebras A and B, a bimodule $X \in {}_A\mathrm{DOM}_B$ is *a normal dual operator A, B-bimodule* (${}_A\mathrm{NDOM}_B$) if there exist a normal Hilbert module $\mathcal{H}$ over A, B and a complete isometry in $\mathrm{NCB}_A(X, \mathrm{B}(\mathcal{H}))_B$.

The original characterization of normal dual operator bimodules [20] was greatly improved in [8, 4.1, 4.2; 9, 5.4, 5.7], from which we shall need the following:

Theorem 2.9 (*Blecher [8], Blecher et al. [9]*). *If X is a dual operator space, then the algebras $A_l(X)$ and $A_r(X)$ in Theorem* 2.3 *are von Neumann algebras and X is a normal dual operator $A_l(X), A_r(X)$-bimodule. Thus, if A and B are von Neumann algebras and $X \in {}_A\mathrm{DOM}_B$, the maps $X \ni x \mapsto ax$ $(a \in A)$ and $x \mapsto xb$ $(b \in B)$ are automatically weak* continuous. If the maps $A \ni a \mapsto ax$ and $B \ni b \mapsto xb$ are also weak* continuous for each $x \in X$, then $X \in {}_A\mathrm{NDOM}_B$.*

Remark 2.10. If a dual operator A, B-bimodule X satisfies the norm semicontinuity condition of Theorem 2.4 then X is a normal dual operator bimodule (see [32, p. 199–200]), that is ${}_A\mathrm{DOM}_B \cap {}_A\mathrm{NOM}_B = {}_A\mathrm{NDOM}_B$.

For an index set $\mathbb{J}$ and an $X \in \mathrm{OS}$, let $\mathrm{R}_{\mathbb{J}}(X)$ be the set $\mathrm{M}_{1,\mathbb{J}}(X)$ of all $1 \times \mathbb{J}$ bounded matrices with the entries in X. Similarly, $\mathrm{C}_{\mathbb{J}}(X) := \mathrm{M}_{\mathbb{J},1}(X)$. (An $\mathbb{I} \times \mathbb{J}$ matrix is bounded if the supremum of the norms of its finite submatrices is finite.)

Definition 2.11. A bimodule $X \in {}_A\mathrm{NOM}_B$ is called *strong* ($X \in {}_A\mathrm{SOM}_B$) if

$$[a_i][x_{ij}](b_j) = \sum_{i \in \mathbb{I},\, j \in \mathbb{J}} a_i x_{ij} b_j \in X \tag{2.2}$$

for all $[a_i] \in \mathrm{R}_{\mathbb{I}}(A)$, $[x_{ij}] \in \mathrm{M}_{\mathbb{I},\mathbb{J}}(X)$, $(b_j) \in \mathrm{C}_{\mathbb{J}}(A)$ and all index sets $\mathbb{I}$ and $\mathbb{J}$.

As shown in [31], it suffices to require condition (2.2) for orthogonal families of projections $(a_i) \subseteq A$ and $(b_j) \subseteq B$. Strong bimodules in $\mathrm{B}(\mathcal{H})$ are characterized as closed in the A, B-topology [33], the definition of which will not be needed here. For our purposes it will suffices to note that a functional ρ on $\mathrm{B}(\mathcal{H})$ is A, B-continuous if and only if $\rho \in \mathrm{B}(\mathcal{H})^{A\sharp B}$, where $\mathrm{B}(\mathcal{H})^{A\sharp B}$ is defined as follows.

Definition 2.12. If A and B are von Neumann algebras and $X \in {}_A\mathrm{BM}_B$, let $X^{A\sharp B}$ be the subspace of the dual $X^\sharp$ of X, consisting of all $\rho \in X^\sharp$ such that for each $x \in X$ the maps $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are weak* continuous.

The argument from [31, 4.6] shows that bounded bimodule homomorphisms are continuous in the A, B-topology.

Occasionally we shall need a version of the operator bipolar theorem. A subset K of a bimodule $X \in {}_A\mathrm{BM}_B$ is called A, B-*absolutely convex* if

$$\sum_{j=1}^{n} a_j x_j b_j \in K$$

for all $x_j \in K$ and $a_j \in A$, $b_j \in B$ satisfying $\sum_{j=1}^{n} a_j a_j^* \leqslant 1$, $\sum_{j=1}^{n} b_j^* b_j \leqslant 1$.

Theorem 2.13 (*Magajna [33, 3.8, 3.9]*). *Let K be an A, B-absolutely convex subset of a bimodule $X \in {}_A\mathrm{SOM}_B$. If K is closed in the A, B-topology, then for each $x \in X \setminus K$ there exist normal cyclic Hilbert modules $\mathcal{H}$ over A and $\mathcal{K}$ over B and a map $\phi \in \mathrm{CB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ such that $\|\phi(y)\| \leqslant 1$ for all $y \in K$ and $\|\phi(x)\| > 1$. If $X \in {}_A\mathrm{NDOM}_B$ and K is weak* closed then ϕ can be chosen weak* continuous.*

For bimodules $U \in {}_A\mathrm{OM}_B$ and $V \in {}_B\mathrm{OM}_A$ we denote by $U {}_A\hat{\otimes}_B V$ the quotient of the maximal operator space tensor product $U \hat{\otimes} V$ by the closed subspace N generated by $\{aub \otimes v - u \otimes bva : \ a \in A, \ b \in B, \ u \in U, \ v \in V\}$. Consider the natural completely isometric isomorphism [10, (1.51)]; [39, 4.1]

$$\imath : \mathrm{CB}(U, V^\sharp) \to (U \hat{\otimes} V)^\sharp, \quad \imath(\phi)(u \otimes v) = \phi(u)(v) \quad (\phi \in \mathrm{CB}(U, V^\sharp))$$

and note that $\imath(\phi)$ annihilates N if and only if $\phi \in \mathrm{CB}_A(U, V^\sharp)_B$, where $V^\sharp$ is the dual A, B-bimodule of V (Definition 2.5). Thus, we have completely isometrically

$$\mathrm{CB}_A(U, V^\sharp)_B = (U {}_A\hat{\otimes}_B V)^\sharp. \tag{2.3}$$

Now we turn to the definition of bimodule duality.

Definition 2.14. Given operator algebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ (containing the identity operators), the *bimodule dual* (with respect to $\mathcal{H}$ and $\mathcal{K}$) of a bimodule $X \in {}_A\mathrm{OM}_B$ is the A', B'-bimodule $X^\natural = \mathrm{CB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, where

$$(a'\phi b')(x) := a'\phi(x)b' \quad (\phi \in X^\natural).$$

If $\mathcal{H}$ and $\mathcal{K}$ are proper, we emphasize this by writing $X^{\natural\mathrm{p}}$ instead of $X^\natural$ and call $X^{\natural\mathrm{p}}$ a *proper bimodule dual* of X.

We could replace in the above definition the inclusions $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ by more general (normal) representations without essentially changing the ideas, only the notation would be more complicated. If $\mathcal{H}$ and $\mathcal{K}$ are separable and proper then, as we already remarked, they are unique up to a unitary equivalence of modules and consequently the proper duals are essentially unique in this case. From (2.3) (and using Theorem 2.9) we deduce by standard arguments:

Proposition 2.15. *$X^\natural$ is a normal dual operator A', B'-bimodule if $X \in {}_A\mathrm{OM}_B$.*

Definition 2.16. For von Neumann algebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathbb{K}$ and a bimodule $X \in {}_A\mathrm{DOM}_B$, the A', B'-bimodule $X_\natural = \mathrm{NCB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ is called the *bimodule predual* of X. If $\mathcal{H}$ and $\mathcal{K}$ are proper then $X_\natural$ is denoted also by $X_{\natural_\mathrm{p}}$.

The following theorem was proved in [35] in the case $B = A$, but the same proof works in general.

Theorem 2.17. *If $X \in {}_A\mathrm{NOM}_B$, then $(X^{\natural_\mathrm{p}})_{\natural_\mathrm{p}}$ is the smallest strong A, B-bimodule containing X. In particular, $(X^{\natural_\mathrm{p}})_{\natural_\mathrm{p}} = X$ if and only if X is strong.*

Now we recall the definition of the (extended) Haagerup tensor product of modules. For two modules $X \in \mathrm{NOM}_B$ and $Y \in {}_B\mathrm{NOM}$ the completion of the algebraic tensor product $X \otimes_B Y$ with the norm

$$h(w) = \inf\{\|\sum_{j=1}^{n} x_j x_j^*\|^{1/2} \|\sum_{j=1}^{n} y_j^* y_j\|^{1/2} : \ w = \sum_{j=1}^{n} x_j \otimes_B y_j\}$$

is the *Haagerup tensor product* $X \overset{h}{\otimes}_B Y$ [10]. A typical element $w \in X \overset{h}{\otimes}_B Y$ can be represented as $w = \sum_{j=1}^{\infty} x_i \otimes_B y_i$, where the two series $\sum_{j=1}^{\infty} x_j x_j^*$ and $\sum_{j=1}^{\infty} y_j^* y_j$ are norm convergent. We write this as

$$w = x \odot_B y, \tag{2.4}$$

where $x \in \mathrm{R}_{\mathbb{J}}(X)$, $y \in \mathrm{C}_{\mathbb{J}}(Y)$ and $\mathbb{J} = \{1, 2, \ldots\}$.

The *extended Haagerup tensor product* $X \overset{eh}{\otimes}_B Y$ consists of all 'formal expressions' (2.4), where $x \in \mathrm{R}_{\mathbb{J}}(X)$ and $y \in \mathrm{C}_{\mathbb{J}}(Y)$ for some (infinite) index set $\mathbb{J}$. To explain the term 'formal expression', we may assume (by the CES theorem, [10, 3.3.1]) that $X, Y, B \subseteq \mathrm{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ and regard $w = x \odot_B y$ as completely bounded map $b' \mapsto xb'y$ from B' into $\mathrm{B}(\mathcal{H})$. From [30, 3.2] we have that

$$x \odot_B y = 0 \Longleftrightarrow \exists \text{ a projection } P \in \mathrm{M}_{\mathbb{J}}(B) \text{ such that } xP = 0 \text{ and } Py = y. \tag{2.5}$$

Thus, $X \overset{eh}{\otimes}_B Y$ is defined as the space of all maps in $\mathrm{CB}(B', \mathrm{B}(\mathcal{H}))$ that can be represented in form (2.4) with $x \in \mathrm{R}_{\mathbb{J}}(X)$ and $y \in \mathrm{C}_{\mathbb{J}}(Y)$ for some cardinal $\mathbb{J}$. (The two sums $\sum_{j\in\mathbb{J}} x_j x_j^*$ and $\sum_{j\in\mathbb{J}} y_j^* y_j$ are now weak* convergent.) If $X \in {}_A\mathrm{SOM}_B$ and $Y \in {}_B\mathrm{SOM}_C$ then $X \overset{eh}{\otimes}_B Y \in {}_A\mathrm{SOM}_C$ and for each $w \in X \overset{eh}{\otimes}_B Y$

$$\|w\|_{\mathrm{cb}} = \inf\{\|x\|\|y\| : \ w = x \odot_B y, \quad x \in \mathrm{R}_{\mathbb{J}}(X), \ \ y \in \mathrm{C}_{\mathbb{J}}(Y)\}. \tag{2.6}$$

For more see [30] and, for alternative approaches in the case $B = \mathbb{C}$, [12,22]. We shall need the following basic property of the symbol $\odot_B$:

$$xb \odot_B y = x \odot_B by, \quad (b \in \mathrm{M}_{\mathbb{J}}(B), \ \ x \in \mathrm{R}_{\mathbb{J}}(X), \ \ y \in \mathrm{C}_{\mathbb{J}}(Y)). \tag{2.7}$$

Remark 2.18. Since for Hilbert space vectors $(\xi_j) \in \mathrm{C}_{\mathbb{J}}(\mathcal{H})$ the sum $\sum_{j\in\mathbb{J}} \|\xi_j\|^2$ is convergent, for a Hilbert A-module $\mathcal{H}$ and $X \in_A \mathrm{SOM}$ we have that

$$X \overset{eh}{\otimes}_A \mathcal{H} = X \overset{h}{\otimes}_A \mathcal{H} \quad \text{and} \quad \mathcal{H}^* \overset{eh}{\otimes}_A X = \mathcal{H}^* \overset{h}{\otimes}_A X.$$

3. Basic duality for normal operator bimodules

In this section A, B and C are von Neumann algebras and the bimodule duality is defined using fixed faithful normal Hilbert modules $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ over A, B, C (resp.).

Definition 3.1. Given $X \in {}_A\mathrm{OM}_B$ and $Y \in {}_B\mathrm{OM}_C$, let $(X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}$ denote the subspace of the A, C-bimodule dual of $X \overset{h}{\otimes}_B Y$ consisting of all $\Omega \in (X \overset{h}{\otimes}_B Y)^{\natural}$ such that the map $B \ni b \mapsto \Omega(x \otimes_B by)$ is weak* continuous for all $x \in X$, $y \in Y$.

A part of the development in this section is based on the following extension of a result of Blecher and Smith [12].

Theorem 3.2. *If $X \in {}_A\mathrm{NOM}_B$ and $Y \in {}_B\mathrm{NOM}_C$ then $(X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}} = X^{\natural} \overset{eh}{\otimes}_{B'} Y^{\natural}$ completely isometrically as A', C'-bimodules.*

Proof. Consider the natural map $\iota : X^{\natural} \overset{eh}{\otimes}_{B'} Y^{\natural} \to (X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}$ defined by

$$\iota(\phi \odot_{B'} \psi)(x \otimes_B y) = \phi(x)\psi(y),$$

where $x \in X$, $y \in Y$, $\phi \in \mathrm{R}_{\mathbb{J}}(X^{\natural})$ and $\psi \in \mathrm{C}_{\mathbb{J}}(Y^{\natural})$. Note that

$$\mathrm{R}_{\mathbb{J}}(X^{\natural}) = \mathrm{CB}_A(X, \mathrm{R}_{\mathbb{J}}(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B \subseteq \mathrm{CB}(X, \mathrm{B}(\mathcal{K}^{\mathbb{J}}, \mathcal{H})),$$

hence $\phi(x) \in \mathrm{B}(\mathcal{K}^{\mathbb{J}}, \mathcal{H})$ and similarly $\psi(y) \in \mathrm{B}(\mathcal{L}, \mathcal{K}^{\mathbb{J}})$. Using (2.5) it can be verified that $\imath$ is a well defined completely contractive homomorphism of A', C'-bimodules. To show that $\imath$ is injective, suppose that $\phi \odot_{B'} \psi$ is in the kernel of $\imath$. This means that

$$\phi(X)\psi(Y) = 0. \tag{3.1}$$

Since $\psi(Y)$ is a B-module, the subspace $[\psi(Y)\mathcal{L}]$ of $\mathcal{K}^{\mathbb{J}}$ is invariant under B, hence the projection $p' \in \mathrm{B}(\mathcal{K}^{\mathbb{J}})$ with the range $[\psi(Y)\mathcal{L}]$ is in $\mathrm{M}_{\mathbb{J}}(B')$. Clearly $p'\psi = \psi$, while (3.1) implies that $\phi p' = 0$. Hence (using (2.7)) $\phi \odot_{B'} \psi = \phi \odot_{B'} p'\psi = \phi p' \odot_{B'} \psi = 0$.

Now, since we have just shown that $\imath$ is injective, it suffices to prove $\imath$ is a completely quotient map. Let

$$\Omega \in \mathrm{M}_n((X \overset{h}{\otimes}_B Y)^{\natural_{B\mathrm{nor}}}) \subseteq \mathrm{CB}_A(X \overset{h}{\otimes}_B Y, \mathrm{B}(\mathcal{L}^n, \mathcal{H}^n))_C$$

be a complete contraction. Then from the well known CSPS theorem [38, p. 17.8; 10, 1.5.7] it can be deduced (in the same way as in [30, proof of 3.9]) that there exist a normal Hilbert B-module $\mathcal{G}$ and complete contractions $\phi \in \mathrm{CB}_A(X, \mathrm{B}(\mathcal{G}, \mathcal{H}^n))_B$ and $\psi \in \mathrm{CB}_B(Y, \mathrm{B}(\mathcal{L}^n, \mathcal{G}))_C$ such that

$$\Omega(x \otimes_B y) = \phi(x)\psi(y) \quad (x \in X, \ \ y \in Y).$$

Since each normal representation of B is contained in a multiple of the identity representation, we may assume that $\mathcal{G} = \mathcal{K}^{\mathbb{J}}$ for some cardinal $\mathbb{J}$. Then

$$\phi \in \mathrm{CB}_A(X, \mathrm{B}(\mathcal{G}, \mathcal{H}^n))_B = \mathrm{M}_{n,\mathbb{J}}(\mathrm{CB}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B) = \mathrm{R}_{\mathbb{J}}(\mathrm{C}_n(X^{\natural}))$$

and

$$\psi \in \mathrm{CB}_B(Y, \mathrm{B}(\mathcal{L}^n, \mathcal{G}))_C = \mathrm{M}_{\mathbb{J},n}(\mathrm{CB}_B(Y, \mathrm{B}(\mathcal{L}, \mathcal{K}))_C) = \mathrm{C}_{\mathbb{J}}(\mathrm{R}_n(Y^{\natural})),$$

hence $\phi \odot_B \psi$ is an element of $\mathrm{C}_n(X^{\natural}) \overset{eh}{\otimes}_B \mathrm{R}_n(Y^{\natural}) = \mathrm{M}_n(X^{\natural} \overset{eh}{\otimes}_B Y^{\natural})$ such that $\|\phi \odot_B \psi\| \leqslant 1$ and $\imath_n(\phi \odot_B \psi) = \Omega$. $\square$

A special case of Theorem 3.2 is the following result of Effros and Exel [16].

Corollary 3.3 (*Effros and Exel [16]*). *$(\mathcal{K}^* \overset{h}{\otimes}_B \mathcal{K})^{\sharp} = B'$ if $\mathcal{K}$ is a normal (faithful) Hilbert B-module.*

Proof. We regard $\mathcal{K}$ as a $B, \mathbb{C}$-bimodule and $\mathcal{K}^*$ as a $\mathbb{C}, B$-bimodule. Since $\mathcal{K}^{\natural} = \mathrm{CB}_B(\mathcal{K}) = B'$ and $(\mathcal{K}^*)^{\natural} = B'$, we have $(\mathcal{K}^* \overset{h}{\otimes}_B \mathcal{K})^{\sharp} = B' \overset{eh}{\otimes}_{B'} B' = B'$. $\square$

As an application of Theorem 3.2 we can express the bimodule dual of $X \in {}_A\mathrm{NOM}_B$ in terms of usual operator space duality, but first we state a definition.

Definition 3.4. If $X \in {}_A\mathrm{OM}_B$, define the Banach B', A'-bimodule structure on $\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K}$ by (using the conventions from Remark 2.7)

$$b'(\xi^* \otimes_A x \otimes_B \eta)a' = \xi^* a' \otimes_A x \otimes_B b'\eta.$$

Part (i) of the following corollary is known in some form [37,5].

Corollary 3.5. *For each $X \in {}_A\mathrm{NOM}_B$ the following natural maps are completely isometric isomorphisms of bimodules* (*and will be regarded as equalities later on*):

(i) $\kappa : X^\natural \to (\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp$, $\quad \kappa(\phi)(\xi^* \otimes_A x \otimes_B \eta) = \langle \phi(x)\eta, \xi\rangle$. *Here the A', B'-bimodule structure on $X^\natural$ is as in Definition* 2.14, *while the bimodule structure on $(\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp$ is dual* (*in the sense of Definition* 2.5) *to that on $\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K}$* (*Definition* 3.4).

(ii) $\iota : \mathcal{H}^* \overset{h}{\otimes}_{A'} X^\natural \overset{h}{\otimes}_{B'} \mathcal{K} \to X^{A\sharp B}$, $\quad \iota(\xi^* \otimes_{A'} \phi \otimes_{B'} \eta) = \langle \phi(x)\eta, \xi\rangle$. *Here the structure of B, A-bimodule on $\mathcal{H}^* \overset{h}{\otimes}_{A'} X^\natural \overset{h}{\otimes}_{B'} \mathcal{K}$ is as in Definition* 3.4 (*but with A and B replaced by A' and B', resp.*), *while $X^{A\sharp B}$ inherits its structure from $X^\sharp$* (*which is dual to that on X, Definition* 2.5(i)). *In fact, for each $\rho \in X^{A\sharp B}$ there exist a set $\mathbb{J}$, unit vectors $\xi \in \mathcal{H}^{\mathbb{J}}$ and $\eta \in \mathcal{K}^{\mathbb{J}}$ and a map $\phi \in \mathrm{M}_{\mathbb{J}}(X^\natural)$ such that $\rho(x) = \langle \phi(x)\eta, \xi\rangle$ $(x \in X)$ and $\|\phi\|_{\mathrm{cb}} = \|\rho\|_{\mathrm{cb}}$.*

(iii) $(X^{A\sharp B})^\sharp = X^{\natural\natural}$.

Proof. The routine verifications that κ and ι are bimodule homomorphisms and that the identifications below are the same as stated in the corollary will be omitted.

(i) That κ is a complete isometry follows from Theorem 3.2 and the associativity of the (extended) Haagerup tensor product. Namely, since $\mathcal{K}^\natural = B'$ (as in the proof of Corollary 3.3) and similarly $(\mathcal{H}^*)^\natural = A'$ and $\mathcal{H}, \mathcal{K}$ are normal, we have the following complete isometries (regarded as equalities):

$$(\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^\sharp = (\mathcal{H}^* \overset{h}{\otimes}_A X)^\natural \overset{eh}{\otimes}_{B'} \mathcal{K}^\natural = (\mathcal{H}^*)^\natural \overset{eh}{\otimes}_{A'} X^\natural = X^\natural.$$

(ii) Regarding A as a $\mathbb{C}, A$-bimodule and B as a $B, \mathbb{C}$-bimodule, we have $A^\natural = \mathcal{H}^*$ and $B^\natural = \mathcal{K}$. Thus by Theorem 3.2

$$X^{A\sharp B} = (A \overset{h}{\otimes}_A X \overset{h}{\otimes}_B B)^{A\sharp B} = A^\natural \overset{eh}{\otimes}_{A'} X^\natural \overset{eh}{\otimes}_{B'} B^\natural = \mathcal{H}^* \overset{eh}{\otimes}_{A'} X^\natural \overset{eh}{\otimes}_{B'} \mathcal{K}.$$

The norm equality $\|\phi\|_{\mathrm{cb}} = \|\rho\|$ follows from the proof of Theorem 3.2.

(iii) This is an immediate consequence of (i) and (ii). $\square$

Corollary 3.6. *For each $X \in {}_A\mathrm{NOM}_B$ the natural homomorphism $X \to X^{\natural\natural}$ is completely isometric.*

Proof. Note that there is a completely contractive projection from $X^\sharp$ onto $X^{A\sharp B}$ (see [31, 4.4] or the proof of Theorem 5.8), hence by Corollary 3.5(iii) $X^{\natural\natural} = (X^{A\sharp B})^\sharp \subseteq X^{\sharp\sharp}$. $\square$

The following result is dual to Theorem 2.17.

Theorem 3.7. *For each $X \in {}_A\mathrm{NDOM}_B$ the natural map $\imath : X \to (X_{\natural\mathrm{p}})^{\natural\mathrm{p}}$ is a completely isometric weak* homeomorphic isomorphism of A, B-bimodules.*

Proof. Set $Y = X_{\natural\mathrm{p}}$. It is not hard to verify that Y is a strong A', B'-subbimodule in $X^\natural$ (see [20, p. 156] if necessary). To prove that the natural A, B-bimodule complete contraction

$$\imath : X \to Y^{\natural\mathrm{p}}, \quad \imath(x)(\phi) = \phi(x) \quad (\phi \in Y)$$

is completely isometric, let $x \in \mathrm{M}_n(X)$ with $\|x\| > 1$. By Theorem 2.13 applied to the normal dual $\mathrm{M}_n(A), \mathrm{M}_n(B)$-bimodule $\mathrm{M}_n(X)$ (with K the unit ball of $\mathrm{M}_n(X)$) there exist cyclic normal Hilbert modules $\tilde{\mathcal{G}}$ over $\mathrm{M}_n(A)$ and $\tilde{\mathcal{L}}$ over $\mathrm{M}_n(B)$ and a weak* continuous contractive bimodule map $\tilde{\phi} : \mathrm{M}_n(X) \to \mathrm{B}(\tilde{\mathcal{L}}, \tilde{\mathcal{G}})$ such that $\|\tilde{\phi}(x)\| > 1$. In fact $\|\tilde{\phi}\|_{\mathrm{cb}} \leqslant 1$ by a result of Smith quoted below as Theorem 7.2. An elementary well-known argument about Hilbert modules over $\mathrm{M}_n(A)$ shows that $\tilde{\mathcal{G}} = \mathcal{G}^n$ and $\tilde{\mathcal{L}} = \mathcal{L}^n$ for some normal Hilbert modules $\mathcal{G}$ over A and $\mathcal{L}$ over B and (since $\tilde{\phi}$ is a homomorphism of $\mathrm{M}_n(A), \mathrm{M}_n(B)$-bimodules) $\tilde{\phi} = \phi_n$, where $\phi \in \mathrm{NCB}_A(X, \mathrm{B}(\mathcal{L}, \mathcal{G}))_B$ (that is, $\tilde{\phi}([x_{ij}]) = [\phi(x_{ij})]$ for all $[x_{ij}] \in \mathrm{M}_n(X)$). Since $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{L}}$ are cyclic over $\mathrm{M}_n(A)$ and $\mathrm{M}_n(B)$ (resp.), $\mathcal{G}$ and $\mathcal{L}$ are n-cyclic over A and B (resp.), which means that (up to a unitary equivalence) $\mathcal{G} \subseteq \mathcal{H}^n$ and $\mathcal{L} \subseteq \mathcal{K}^n$, where $\mathcal{H}$ and $\mathcal{K}$ are proper modules used in the definition of duality. Then ϕ may be regarded as an element of $\mathrm{NCB}_A(X, \mathrm{M}_n(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B = \mathrm{M}_n(Y)$ and $\|\phi\|_{\mathrm{cb}} \leqslant 1$. Since $\|\phi_n(x)\| > 1$, it follows that $\|\imath(x)\| > 1$ and $\imath$ must be completely isometric.

Next note that $\imath$ is weak* continuous on the unit ball, hence a weak* homeomorphism onto the weak* closed subspace $\imath(X)$ in $Y^{\natural\mathrm{p}}$ by the Krein–Smulian theorem. Indeed, if (x_j) is a bounded net in X weak* converging to an $x \in X$, then for each $\phi \in Y$ $(= X_{\natural\mathrm{p}})$ the net $(\phi(x_j))$ converges to $\phi(x)$ in the weak* topology of $\mathrm{B}(\mathcal{K}, \mathcal{H})$, hence

$$\langle \imath(x_j), \xi^* \otimes_{A'} \phi \otimes_{B'} \eta \rangle = \langle \phi(x_j)\eta, \xi \rangle \to \langle \phi(x)\eta, \xi \rangle$$

for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Since elements of the form $\xi^* \otimes_{A'} \phi \otimes_{B'} \eta$ generate the predual $\mathcal{H}^* \overset{h}{\otimes}_{A'} Y \overset{h}{\otimes}_{B'} \mathcal{K}$ of $Y^{\natural\mathrm{p}}$, this proves that $\imath$ is weak* continuous.

Now we may identify X with $\imath(X)$ in $Y^{\natural\mathrm{p}}$. If $X \neq Y^{\natural\mathrm{p}}$, then by Theorem 2.13 there exists a nonzero $\phi \in (Y^{\natural\mathrm{p}})_{\natural\mathrm{p}}$ such that $\phi(X) = 0$. But, since Y is a strong A',

B'-bimodule, $(Y^{\natural_{\mathrm{p}}})_{\natural_{\mathrm{p}}} = Y$ by Theorem 2.17. Thus $\phi \in Y$ and therefore $\phi(X) = 0$ implies $\phi = 0$, since $Y = X_{\natural_{\mathrm{p}}}$. This contradiction proves that $X = Y^{\natural_{\mathrm{p}}}$. $\square$

We remark without proof that the restriction to proper duals in Theorem 3.7 is necessary, without this restriction the map ι need not be isometric.

Definition 3.8. A bimodule $X \in {}_A\mathrm{NOM}_B$ is A, B*-reflexive* if the natural complete isometry $X \to X^{\natural\natural}$ is surjective.

Here is a generalization of the classical characterization of reflexivity.

Proposition 3.9. *A bimodule $X \in {}_A\mathrm{NOM}_B$ is A, B-reflexive if and only if its unit ball B_X is compact in the topology induced by $X^{A\sharp B}$.*

Proof. By Corollary 3.5(i) $X^{\natural\natural} = (X^{A\sharp B})^{\sharp}$. By classical arguments the unit ball of $(X^{A\sharp B})^{\sharp}$ is compact in the topology induced by $X^{A\sharp B}$, with B_X a dense subset. $\square$

As an immediate application of Proposition 3.9 one can deduce that strong sub-bimodules of A, B-reflexive normal operator bimodules are A, B-reflexive and that if $X \in {}_A\mathrm{NOM}_B$ is C, D-reflexive for some von Neumann subalgebras $C \subseteq A$ and $D \subseteq B$, then X is A, B-reflexive (since the topology induced by $X^{A\sharp B}$ is weaker than that by $X^{C\sharp D}$).

Now we consider the (non) reflexivity of the basic bimodule $\mathrm{B}(\mathcal{K}, \mathcal{H})$.

Example 3.10. The bimodule $\mathrm{B}(\mathcal{K}, \mathcal{H})$ is A, B-reflexive if and only if at least one of the algebras A or B is atomic and finite.

To prove this, note that by Proposition 3.9 the A, B-reflexivity does not depend on the choice of $\mathcal{H}$ and $\mathcal{K}$ in the definition of duality, hence we may assume that $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ are in the standard form, so of the same type as A' and B', respectively.

If, say B', is atomic and finite then by Magajna [35, 3.4]

$$\mathrm{B}(\mathcal{K}, \mathcal{H})^{\natural} = \mathrm{CB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = \mathrm{NCB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{eh}{\otimes} B'.$$

Since the unit ball of $\mathrm{M}_n(A' \overset{h}{\otimes} B')$ is dense in that of $\mathrm{M}_n(A' \overset{eh}{\otimes} B')$ in the A', B'-topology (which can be shown by approximating elements of $\mathrm{M}_n(A' \overset{eh}{\otimes} B') = \mathrm{C}_n(A') \overset{eh}{\otimes} \mathrm{R}_n(B')$ by finite sums similarly as in [31, p. 33]), it follows that

$$\mathrm{CB}_{A'}(A' \overset{eh}{\otimes} B', \mathrm{B}(\mathcal{K}, \mathcal{H}))_{B'} = \mathrm{CB}_{A'}(A' \overset{h}{\otimes} B', \mathrm{B}(\mathcal{K}, \mathcal{H}))_{B'} = \mathrm{B}(\mathcal{K}, \mathcal{H}), \tag{3.2}$$

hence $\mathrm{B}(\mathcal{K}, \mathcal{H})^{\natural\natural} = \mathrm{B}(\mathcal{K}, \mathcal{H})$ and $\mathrm{B}(\mathcal{K}, \mathcal{H})$ is A, B-reflexive.

On the other hand, by Effros and Kishimoto [17] $\mathrm{B}(\mathcal{K}, \mathcal{H})^{\natural} = \mathrm{CB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{\sigma h}{\otimes} B' =: V$, and V contains $\mathrm{NCB}_A(\mathrm{B}(\mathcal{K}, \mathcal{H}))_B = A' \overset{eh}{\otimes} B' =: U$. Now $V^{\natural} =$

$\mathrm{B}(\mathcal{K},\mathcal{H})^{\natural\natural}$ and $U^{\natural} = \mathrm{CB}_{A'}(A' \overset{h}{\otimes} B', \mathrm{B}(\mathcal{K},\mathcal{H}))_{B'} = \mathrm{B}(\mathcal{K},\mathcal{H})$. If $\mathrm{B}(\mathcal{K},\mathcal{H})^{\natural\natural} = \mathrm{B}(\mathcal{K},\mathcal{H})$, the two strong A', B'-bimodules U and V have the same bimodule dual $\mathrm{B}(\mathcal{K},\mathcal{H})$, hence $U = V$ by Theorem 2.17. It follows that $C \overset{\sigma h}{\otimes} D = \mathrm{CB}_{C'}(\mathrm{B}(\mathcal{K},\mathcal{H}))_{D'} = \mathrm{NCB}_{C'}(\mathrm{B}(\mathcal{K},\mathcal{H}))_{D'} = C \overset{eh}{\otimes} D$ for all von Neumann algebras $C \subseteq A'$ and $D \subseteq B'$. If neither A' nor B' is atomic and finite, we can choose C and D both isomorphic to $C = L_\infty[0,1]$. But, with this choice, $C \overset{\sigma h}{\otimes} C \neq C \overset{eh}{\otimes} C$ since there exist nonnormal completely bounded C-bimodule maps on $\mathrm{B}(L_2[0,1])$.

Now we are going to consider very briefly the adjoints of bimodule maps. Again, for maps between strong bimodules the results resemble the classical ones, but there is a difference (Proposition 3.12(iii) below).

The *bimodule adjoint* of a map $T \in \mathrm{CB}_A(X,Y)_B$ is the A', B'-bimodule map

$$T^{\natural} : Y^{\natural} \to X^{\natural}, \quad T^{\natural}(\psi) = \psi \circ T \quad (\psi \in Y^{\natural}). \tag{3.3}$$

If $X^{\natural}$ and $Y^{\natural}$ are proper bimodule duals, we write $T^{\natural\mathrm{p}}$ instead of $T^{\natural}$.

The following proposition can be deduced using Theorem 2.17 by standard arguments, so we omit its proof.

Proposition 3.11. *If $X, Y \in {}_A\mathrm{NOM}_B$ with Y strong and $T \in \mathrm{NCB}_{A'}(Y^{\natural\mathrm{p}}, X^{\natural\mathrm{p}})_{B'}$, then there exists a unique $S \in \mathrm{CB}_A(X,Y)_B$ such that $T = S^{\natural\mathrm{p}}$.*

Note that $\|T^{\natural}\|_{\mathrm{cb}} \leqslant \|T\|_{\mathrm{cb}}$. If $X, Y \in {}_A\mathrm{NOM}_B$ and $T \in \mathrm{CB}_A(X,Y)_B$ then, using the identification $X^{\natural} = (\mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K})^{\sharp}$ from Corollary 3.5(i), $T^{\natural}$ can be expressed as the usual adjoint of another completely bounded map T_h:

$$T^{\natural} = T_h^{\sharp}, \text{ where } T_h = 1_{\mathcal{H}^*} \otimes_A T \otimes_B 1_{\mathcal{K}} : \mathcal{H}^* \overset{h}{\otimes}_A X \overset{h}{\otimes}_B \mathcal{K} \to \mathcal{H}^* \overset{h}{\otimes}_A Y \overset{h}{\otimes}_B \mathcal{K}. \tag{3.4}$$

Proposition 3.12. *Let $X, Y \in {}_A\mathrm{SOM}_B$ and $T \in \mathrm{CB}_A(X,Y)_B$. Then*:

(i) $\|T^{\natural}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$;

(ii) *T is a complete isometry if and only if $T^{\natural}$ is a completely quotient map.*

(iii) *$T^{\natural\mathrm{p}}$ is a complete isometry if and only if for each $n \in \mathbb{N}$ the image $T(B_{\mathrm{M}_n(X)})$ of the unit ball of $\mathrm{M}_n(X)$ is dense in $B_{\mathrm{M}_n(Y)}$ in the A, B-topology.*

(iv) *If $T^{\natural\mathrm{p}}$ is a complete isometry and T is injective, then T is a completely isometric surjection.*

Proof. Parts (i)–(iii) can be deduced by classical reasoning using the operator bipolar Theorem 2.13 and Corollary 3.6. (Alternatively, using (3.4), (i) and (ii) can also be deduced from the corresponding properties of the usual completely bounded adjoint operators, but we omit the details.) To prove (iv), observe that since T is injective, the same holds for T_h in (3.4). (Indeed, if $T_j : X_j \to Y_j$ are injective bimodule

maps then, using (2.5), $T_1 \otimes_B T_2 : X_1 \overset{eh}{\otimes}_B X_2 \to Y_1 \overset{eh}{\otimes}_B Y_2$ can easily be proved to be injective.) Then, by classical duality and (3.4) $T^{\natural_{\mathrm{p}}}$ has dense range. On the other hand, since $T^{\natural_{\mathrm{p}}}$ is a weak* continuous isometry and the ball $B_{Y^{\natural_{\mathrm{p}}}}$ is weak* compact, $B_{T^{\natural_{\mathrm{p}}}(Y^{\natural_{\mathrm{p}}})} = T^{\natural_{\mathrm{p}}}(B_{Y^{\natural_{\mathrm{p}}}})$ must be weak* compact. Now the Krein–Smulian theorem shows that $T^{\natural_{\mathrm{p}}}(Y^{\natural_{\mathrm{p}}})$ is weak* closed, hence it follows that $T^{\natural_{\mathrm{p}}}$ is surjective. Thus $T^{\natural_{\mathrm{p}}}$ is a weak* homeomorphism of the unit balls, hence $R := (T^{\natural_{\mathrm{p}}})^{-1}$ is weak* continuous by the Krein–Smulian theorem. By Proposition 3.11 there exists an $S \in \mathrm{CB}_A(Y, X)_B$ such that $R = S^{\natural_{\mathrm{p}}}$. From $S^{\natural_{\mathrm{p}}} = (T^{\natural_{\mathrm{p}}})^{-1}$ we conclude that $T = S^{-1}$; moreover, since S and T are complete contractions, both must be completely isometric. $\square$

We conclude this section with some applications to Hilbert W*-modules and correspondences. These will not be needed later in the paper. Basic facts about such modules can be found in many sources (e.g. [10] or [42]). We only recall that if B and C are von Neumann algebras, a W*-correspondence from B to C is a self-dual right Hilbert C*-module F over C together with a normal representation of B in the von Neumann algebra $\mathrm{L}(F)$ of all adjointable operators on F, hence $F \in {}_B\mathrm{NDOM}_C$. In this case Theorem 3.2 can be slightly improved.

Proposition 3.13. *If F is a W^*-correspondence from B to C, then*

$$(X \overset{h}{\otimes}_B F)^{\natural_{B\mathrm{nor}}} = (X \overset{h}{\otimes}_B F)^{\natural} = X^{\natural} \overset{eh}{\otimes}_{B'} F^{\natural}$$

for each $X \in {}_A\mathrm{NOM}_B$.

Proof. By Theorem 3.2 it suffices to prove the first equality. We have to show that for each $\theta \in (X \overset{h}{\otimes}_B F)^{\natural}$ and $x \in X$ the map $B \ni b \mapsto \theta(x \otimes_B by) \in \mathrm{B}(\mathcal{L}, \mathcal{H})$ is normal. Consider the C-module map $\theta_x : F \to \mathrm{B}(\mathcal{L}, \mathcal{H})$, $\theta_x(y) = \theta(x \otimes_B y)$. We recall that F is an orthogonally complemented submodule in $\mathrm{C}_{\mathbb{J}}(C)$ for some cardinal $\mathbb{J}$ [10, 8.5.25], hence $\mathrm{L}(F)$ can be regarded as a w*-closed self-adjoint subalgebra in $\mathrm{L}(\mathrm{C}_{\mathbb{J}}(C)) = \mathrm{M}_{\mathbb{J}}(C)$. Extending θ_x to a map $\sigma \in \mathrm{CB}(\mathrm{C}_{\mathbb{J}}(C), \mathrm{B}(\mathcal{L}, \mathcal{H}))_C = \mathrm{R}_{\mathbb{J}}(\mathrm{B}(\mathcal{L}, \mathcal{H}))$ (the last equality can be proved by using (2.3), or see the proof in [35, 5.1]), σ is of the form $\sigma(y) = Ty$ for some $T \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}(\mathcal{L}, \mathcal{H}))$. It follows that $\theta(x, by) = \sigma(by) = Tby$, which is w*-continuous in b. $\square$

Corollary 3.14. *If $X \in {}_A\mathrm{SOM}_B$ and F is a W^*-correspondence from B to C, then $(X \overset{eh}{\otimes}_B F)^{\natural} = X^{\natural} \overset{eh}{\otimes}_{B'} F^{\natural}$.*

Proof (Sketch). First note that the unit ball of $\mathrm{M}_n(X \overset{h}{\otimes}_B F)$ is dense in the $\mathbb{C}, C$-topology in unit ball of $\mathrm{M}_n(X \overset{eh}{\otimes}_B F) = \mathrm{C}_n(X) \overset{eh}{\otimes}_B \mathrm{R}_n(F)$ for each n. (This follows by the argument from [31, p. 33], using the polar decomposition of elements y in a Hilbert C*-module of the form $\mathrm{C}_{\mathbb{J}}(\mathrm{R}_n(F))$, with $|y| \in \mathrm{M}_n(C)$.) By automatic continuity

of C-module maps it follows that $(X \overset{eh}{\otimes}_B F)^\natural = (X \overset{h}{\otimes}_B F)^\natural$ and now Proposition 3.13 concludes the proof. $\square$

We shall study the bimodule dual of $X \overset{eh}{\otimes}_B Y$ in greater generality in the next section. Here we note that by Blecher [6, 3.1] for W*-correspondences, $E \overset{eh}{\otimes}_B F$ is equal to the usual (self-dual) tensor product $E \overline{\otimes}_B F$. Thus, Corollary 3.14 implies that tensor product of correspondences behaves nicely under the bimodule duality, which is observed also in [36]. However, in [36] the duality is defined in a different way, but we shall show in the following example that the two ways are equivalent.

Example 3.15. To compute the bimodule dual of a W*-correspondence E from A to B we use Corollary 3.5(i), (2.3) and the well-known equality $\mathcal{H}^* \overset{h}{\otimes} X = \mathcal{H}^* \hat{\otimes} X$ [10, 1.5.14] to get

$$E^\natural = (\mathcal{H}^* \overset{h}{\otimes}_A E \overset{h}{\otimes}_B \mathcal{K})^\sharp = \mathrm{B}(\mathcal{H}^*, (E \overset{h}{\otimes}_B \mathcal{K})^*)_A.$$

The latter space can be naturally identified with $\mathrm{B}_A(\mathcal{H}, E \overset{h}{\otimes}_B \mathcal{K})$ (see (2.1)), which is essentially the definition of the dual in [36, 3.1]. Note that $\mathrm{B}_A(\mathcal{H}, E \overset{h}{\otimes}_B \mathcal{K})$ is a W*-correspondence from B' to A' for the A'-valued inner product $\langle x, y\rangle_{A'} = x^* y$ and the B'-module action $b'(x \otimes_B \eta) = x \otimes_B b'\eta$ [42]. In the special case when $A = \mathbb{C} = \mathcal{H}$, we have that $E^\natural \cong E \overset{h}{\otimes}_B \mathcal{K}$, hence

$$E^{\natural\natural} \cong \mathrm{CB}_{B'}(E \hat{\otimes}_B \mathcal{K}, \mathcal{K}) = (\mathcal{K}^* \hat{\otimes}_{B'} (E \hat{\otimes}_B \mathcal{K}))^\sharp = (E \hat{\otimes}_B B_\sharp)^\sharp = \mathrm{CB}(E, B)_B = E$$

since E is self-dual. By the comment following Proposition 3.9 this shows that every W*-correspondence is reflexive as an operator bimodule.

4. The bimodule dual of the extended Haagerup tensor product of bimodules

Due to the important role of the extended module Haagerup tensor product, it is worthwhile to compute its bimodule dual. Effros and Ruan [22] defined the *normal Haagerup tensor product* of dual operator spaces by $U^\sharp \overset{\sigma h}{\otimes} V^\sharp := (U \overset{eh}{\otimes} V)^\sharp$. Using that each bimodule $X \in {}_A\mathrm{NDOM}_B$ is of the form $X = (X_{\natural_\mathrm{p}})^{\natural_\mathrm{p}}$ by Theorem 3.7, we may define the module version of this product.

Definition 4.1. For $X \in {}_A\mathrm{NDOM}_B$ and $Y \in {}_B\mathrm{NDOM}_C$ let

$$X \overset{\sigma h}{\otimes}_B Y = (X_{\natural_\mathrm{p}} \overset{eh}{\otimes}_{B'} Y_{\natural_\mathrm{p}})^{\natural_\mathrm{p}},$$

where $X_{\natural_\mathrm{p}} \overset{eh}{\otimes}_{B'} Y_{\natural_\mathrm{p}}$ is regarded as an A', C'-bimodule.

The bimodule $X \overset{\sigma h}{\otimes}_B Y$ can be described in the following way, which shows in particular that, as an operator space, $X \overset{\sigma h}{\otimes}_B Y$ is independent of A and C.

Theorem 4.2. *$X \overset{\sigma h}{\otimes}_B Y = (X \overset{\sigma h}{\otimes} Y)/N$, where N is the weak* closed subspace of $X \overset{\sigma h}{\otimes} Y$ generated by all elements of the form $xb \otimes y - x \otimes by$ $(x \in X,\ y \in Y,\ b \in B)$.*

Proof. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be proper Hilbert modules over A, B and C (resp.) in terms of which the duals are defined. By Corollary 3.5(i)

$$X \overset{\sigma h}{\otimes}_B Y = (X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}} = (\mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} \mathcal{L})^{\sharp}. \tag{4.1}$$

By [12,21] $\mathcal{K} \overset{eh}{\otimes} \mathcal{K}^* = (\mathcal{K}^* \overset{h}{\otimes} \mathcal{K})^{\sharp} = \mathrm{B}(\mathcal{K})$, hence $B' \subseteq \mathcal{K} \overset{eh}{\otimes} \mathcal{K}^*$ and

$$U := \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} \mathcal{L} = \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} B' \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} \mathcal{L} \tag{4.2}$$

is an operator subspace of

$$\begin{aligned} V :&= \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} \mathrm{B}(\mathcal{K}) \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} \mathcal{L} \\ &= \mathcal{H}^* \overset{eh}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} \mathcal{K} \overset{eh}{\otimes} \mathcal{K}^* \overset{eh}{\otimes}_{B'} Y_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{C'} \mathcal{L}. \end{aligned}$$

Note that $X = (\mathcal{H}^* \overset{h}{\otimes}_{A'} X_{\natural_{\mathrm{p}}} \overset{h}{\otimes}_{B'} \mathcal{K})^{\sharp}$ for each $X \in {}_A\mathrm{NDOM}_B$. (Namely, by Theorem 3.7 $X = (X_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}}$; now apply Corollary 3.5(i).) It follows that $V = X_{\sharp} \overset{eh}{\otimes} Y_{\sharp}$. From (4.1) and (4.2) we have $X \overset{\sigma h}{\otimes}_B Y = U^{\sharp}$. The adjoint of the inclusion $U \to V$ is the weak* continuous completely quotient map

$$q : X \overset{\sigma h}{\otimes} Y = V^{\sharp} \to U^{\sharp} = X \overset{\sigma h}{\otimes}_B Y$$

with $\ker q = U^{\perp}$, the annihilator of U in $V^{\sharp}$. It remains to prove that $U^{\perp} = N$ or equivalently, since N is weak* closed, that $U = N_{\perp}$ $(\subseteq V)$.

A general element v of V has the form $v = \xi^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{C'} \eta$, where

$$\xi \in \mathrm{C}_{\mathbb{J}}(\mathcal{H}),\ \ \eta \in \mathrm{C}_{\mathbb{J}}(\mathcal{L}),\ \ \phi = [\phi_{ij}] \in \mathrm{M}_{\mathbb{J}}(X_{\natural_{\mathrm{p}}}),\ \ \psi = [\psi_{ij}] \in \mathrm{M}_{\mathbb{J}}(Y_{\natural_{\mathrm{p}}}),\ \ T \in \mathrm{M}_{\mathbb{J}}(\mathrm{B}(\mathcal{K}))$$

for some cardinal $\mathbb{J}$. We have that $v \in N_\perp$ if and only if

$$\langle v, xb \otimes y - x \otimes by \rangle = 0 \quad \text{for all } x \in X, \ \ y \in Y, \ \ b \in B.$$

This can be written as $\langle (\phi(xb)T\psi(y) - \phi(x)T\psi(by))\eta, \xi \rangle = 0$ or

$$\langle \phi(X)(bT - Tb)\psi(Y)\eta, \xi \rangle = 0. \tag{4.3}$$

Since $[\psi(Y)\eta]$ is a B-submodule of $\mathrm{C}_{\mathbb{J}}(\mathcal{K}) = \mathcal{K}^{\mathbb{J}}$, we have that $[\psi(Y)\eta] = q'\mathcal{K}^{\mathbb{J}}$ for a projection $q' \in \mathrm{M}_{\mathbb{J}}(B')$. Similarly $[\phi(X)^*\xi] = p'\mathcal{K}^{\mathbb{J}}$ for some projection $p' \in \mathrm{M}_{\mathbb{J}}(B')$, and (4.3) is equivalent to the requirement that $p'(bT - Tb)q' = 0$ for all $b \in B$ or

$$p'Tq' \in \mathrm{M}_{\mathbb{J}}(B'). \tag{4.4}$$

Let $e' \in \mathrm{M}_{\mathbb{J}}(A')$ and $f' \in \mathrm{M}_{\mathbb{J}}(C')$ be the projections with ranges $[A\xi]$ and $[C\eta]$ (resp.). From $q'\psi(y)\eta = \psi(y)\eta$ $(y \in Y)$ we have that $q'^{\perp}[\psi(Y)C\eta] = q'^{\perp}[\psi(Y)\eta] = 0$ (since ψ is a C-module map), hence $q'^{\perp}\psi(Y)f' = 0$. This means that

$$q'^{\perp}\psi f' = 0; \quad \text{similarly} \quad e'\phi p'^{\perp} = 0. \tag{4.5}$$

Finally, it follows that

$$\begin{aligned} v &= \xi^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{C'} \eta \\ &= (e'\xi)^* \odot_{A'} \phi \odot_{B'} T \odot_{B'} \psi \odot_{B'} f'\eta \\ &= \xi^* \odot_{A'} e'\phi \odot_{B'} T \odot_{B'} \psi f' \odot_{C'} \eta \\ &= \xi^* \odot_{A'} e'\phi p' \odot_{B'} T \odot_{B'} q'\psi f' \odot_{C'} \eta \quad \text{(by (4.5))} \\ &= \xi^* \odot_{A'} e'\phi \odot_{B'} p'Tq' \odot_{B'} \psi f' \odot_{C'} \eta \\ &\in U \ \text{(by (4.4) and (4.2))}. \end{aligned}$$

This (reversible) computation proves that $U = N_\perp$. $\square$

If A and C are von Neumann algebras on a Hilbert space $\mathcal{H}$, the space $A \overset{\sigma h}{\otimes} C$ was identified by Effros and Kishimoto [17] with $\mathrm{CB}_{A'}(\mathrm{B}(\mathcal{H}))_{C'}$. If B is a common von Neumann subalgebra in A and C, we have a similar identification for $A \overset{\sigma h}{\otimes}_B C$ (Theorem 4.4), but for this we first need to extend a result from [12, p. 131].

Proposition 4.3. *Given von Neumann algebras $T \subseteq \mathrm{B}(\mathcal{H}_T)$, A, B, Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and normal representations $A \xrightarrow{\alpha} \mathrm{B}(\mathcal{H})$, $A \xrightarrow{\pi} T$, $B \xrightarrow{\beta} \mathrm{B}(\mathcal{K})$, $B \xrightarrow{\sigma} T$, we have*

$$\mathrm{B}_A(\mathcal{H}_T, \mathcal{H}) \overset{eh}{\otimes}_{T'} \mathrm{B}_B(\mathcal{K}, \mathcal{H}_T) = \mathrm{NCB}_A(T, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B \tag{4.6}$$

completely isometrically by letting each $a' \odot_{T'} b'$ *to act on* T *as*

$$(a' \odot_{T'} b')(t) = a'tb' \quad (t \in T,\ \ a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}_A(\mathcal{H}_T, \mathcal{H})),\ \ b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}_B(\mathcal{K}, \mathcal{H}_T))).$$

Proof. It is perhaps well known (and easy) that for two Hilbert spaces $\mathcal{G} = \mathrm{C}_{\mathbb{I}}(\mathbb{C})$ and $\mathcal{L}^* = \mathrm{R}_{\mathbb{J}}(\mathbb{C})$ and any operator space X we have $\mathrm{C}_{\mathbb{I}}(\mathbb{C}) \overset{eh}{\otimes} X \overset{eh}{\otimes} \mathrm{R}_{\mathbb{J}}(\mathbb{C}) = \mathrm{M}_{\mathbb{I},\mathbb{J}}(X) = \mathrm{M}_{\mathbb{I},\mathbb{J}}(\mathrm{NCB}(X^\sharp, \mathbb{C})) = \mathrm{NCB}(X^\sharp, \mathrm{M}_{\mathbb{I},\mathbb{J}}(\mathbb{C}))$, hence

$$\mathcal{G} \overset{eh}{\otimes} X \overset{eh}{\otimes} \mathcal{L}^* = \mathrm{NCB}(X^\sharp, \mathrm{B}(\mathcal{L}, \mathcal{G})) \tag{4.7}$$

completely isometrically.

In the case $A = \mathbb{C} = B$ the proof of the proposition consists of the following computation:

$$\begin{aligned}
\mathrm{B}(\mathcal{H}_T, \mathcal{H}) \overset{eh}{\otimes}_{T'} \mathrm{B}(\mathcal{K}, \mathcal{H}_T) &= (\mathcal{H} \overset{eh}{\otimes} \mathcal{H}_T^*) \overset{eh}{\otimes}_{T'} (\mathcal{H}_T \overset{eh}{\otimes} \mathcal{K}^*) \\
&= \mathcal{H} \overset{eh}{\otimes} (\mathcal{H}_T^* \overset{eh}{\otimes}_{T'} \mathcal{H}_T) \overset{eh}{\otimes} \mathcal{K}^* \\
&= \mathcal{H} \overset{eh}{\otimes} T_\sharp \overset{eh}{\otimes} \mathcal{K}^* \quad \text{(by Corollary 3.3)} \\
&= \mathrm{NCB}(T, \mathrm{B}(\mathcal{K}, \mathcal{H})) \ \text{(by (4.7))}.
\end{aligned}$$

In general, we have now only to show that each $\theta \in \mathrm{NCB}_A(T, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, just proved to be of the form $\theta = a' \odot_{T'} b'$ for some $a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}(\mathcal{H}_T, \mathcal{H}))$ and $b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}(\mathcal{K}, \mathcal{H}_T))$, has this form with the addition that $a' \in \mathrm{R}_{\mathbb{J}}(\mathrm{B}_A(\mathcal{H}_T, \mathcal{H}))$ and $b' \in \mathrm{C}_{\mathbb{J}}(\mathrm{B}_B(\mathcal{K}, \mathcal{H}_T))$; for this see the proof of [29, 1.2]. $\square$

In (4.6) $E := \mathrm{B}_A(\mathcal{H}_T, \mathcal{H})$ and $F := \mathrm{B}_B(\mathcal{K}, \mathcal{H}_T)$ are right Hilbert W*-modules over $\pi(A)'$ and $\beta(B)'$ (resp.), but the tensor product is over T' (not over $\pi(A)'$). In the special case, when $A = T$, $\pi = \mathrm{id}$ and β is the inclusion, (4.6) can be interpreted as $E \overline{\otimes}_{T'} F = \mathrm{CB}_{T'}(E^*, F)$, a result of Denizeau and Havet as stated in [6, 3.3]. Since this will not be needed here, we shall not explain it further.

The following theorem is a generalization of [17, 2.5].

Theorem 4.4. *Let* $B \to A \subseteq \mathrm{B}(\mathcal{H})$ *and* $B \to C \subseteq \mathrm{B}(\mathcal{L})$ *be normal* $*$*-homomorphisms of von Neumann algebras* (*so that* A *and* C *are* B*-bimodules*). *Then*

$$A \overset{\sigma h}{\otimes}_B C = \mathrm{CB}_{A'}(\mathrm{B}_B(\mathcal{L}, \mathcal{H}), \mathrm{B}(\mathcal{L}, \mathcal{H}))_{C'}.$$

More precisely, the equality here means the completely isometric weak homeomorphism of* A, C*-bimodules that sends* $a \otimes_B c$ *to the map* $x \mapsto axc$ $(x \in \mathrm{B}_B(\mathcal{L}, \mathcal{H}))$.

Proof. Let $\mathcal{K}$ be a proper Hilbert B-module. Regarding A as a $\mathbb{C}, B$-bimodule and C as a $B, \mathbb{C}$-bimodule, we have as special cases of Proposition 4.3:

$$A_{\natural_{\mathrm{p}}} = \mathrm{NCB}(A, \mathcal{K}^*)_B = \mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(\mathcal{K}, \mathcal{H}), \;\; C_{\natural_{\mathrm{p}}} = \mathrm{NCB}_B(C, \mathcal{K}) = \mathrm{B}_B(\mathcal{L}, \mathcal{K}) \overset{eh}{\otimes}_{C'} \mathcal{L},$$

hence

$$A \overset{\sigma h}{\otimes}_B C = (A_{\natural_{\mathrm{p}}} \overset{eh}{\otimes}_{B'} C_{\natural_{\mathrm{p}}})^\sharp = \left(\mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(\mathcal{K}, \mathcal{H}) \overset{eh}{\otimes}_{B'} \mathrm{B}_B(\mathcal{L}, \mathcal{K}) \overset{eh}{\otimes}_{C'} \mathcal{L}\right)^\sharp.$$

Since by Proposition 4.3

$$\mathrm{B}_B(\mathcal{K}, \mathcal{H}) \overset{eh}{\otimes}_{B'} \mathrm{B}_B(\mathcal{L}, \mathcal{K}) = \mathrm{NCB}_B(B, \mathrm{B}(\mathcal{L}, \mathcal{H}))_B = \mathrm{B}_B(\mathcal{L}, \mathcal{H}),$$

it follows (by using Remark 2.18, the commutativity and associativity of $\hat{\otimes}$, the identities $\mathcal{H}^* \overset{h}{\otimes} V = \mathcal{H}^* \hat{\otimes} V$, $V \overset{h}{\otimes} \mathcal{L} = V \hat{\otimes} \mathcal{L}$ and (2.3)) that

$$\begin{aligned} A \overset{\sigma h}{\otimes}_B C &= (\mathcal{H}^* \overset{eh}{\otimes}_{A'} \mathrm{B}_B(\mathcal{L}, \mathcal{H}) \overset{eh}{\otimes}_{C'} \mathcal{L})^\sharp \\ &= (\mathcal{H}^* \hat{\otimes}_{A'} \mathrm{B}_B(\mathcal{L}, \mathcal{H}) \hat{\otimes}_{C'} \mathcal{L})^\sharp \\ &\cong (\mathrm{B}_B(\mathcal{L}, \mathcal{H})\, {}_{A'}\hat{\otimes}_{C'} (\mathcal{L} \hat{\otimes} \mathcal{H}^*))^\sharp \\ &= \mathrm{CB}_{A'}(\mathrm{B}_B(\mathcal{L}, \mathcal{H}), \mathrm{B}(\mathcal{L}, \mathcal{H}))_{C'}. \qquad \square \end{aligned}$$

5. The normal part of an operator bimodule

In this section $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$ will be C^*-algebras, $\Phi : A \to \mathrm{B}(\tilde{\mathcal{H}})$, $\Psi : B \to \mathrm{B}(\tilde{\mathcal{K}})$ the universal representations and $\tilde{A} = \overline{\Phi(A)}$ and $\tilde{B} = \overline{\Psi(B)}$ the von Neumann envelopes of A and B, respectively.

We first recall some basic facts about the universal representation Φ of a C*-algebra A (see [27, Section 10.1] for more details if necessary). Since Φ is the direct sum of all cyclic representations of A obtained from the GNS construction, each $\rho \in A^\sharp$ is of the form $\rho(a) = \langle \Phi(a)\eta, \xi \rangle$ for some vectors $\xi, \eta \in \tilde{\mathcal{H}}$, therefore $\rho\Phi^{-1}$ has a unique normal extension to $\tilde{A}$. It follows that $\tilde{A} = A^{\sharp\sharp}$ and that for each $T \in \mathrm{B}(A, \mathrm{B}(\mathcal{L}))$ the map $T\Phi^{-1}$ has a unique weak* continuous extension $\tilde{T} : \tilde{A} \to \mathrm{B}(\mathcal{L})$. In particular, with $T = i_A : A \to \mathrm{B}(\mathcal{H})$ the inclusion, the map $\tilde{i}_A : \tilde{A} \to \overline{A}$ is a normal $*$-homomorphism, hence

$$\ker \tilde{i}_A = P^\perp \tilde{A} \quad (\text{and similarly} \quad \ker \tilde{i}_B = Q^\perp \tilde{B}) \tag{5.1}$$

for some central projections $P \in \tilde{A}$ (and $Q \in \tilde{B}$). Finally, recall that a map $T \in \mathrm{B}(A, \mathrm{B}(\mathcal{L}))$ is weak* continuous if and only if $T(a) = \tilde{T}(P\Phi(a))$ for all $a \in A$.

Now we are going to explain how a dual Banach A, B-bimodule is in a canonical way an $\tilde{A}, \tilde{B}$-bimodule.

Definition 5.1. Given $X = V^\sharp \in {}_A\mathrm{DBM}_B$ (as in Definition 2.5), for each $x \in X$ and $v \in V$ let $\omega_{x,v} \in A^\sharp$ and $\rho_{x,v} \in B^\sharp$ be defined by

$$\omega_{x,v}(a) = \langle ax, v\rangle \quad \text{and} \quad \rho_{x,v}(b) = \langle xb, v\rangle$$

and let $\tilde{\omega}_{x,v}$ and $\tilde{\rho}_{x,v}$ be the weak* continuous extensions of $\omega_{x,v}\Phi^{-1}$ and $\rho_{x,v}\Psi^{-1}$ to $\tilde{A}$ and $\tilde{B}$, respectively. Then for $a \in \tilde{A}$, $b \in \tilde{B}$ and $x \in X$ define ax and xb by

$$\langle ax, v\rangle = \tilde{\omega}_{x,v}(a) \quad \text{and} \quad \langle xb, v\rangle = \tilde{\rho}_{x,v}(b). \tag{5.2}$$

It will turn out that this defines an $\tilde{A}, \tilde{B}$ bimodule structure on X, which will be called *the canonical* $\tilde{A}, \tilde{B}$*-bimodule* structure on X.

Relations (5.2) mean that if $a \in \tilde{A}$, $b \in \tilde{B}$ and (a_i), (b_j) are nets in A and B (resp.) such that $(\Phi(a_i))$ and $(\Psi(b_j))$ weak* converge to a and b (resp.), then

$$ax = \lim_i a_i x \quad \text{and} \quad xb = \lim_j x b_j \tag{5.3}$$

in the weak* topology of X.

Remark 5.2. Recall (Theorems 2.3, 2.9) that on a dual operator space X each operator left A-module structure is given by a $*$-homomorphism π from A into the von Neumann algebra $A_l(X)$. The above structure of a left $\tilde{A}$-module then necessary comes from the normal extension $\tilde{\pi} : \tilde{A} \to A_l(X)$ of π. A similar conclusion holds for right modules and X is automatically a normal dual operator $\tilde{A}, \tilde{B}$-bimodule. If X is a general dual Banach bimodule, however, we need to prove that

$$(ax)b = a(xb) \quad (a \in \tilde{A},\ \ b \in \tilde{B},\ \ x \in X). \tag{5.4}$$

Proposition 5.3. (i) *If* $X \in {}_A\mathrm{DBM}_B$ *then relations* (5.2) *introduce to* X *the structure of a Banach* $\tilde{A}, \tilde{B}$*-bimodule. Moreover, if* $X \in {}_A\mathrm{DOM}_B$ *then* X *is a normal dual operator* $\tilde{A}, \tilde{B}$*-bimodule.*

(ii) *Each weak* continuous* A, B*-bimodule map* T *between dual Banach* A, B*-bimodules is automatically an* $\tilde{A}, \tilde{B}$*-bimodule map.*

Proof. (i) The relations $(a_1a_2)x = a_1(a_2x)$ and $x(b_1b_2) = (xb_1)b_2$ $(a_k \in \tilde{A},\ \ b_k \in \tilde{B})$ follow easily from (5.3). To prove (5.4), chose nets $(a_i) \subseteq A$ and $(b_j) \subseteq B$ so that $(\Phi(a_i))$ and $(\Psi(b_j))$ weak* converge to $a \in \tilde{A}$ and $b \in \tilde{B}$ (resp.). Then, since the right multiplication by b_j on X is weak* continuous,

$$(ax)b_j = (\lim_i a_i x)b_j = \lim_i (a_i x b_j) = \lim_i (a_i(xb_j)) = a(xb_j).$$

Therefore $(ax)b = \lim_j((ax)b_j) = \lim_j(a(xb_j))$ and we would like to show that this is equal to $a(xb)$ or, equivalently, that

$$\lim_j \langle a(xb_j), v\rangle = \langle a(xb), v\rangle = \tilde{\omega}_{xb,v}(a)$$

for each $v \in V = X_\sharp$. For this, it suffices to show that (for $a \in \tilde{A}$) the functional $\tilde{B} \ni b \mapsto \tilde{\omega}_{xb,v}(a)$ is normal, which in turn is a consequence of weak compactness of bounded operators from C*-algebras to preduals of von Neumann algebras [1]. Namely, the weak compactness of the operator $T : A \to B^\sharp$, $T(a)(b) = \theta(a, b)$, where $\theta(a, b) = \omega_{xb,v}(a) = \rho_{ax,v}(b)$, implies that the left and the right canonical extensions of θ to $\tilde{A} \times \tilde{B}$ agree (see [14, p. 12]). This means that $\tilde{\omega}_{xb,v}(a) = \tilde{\rho}_{ax,v}(b)$, which is a normal functional in the variable $b \in \tilde{B}$.

If $X \in {}_A\mathrm{DOM}_B$ then, as we have noted in Remark 5.2, X is a normal dual operator $\tilde{A}, \tilde{B}$-bimodule.

(ii) This is a consequence of (5.3) and the weak* continuity of T. $\square$

Remark 5.4. Given $X \in {}_A\mathrm{BM}_B$, $X^\sharp$ is a dual Banach B, A-bimodule (in the sense of Definition 2.5), hence by Proposition 5.3 $X^\sharp$ is canonically a $\tilde{B}, \tilde{A}$-bimodule. Now on $X^{\sharp\sharp}$ we have two $\tilde{A}, \tilde{B}$-bimodule structures:

(i) The dual $\tilde{A}, \tilde{B}$-bimodule in the sense of Definition 2.5, that is $\langle aFb, \theta\rangle = \langle F, b\theta a\rangle$ ($a \in \tilde{A}$, $b \in \tilde{B}$, $\theta \in X^\sharp$, $F \in X^{\sharp\sharp}$); we denote this bimodule by $X_d^{\sharp\sharp}$.

(ii) The canonical $\tilde{A}, \tilde{B}$-bimodule as in Definition 5.1, that is $aF = \lim_i a_i F$ and $Fb = \lim_j Fb_j$ in the weak* topology of $X^{\sharp\sharp}$, where $(a_i) \subseteq A$ and $(b_j) \subseteq B$ are nets such that $(\Phi(a_i)) \to a$ and $(\Psi(b_j)) \to b$ and where $X^{\sharp\sharp}$ (as an A, B-bimodule) is dual to the B, A-bimodule $X^\sharp$.

If $X_d^{\sharp\sharp}$ is a normal $\tilde{A}, \tilde{B}$-bimodule, then $X_d^{\sharp\sharp} = X^{\sharp\sharp}$ by continuity since $X_d^{\sharp\sharp}$ and $X^{\sharp\sharp}$ agree as A, B-bimodules.

Proposition 5.5. *If $X \in {}_A\mathrm{OM}_B$, then $X_d^{\sharp\sharp}$ is a normal dual operator $\tilde{A}, \tilde{B}$-bimodule, hence $X_d^{\sharp\sharp} = X^{\sharp\sharp}$.*

Proof. There exist a Hilbert space $\mathcal{L}$, representations $\pi : A \to \mathrm{B}(\mathcal{L})$ and $\sigma : B \to \mathrm{B}(\mathcal{L})$ and a completely isometric A, B-bimodule embedding $X \subseteq \mathrm{B}(\mathcal{L})$. Then $X_d^{\sharp\sharp} \subseteq \mathrm{B}(\mathcal{L})_d^{\sharp\sharp} = \widetilde{\mathrm{B}(\mathcal{L})}$ (the universal von Neumann envelope of $\mathrm{B}(\mathcal{L})$), hence it suffices to prove that $\widetilde{\mathrm{B}(\mathcal{L})}$ is a normal $\tilde{A}, \tilde{B}$-bimodule, where

$$\langle ax, \theta\rangle = \langle x, \theta a\rangle \quad \text{and} \quad \langle xb, \theta\rangle = \langle x, b\theta\rangle \quad (a \in \tilde{A},\ b \in \tilde{B},\ \theta \in \mathrm{B}(\mathcal{L})^\sharp,\ x \in \widetilde{\mathrm{B}(\mathcal{L})}).$$

Here $\langle x, \theta a\rangle$ means $\widetilde{\theta a}(x)$, where $\widetilde{\theta a}$ is the normal extension of the functional $\theta a \in \mathrm{B}(\mathcal{L})^\sharp$ to $\widetilde{\mathrm{B}(\mathcal{L})}$. But, since the multiplication $\tilde{A} \times \widetilde{\mathrm{B}(\mathcal{L})} \ni (a, x) \mapsto ax$ is separately weak* continuous in both variables (for ax is just the internal product $\pi^{\sharp\sharp}(a)x$ in $\widetilde{\mathrm{B}(\mathcal{L})}$

and $\pi^{\sharp\sharp} : \tilde{A} = A^{\sharp\sharp} \to \mathrm{B}(\mathcal{L})^{\sharp\sharp} = \widetilde{\mathrm{B}(\mathcal{L})}$ is normal), we have that $\widetilde{\theta a}(x) = \tilde{\theta}(ax)$, where $\tilde{\theta}$ is the weak* continuous extension of $\theta \in \mathrm{B}(\mathcal{L})^{\sharp}$ to $\widetilde{\mathrm{B}(\mathcal{L})}$. It follows that $\langle x, \theta a\rangle = \tilde{\theta}(ax)$ and, since the map $\tilde{A} \ni a \mapsto \tilde{\theta}(ax)$ is weak* continuous, $\widetilde{\mathrm{B}(\mathcal{L})}$ is a normal left $\tilde{A}$-module. Similarly $\widetilde{\mathrm{B}(\mathcal{L})}$ is a normal right $\tilde{B}$-module. The identity $X_d^{\sharp\sharp} = X^{\sharp\sharp}$ follows now from Remark 5.4. $\square$

Definition 5.6. The *normal part* of a bimodule $X \in {}_A\mathrm{OM}_B$, denoted by X_{n}, is the norm closure of $\imath(X)$, where $\imath : X \to X^{\natural\natural}$ is the natural complete contraction.

The name 'normal part' may be justified by the universal property of X_{n} stated in part (i) of the following proposition.

Proposition 5.7. *Let A and B be von Neumann algebras and $X \in {}_A\mathrm{OM}_B$. Then*:

(i) *$X_{\mathrm{n}} \in {}_A\mathrm{NOM}_B$ and the canonical map $\imath \in \mathrm{CB}_A(X, X_{\mathrm{n}})_B$ has the following properties*: (1) $\|\imath\|_{\mathrm{cb}} \leqslant 1$ *and* (2) *for each $Y \in {}_A\mathrm{NOM}_B$ and $T \in \mathrm{CB}_A(X, Y)_B$ there exists a unique $T_{\mathrm{n}} \in \mathrm{CB}_A(X_{\mathrm{n}}, Y)_B$ with $\|T_{\mathrm{n}}\| \leqslant \|T\|$ and $T_{\mathrm{n}}\imath = T$. Moreover, if $X_0 \in {}_A\mathrm{NOM}_B$ and a map $\imath_0 \in \mathrm{CB}_A(X, X_0)_B$ also has properties* (1) *and* (2) (*with $\imath$ replaced by $\imath_0$*), *then there exists a completely isometric A, B-bimodule isomorphism $\sigma : X_{\mathrm{n}} \to X_0$ such that $\imath_0 = \sigma\imath$.*

(ii) *If $Y \in {}_A\mathrm{NOM}_B$ and $\phi \in \mathrm{CB}(X, Y)$ is weakly A, B-continuous in the sense that $\rho\phi \in X^{A\sharp B}$ for each $\rho \in Y^{A\sharp B}$, then there exists a* (*unique*) *map $\phi_{\mathrm{n}} \in \mathrm{CB}(X_{\mathrm{n}}, Y)$ such that $\phi_{\mathrm{n}}\imath = \phi$, and we have that $\|\phi_{\mathrm{n}}\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$ and ϕ_{n} is weakly A, B-continuous. In particular $X^{A\sharp B} = (X_{\mathrm{n}})^{A\sharp B}$ completely isometrically.*

Proof. (i) By Proposition 2.15 $X^{\natural\natural}$ (hence also X_{n}) is a normal operator A, B-bimodule. If $\imath_Y : Y \to Y^{\natural\natural}$ is the canonical inclusion (completely isometric by Corollary 3.6 since Y is normal), then $\imath_Y T = T^{\natural\natural}\imath_X$, hence we may simply set $T_{\mathrm{n}} = T^{\natural\natural}|X_{\mathrm{n}}$. The rest of (i) is evident, by elementary categorical arguments.

(ii) If $\rho \in X^{\sharp}$, then ρ is a weak* continuous functional on the normal dual operator $\tilde{A}, \tilde{B}$-bimodule $X^{\sharp\sharp}$, hence it follows from Corollary 3.5(ii) that there exist an index set $\mathbb{J}$, unit vectors $\xi \in \tilde{\mathcal{H}}^{\mathbb{J}}$, $\eta \in \tilde{\mathcal{K}}^{\mathbb{J}}$ and a map $\psi \in \mathrm{CB}_{\tilde{A}}(X^{\sharp\sharp}, \mathrm{B}(\tilde{\mathcal{K}}^{\mathbb{J}}, \tilde{\mathcal{H}}^{\mathbb{J}}))_{\tilde{B}}$ such that

$$\rho(x) = \langle \psi(x)\eta, \xi\rangle \quad (x \in X^{\sharp\sharp})$$

and $\|\psi\|_{\mathrm{cb}} = \|\rho\|_{\mathrm{cb}}$. If in addition $\rho \in X^{A\sharp B}$ then, since the functionals $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are normal, it follows by Kadison and Ringrose [27, 10.1.13] that

$$\rho(x) = \rho(PxQ) = \langle \psi(PxQ)\eta, \xi\rangle = \langle P\psi(x)Q\eta, P\xi\rangle \quad (x \in X).$$

We may regard the map $X \ni x \mapsto P\psi(x)Q$ as an A, B-bimodule map ψ_0 from X into the normal operator A, B-bimodule $\mathrm{B}(Q\tilde{\mathcal{K}}^{\mathbb{J}}, P\tilde{\mathcal{H}}^{\mathbb{J}})$, hence by part (i) there exists a map $\psi_1 \in \mathrm{CB}_A(X_{\mathrm{n}}, \mathrm{B}(Q\tilde{\mathcal{K}}^{\mathbb{J}}, P\tilde{\mathcal{H}}^{\mathbb{J}})_B$ such that $\psi_0 = \psi_1\imath$ and $\|\psi_1\|_{\mathrm{cb}} \leqslant \|\psi\|_{\mathrm{cb}}$.

With $\rho_{\mathrm{n}} \in (X_{\mathrm{n}})^{A\sharp B}$ defined by

$$\rho_{\mathrm{n}}(v) = \langle \psi_1(v) Q\eta, P\xi \rangle \quad (v \in X_{\mathrm{n}}),$$

we clearly have that $\rho = \rho_{\mathrm{n}}\imath$ and $\|\rho_{\mathrm{n}}\| \leqslant \|\psi_1\|_{\mathrm{cb}} \leqslant \|\rho\|$. The reverse inequality, $\|\rho\| \leqslant \|\rho_{\mathrm{n}}\|$, follows from $\rho = \rho_{\mathrm{n}}\imath$ since $\|\imath\|_{\mathrm{cb}} \leqslant 1$. Since $\imath(X)$ is dense in X_{n}, ρ_{n} is unique.

For a more general weakly A, B-continuous map $\phi \in \mathrm{CB}(X, Y)$, we regard Y as a normal operator A, B subbimodule in $\mathrm{B}(\mathcal{L}, \mathcal{G})$ for some normal Hilbert modules $\mathcal{G}$ and $\mathcal{L}$ over A and B, respectively. Since $\omega\phi \in X^{A\sharp B}$ for each $\omega \in \mathrm{B}(\mathcal{L}, \mathcal{G})_\sharp$, we have from the previous paragraph that $\omega\phi(\ker \imath) = 0$. Thus, $\phi(\ker \imath) = 0$ and therefore there exists a unique map $\phi_{\mathrm{n}} : X_{\mathrm{n}} \to Y$ such that $\phi = \phi_{\mathrm{n}}\imath$. We shall omit the verification that this map ϕ_{n} satisfies all the requirements. $\square$

Finally, we can describe the module bidual $X^{\natural\natural}$ and the normal part X_{n} of an operator bimodule X in a useful alternative way.

Theorem 5.8. *Let A, B be von Neumann algebras and $X \in {}_A\mathrm{OM}_B$. Regard X as an A, B-subbimodule in $X^{\sharp\sharp}$ and let $P \in \tilde{A}$, $Q \in \tilde{B}$ be the central projections as in* (5.1). *Then $X^{\natural\natural} = PX^{\sharp\sharp}Q$ and X_{n} is the norm closure of PXQ in $X^{\sharp\sharp}$. For $x \in \mathrm{M}_m(X)$* (*with $\imath : X \to X_{\mathrm{n}}$ the canonical map*) *we have that*

$$\|\imath_m(x)\| = \inf\left(\sup_j \|a_j x b_j\|\right), \tag{5.5}$$

where the infimum is taken either over all nets (a_j) and (b_j) in the unit balls of A and B (*respectively*) *that strongly converge to* 1 *or over all nets of projections $(a_j) \subseteq A$ and $(b_j) \subseteq B$ converging to* 1.

Proof. Since $X^{A\sharp B}$ consists of all $\rho \in X^\sharp$ such that the two maps $A \ni a \mapsto \rho(ax)$ and $B \ni b \mapsto \rho(xb)$ are normal and since a functional ω on A is normal if and only if $\rho = P\rho$ (and similarly for B), it follows that $X^{A\sharp B} = QX^\sharp P$. Since the $\tilde{A}, \tilde{B}$-bimodule $X^{\sharp\sharp}$ is dual to the $\tilde{B}, \tilde{A}$-bimodule $X^\sharp$ by Proposition 5.5, this implies that $(X^{A\sharp B})^\sharp = PX^{\sharp\sharp}Q$. By Proposition 5.7 we have that $X^\natural = (X_{\mathrm{n}})^\natural$ and $X^{A\sharp B} = (X_{\mathrm{n}})^{A\sharp B}$, hence (applying Corollary 3.5(iii) to X_{n})

$$X^{\natural\natural} = (X_{\mathrm{n}})^{\natural\natural} = ((X_{\mathrm{n}})^{A\sharp B})^\sharp = (X^{A\sharp B})^\sharp = PX^{\sharp\sharp}Q. \tag{5.6}$$

Now it follows from the definition that X_{n} is just the norm closure of PXQ.

If (a_j) and (b_j) are nets in the unit balls of A and B (resp.) converging to 1 in the strong operator topology, then $\|\imath_m(x)\| = \sup_j \|a_j \imath_m(x) b_j\| \leqslant \sup_j \|a_j x b_j\|$ since X_{n} is normal and $\|\imath_m\|_{\mathrm{cb}} \leqslant 1$. This proves the inequality $\leqslant$ in (5.5). To prove the reverse inequality, choose nets (a_j) and (b_j) in the unit balls of A and B so that $(\Phi(a_j))$ and $(\Psi(b_j))$ strongly converge to P and Q, respectively. (Note that then (a_j) and (b_j) must

converge to 1 since the normal extensions of Φ^{-1} and Ψ^{-1} are strongly continuous on bounded sets and map P and Q to 1.) Since $\|\iota_m(x)\| = \|PxQ\|$ and $X^{\sharp\sharp}$ is a normal operator $\tilde{A}, \tilde{B}$-bimodule,

$$\|\iota_m(x)\| = \|PxQ\| = \sup_j \|\Phi(a_j)x\Psi(b_j)\| = \sup_j \|a_j x b_j\|.$$

We may replace in this equality each a_j (resp. b_j) with the range projection $R(a_j) \in A$ (resp. $R(b_j) \in B$) since $a_j \leqslant R(a_j) \leqslant 1$. $\square$

6. Central bimodules

In this section we consider normality for central bimodules. A slightly more general version of central bimodules than defined below is studied also in [8].

Definition 6.1. A bimodule X over an abelian operator algebra C is called *central* if $cx = xc$ for all $x \in X$ and $c \in C$. The classes of central C-bimodules among, operator and normal operator bimodules are denoted by COM_C and CNOM_C, respectively.

Remark 6.2. If C is a C*-subalgebra of the center of a C*-algebra A, J is a closed ideal in C and $X \subseteq A$, then $d(x, [JA]) = d(x, [JX])$ for each $x \in X$, where $d(x, S)$ denotes the distance of x to a set S. This, probably well known fact, follows by choosing an approximate identity (e_j) for J and noting that (since (e_j) is also an approximate identity for $[JA]$) $d(x, [JA]) = \lim_j \|(x - e_j x)\| \geqslant d(x, [JX])$ (see [27, p. 300]).

Remark 6.3. For an abelian C*-algebra C we denote by Δ the spectrum of C and by C_t the kernel of a character $t \in \Delta$. For a bimodule $X \in \mathrm{COM}_C$ we consider the quotients $X(t) = X/[C_t X]$. Given $n \in \mathbb{N}$ and $x \in \mathrm{M}_n(X)$ we denote by $x(t)$ the coset of x in $\mathrm{M}_n(X)/[C_t \mathrm{M}_n(X)]$. We shall need to know that the function

$$\Delta \ni t \mapsto \|x(t)\| \tag{6.1}$$

is upper semicontinuous and that

$$\|x\| = \sup_{t \in \Delta} \|x(t)\|. \tag{6.2}$$

This is known from [15, p. 37, 41; 40, p. 71], but (to avoid Banach bundles) we provide now a different short argument. We may assume that $X, C \subseteq \mathrm{B}(\mathcal{L})$ for some Hilbert space $\mathcal{L}$. Since X is central, $X \subseteq C'$, hence $\mathrm{M}_n(X) \subseteq \mathrm{M}_n(C') =: A$ and C is identified with the center of A. Using Remark 6.2 we have that $\|x(t)\| = d(x, [C_t \mathrm{M}_n(X)]) = d(x, [C_t A])$, which is just the norm of the coset of x in $A/[C_t A]$. Now (6.2) and the

continuity of function (6.1) follow from [25, p. 232]. We shall call the embedding

$$X \to \oplus_{t\in\Delta} X(t), \quad x \mapsto (x(t))_{t\in\Delta}$$

the *canonical decomposition* of X.

Throughout the rest of the section C is an abelian von Neumann algebra.

Lemma 6.4. *A bimodule $X \in \mathrm{COM}_C$ is normal if and only if pX is a normal pC-bimodule for each σ-finite projection $p \in C$. If C is σ-finite, then X is normal if and only if*

$$\lim_j \|p_j x\| = \|x\| \tag{6.3}$$

for each $x \in \mathrm{M}_n(X)$ ($n \in \mathbb{N}$) and each sequence of projections $p_j \in C$ increasing to 1.

Proof. We may assume that C is σ-finite, for in general C is a direct sum of σ-finite subalgebras and X (being central) also decomposes in the corresponding ℓ_∞-direct sum. Then by Theorem 2.4 we have to prove that for each $n \in \mathbb{N}$, each $x \in \mathrm{M}_n(X)$ and sequence (e_j) of projections in $\mathrm{M}_n(C)$ increasing to 1 the sequence $(\|e_j x\|)$ converges to $\|x\|$. Suppose the contrary, that for an x and a sequence of projections (e_j) we have

$$\|e_j x\| \leqslant M \quad \text{for some constant} \quad M < \|x\|.$$

Let τ be the canonical normal central trace on $\mathrm{M}_n(C)$, the values of which on projections of $\mathrm{M}_n(C)$ are of the form $\frac{k}{n}p$, where $p \in C$ is a projection and $k \in \{0, 1, \ldots, n\}$. For each j set $\Delta_j = \{t \in \Delta : \ \tau(e_j)(t) = 1\}$, a clopen subset of Δ, and let $p_j \in C$ be the characteristic function of Δ_j. Since the sequence (e_j) increases to 1 and τ is weak* continuous, $\Lambda := \bigcup_j \Delta_j$ is dense in Δ. (Otherwise $\Lambda_0 := \Delta \setminus \overline{\Lambda}$ would be a nonempty open set such that $\tau(e_j)(t) \leqslant 1 - 1/n$ for all j, which is impossible since $e_j \nearrow 1$.) It follows that the sequence (p_j) also increases to 1. For $t \in \Delta_j$, $e_j(t) \in \mathrm{M}_n(C)(t) = \mathrm{M}_n(\mathbb{C})$ is a projection with the normalized trace equal to 1, hence $e_j(t) = 1$. This implies that $e_j p_j = p_j$, hence $\|p_j x\| \leqslant \|e_j x\| \leqslant M < \|x\|$ for all j; but this is in contradiction with assumption (6.3). $\square$

Proposition 6.5. *A bimodule $X \in \mathrm{COM}_C$ is normal if and only if for each $n \in \mathbb{N}$ and each $x \in \mathrm{M}_n(X)$ the function $\Delta \ni t \mapsto \|x(t)\|$ is continuous.*

Proof. If X is normal, then we may assume that $X \subseteq C'$, the commutant of C in $\mathrm{B}(\mathcal{H})$ for a normal Hilbert C-module $\mathcal{H}$, hence $\mathrm{M}_n(X)$ is contained in the commutant of

C in $\mathrm{B}(\mathcal{H}^n)$ and the continuity of (6.1) follows from Remark 6.2 and [25, p. 233, Lemma 10].

For the converse, by Lemma 6.4 we may assume that C is σ-finite and we have to prove the condition (6.3). Let Δ_j be the clopen subset of Δ corresponding to p_j, where p_j are projections as in Lemma 6.4. Since $p_j \nearrow 1$, $\bigcup_j \Delta_j$ is dense in Δ. Thus, using (6.2), the continuity of the functions $t \mapsto \|x(t)\|$ implies that $\|x\| = \lim_j \sup_{t\in\Delta_j} \|x(t)\| = \lim_j \|p_j x\|$. $\square$

Proposition 6.6. *Let $X \in \mathrm{CNOM}_C$ be a strong bimodule and $Y \subseteq X$ a subbimodule. Then the quotient X/Y is a normal operator bimodule if and only if Y is strong and in this case X/Y is also strong.*

Proof. It was observed in [32, p. 204] that X/Y is normal only if Y is strong. For the converse, assume that C is σ-finite and that the condition in Lemma 6.4 for normality is not satisfied. Then there exist an $\dot{x} \in \mathrm{M}_n(X/Y)$, a sequence of projections (p_j) in C increasing to 1 and a constant $M < \|\dot{x}\|$ such that $\|p_j\dot{x}\| < M$ for all j. Put $q_0 = p_0$ and $q_j = p_j - p_{j-1}$ if $j \geqslant 1$. Let $x \in \mathrm{M}_n(X)$ be any representative of the coset $\dot{x}$. By definition of the quotient norm for each j there exists an element $y_j = q_j y_j \in \mathrm{M}_n(Y)$ such that $\|q_j x - y_j\| < M$. Since the sequence (y_j) is bounded and Y is strong, the sum

$$y := \sum_{j=0}^{\infty} q_j y_j = \sum_{j=0}^{\infty} q_j (y_j q_j)$$

defines an element of Y. But then the estimate

$$\|x - y\| = \|\sum_j q_j(x-y)q_j\| = \sup_j \|q_j(x-y)\| \leqslant \sup_j \|p_j(x-y_j)\| < M$$

implies that $\|\dot{x}\| < M$, which is in contradiction with the choice of M.

To verify that X/Y is a strong left C-module (hence a strong C-bimodule since it is central), let (p_j) be an orthogonal family of projections in C and $(\dot{x}_j)$ a family of elements in X/Y such that the sum $\sum_j \dot{x}_j^* \dot{x}_j$ converges in the strong operator topology of some $\mathrm{B}(\mathcal{H})$ containing X/Y as a normal operator C-bimodule. We can choose for each $\dot{x}_j$ a representative $x_j \in X$ so that the set $(x_j)_j$ is bounded, and then $x := \sum_j p_j x_j = \sum_j p_j x_j p_j \in X$. Since the quotient map $Q : X \to X/Y$ is a bounded C-bimodule map (hence continuous in the C-topology), it follows that $\sum_j p_j \dot{x}_j = \sum_j p_j Q(x_j) = Q(x)$, which shows that $\sum_j p_j \dot{x}_j \in X/Y$. $\square$

For central bimodules we can now improve Proposition 3.12.

Corollary 6.7. *If $X, Y \in \mathrm{CNOM}_C$ are strong and $T \in \mathrm{CB}_C(X, Y)$, then T is completely isometric* (*respectively, completely quotient*) *if and only if $T^{\natural\mathfrak{p}}$ is completely quotient* (*respectively, completely isometric*).

Proof. By Proposition 3.12 it remains to prove that T is completely quotient if $T^{\natural\mathfrak{p}}$ is completely isometric. By Proposition 6.6 $X/\ker T$ is a strong central C-bimodule, hence we consider the induced map $\tilde{T} : X/\ker T \to Y$. Since $\tilde{T}^{\natural\mathfrak{p}} : Y^{\natural\mathfrak{p}} \to (X/\ker T)^{\natural\mathfrak{p}} \subseteq X^{\natural\mathfrak{p}}$ is essentially $T^{\natural\mathfrak{p}}$, hence completely isometric, and $\tilde{T}$ is injective, it follows from Proposition 3.12(iv) that $\tilde{T}$ is a completely isometric surjection, hence T is a completely quotient map. $\square$

Definition 6.8. For a function $f : \Delta \to \mathbb{R}$, let essup f be the infimum of all $c \in \mathbb{R}$ such that the set $\{t \in \Delta : \ f(t) > c\}$ is meager (= contained in a countable union of closed sets with empty interiors).

The *essential direct sum*, ess $\oplus_{t\in\Delta} X(t)$, of a family of Banach spaces $(X(t))_{t\in\Delta}$ is defined as the quotient of the ℓ_∞-direct sum $\oplus_{t\in\Delta} X(t)$ by the zero space of the seminorm $x \mapsto \operatorname{essup}\|x(t)\|$. Then ess $\oplus_{t\in\Delta} X(t)$ with the norm $\dot{x} \mapsto \operatorname{essup}\|x(t)\|$ is a Banach space and we denote by $e : \oplus_{t\in\Delta} X(t) \to \operatorname{ess} \oplus_{t\in\Delta} X(t)$ the quotient map. If $(X(t))_{t\in\Delta}$ is a family of operator spaces, then ess $\oplus_{t\in\Delta} X(t)$ is an operator space by the identification

$$\mathrm{M}_n(\operatorname{ess} \oplus_{t\in\Delta} X(t)) = \operatorname{ess} \oplus_{t\in\Delta} \mathrm{M}_n(X(t)).$$

Theorem 6.9. *Given a bimodule $X \in \mathrm{COM}_C$ with the canonical decomposition $\kappa : X \to \oplus_{t\in\Delta} X(t)$, its normal part X_{n} is just the closure of $e\kappa(X)$ in* ess $\oplus_{t\in\Delta} X(t)$.

Proof. First, to show that $e\kappa(X)$ is a normal operator C-module, by Lemma 6.4 we may assume that C is σ-finite and it suffices to prove that for each sequence of projections $p_j \in C$ increasing to 1 and each $x \in \mathrm{M}_n(X)$ the equality

$$\operatorname{essup}\|x(t)\| = \lim_j \operatorname{essup}\|p_j(t)x(t)\| \tag{6.4}$$

holds. With Δ_j the clopen subset of Δ corresponding to p_j, $\bigcup_j \Delta_j$ is dense in Δ. Since the function $\Delta \ni t \mapsto \|x(t)\|$ is upper semi-continuous (hence Borel), it agrees outside a meager set with a continuous function f on Δ by [27, p. 323]. Then $\operatorname{essup}\|x(t)\| = \sup f(t)$, $\operatorname{essup}\|p_j(t)x(t)\| = \sup_t p_j(t)f(t)$ and $\lim_j \sup_t p_j(t)f(t) = \sup f(t)$ by continuity (since $\bigcup_j \Delta_j$ is dense in Δ). This implies (6.4).

It remains to show that the closure of $e\kappa(X)$ has the universal property of X_{n} from Proposition 5.7(i). Let $Y \in {}_C\mathrm{NOM}_C$ and $T \in \mathrm{CB}_C(X, Y)$ with $\|T\|_{\mathrm{cb}} < 1$. We have to show that T can be factorized through $e\kappa(X)$. Replacing Y by the closure of $T(X)$, we may assume that Y is central. Let $x \in \mathrm{M}_n(X)$ and set $y = T_n(x)$. Since $\|T\|_{\mathrm{cb}} < 1$ and T is a C-module map, $\|y(t)\| \leqslant \|x(t)\|$ for each $t \in \Delta$. Set

$$c = \|(e\kappa)_n(x)\| = \operatorname{essup}_t \|x(t)\| \quad \text{and} \quad V = \{t \in \Delta : \ \|y(t)\| > c\}.$$

Since Y is normal, the function $t \mapsto \|y(t)\|$ is continuous by Proposition 6.5, hence V is open. But for each $t \in V$ we have that $c < \|y(t)\| \leqslant \|x(t)\|$, hence V must be meager

by the definition of c, hence $V = \emptyset$ by Baire's theorem for locally compact spaces. Thus, $\|y(t)\| \leqslant c$ for all $t \in \Delta$, which means that $\|T_n(x)\| = \|y\| = \sup_t \|y(t)\| \leqslant c = \|(e\kappa)_n(x)\|$. This estimate shows that there exists a unique complete contraction $S : (e\kappa)(X) \to Y$ such that $T = S \circ (e\kappa)$. $\square$

7. Operator bimodules of a normal representable bimodule

We begin this section by introducing various classes of Banach bimodules admitting operator bimodule structures.

Definition 7.1. (i) [40] A bimodule $X \in {}_A\mathrm{BM}_B$ is *representable* ($X \in {}_A\mathrm{RM}_B$) if for some Hilbert module $\mathcal{H}$ over A, B there is an isometry in $\mathrm{B}_A(X, \mathrm{B}(\mathcal{H}))_B$; in other words, X can be represented isometrically in $\mathrm{B}(\mathcal{H})$ as an operator A, B-bimodule.

(ii) If in (i) A and B are von Neumann algebras and $\mathcal{H}$ is normal over A and B, then X is called a *normal representable* bimodule; the class of all such bimodules is denoted by ${}_A\mathrm{NRM}_B$.

(iii) If $X \in {}_A\mathrm{DBM}_B$ and for some normal Hilbert module $\mathcal{H}$ over A and B there exists an isometry in $\mathrm{N}_A(X, \mathrm{B}(\mathcal{H}))_B$, then X is called a *normal dual representable A, B-bimodule* ($X \in {}_A\mathrm{NDRM}_B$).

An abstract characterization of normal dual representable bimodules is given in [11, 4.14], but it will not be needed here.

For a representable bimodule $X \in {}_A\mathrm{RM}_B$, we define the *proper dual* as $X^{\natural\mathrm{p}} = \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, where $\mathcal{H}$ and $\mathcal{K}$ are fixed proper modules over A and B, respectively. Now a bimodule $X \in {}_A\mathrm{OM}_B$ has two proper duals: in the class ${}_A\mathrm{OM}_B$ and in the class ${}_A\mathrm{RM}_B$. But they agree in ${}_A\mathrm{RM}_B$ by the following result of Smith.

Theorem 7.2 (*Smith [43, 2.1, 2.2]*). *If $\mathcal{G}$ and $\mathcal{L}$ are locally cyclic Hilbert modules over A and B (respectively), then $\|\phi\|_{\mathrm{cb}} = \|\phi\|$ for each $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{L}, \mathcal{G}))_B$ and $X \in {}_A\mathrm{OM}_B$.*

By [32] or [40] the identities

$$\|x\|_{{}_A\mathrm{m}_B} = \sup \|axb\| \quad (x \in \mathrm{M}_n(X),\ \ n = 1, 2, \ldots), \tag{7.1}$$

where the supremum is over all a and b in the unit balls of $\mathrm{R}_n(A)$ and $\mathrm{C}_n(B)$ (respectively), define on X the *minimal operator A, B-bimodule structure*, denoted by $\mathrm{MIN}_A(X)_B$. If $X \in {}_A\mathrm{NRM}_B$, then by considering an isometric representation of X as a normal subbimodule in some $\mathrm{B}(\mathcal{L})$, we see that $X^{A\sharp B}$ has enough functionals to make the natural contraction $\iota : X \to (X^{A\sharp B})^\sharp$ isometric. Thus, with $Y = \mathrm{MIN}_A(X)_B$, the completely contractive isometry $\iota : Y \to (Y^{A\sharp B})^\sharp$ must be completely isometric (otherwise ι would induce on X an operator A, B-bimodule norm structure smaller than $Y = \mathrm{MIN}_A(X)_B$). Since $(Y^{A\sharp B})^\sharp = Y^{\natural\mathrm{p}\natural\mathrm{p}}$ by (5.6) and $Y^{\natural\mathrm{p}\natural\mathrm{p}}$ is normal by

Proposition 2.15, it follows that $\mathrm{MIN}_A(X)_B$ is a normal operator A, B-bimodule. Further,

$$\|x\|_{A\mathrm{m}_B} = \sup\{\|\phi_n(x)\| : \ \phi \in X^{\natural\mathrm{p}}, \ \|\phi\| \leqslant 1\} \quad (x \in \mathrm{M}_n(X), \ n = 1, 2, \ldots). \tag{7.2}$$

Definition 7.3. (i) Given $X \in {}_A\mathrm{RM}_B$, the *maximal operator bimodule norms* are defined by

$$\|x\|_{A\mathrm{M}_B} = \sup \|T_n(x)\| \quad (x \in \mathrm{M}_n(X), \ n = 1, 2, \ldots), \tag{7.3}$$

where the supremum is over all contractions $T \in \mathrm{B}_A(X, \mathrm{M}_m(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B$, with $m \in \mathbb{N}$ and $\mathcal{H}$, $\mathcal{K}$ the Hilbert spaces of the universal representations of A and B, respectively. Denote the operator bimodule so obtained by $\mathrm{MAX}_A(X)_B$.

(ii) If $X \in {}_A\mathrm{NRM}_B$, the *maximal normal operator bimodule norms*, denoted by $\|x\|_{A\mathrm{MN}_B}$, are defined by the same formula (7.3), but with $\mathcal{H}$ and $\mathcal{K}$ (fixed) proper Hilbert modules over A and B. This operator bimodule is denoted by $\mathrm{MAXN}_A(X)_B$.

(iii) If $X \in {}_A\mathrm{NDRM}_B$, the *maximal normal dual operator bimodule norms*, denoted by $\|x\|_{A\mathrm{MND}_B}$, are defined in the same way as $\|x\|_{A\mathrm{MN}_B}$, except that we now require in addition that the maps T in (7.3) are weak* continuous. Denote this operator bimodule by $\mathrm{MAXND}_A(X)_B$.

Given $X \in {}_A\mathrm{NRM}_B$ and $x \in \mathrm{M}_n(X)$, since each normal operator A, B-bimodule Y is contained in a bimodule of the form $\mathrm{B}(\mathcal{K}^{\mathbb{J}}, \mathcal{H}^{\mathbb{I}})$ with $\mathcal{H}$ and $\mathcal{K}$ fixed proper modules over A and B (resp.), we deduce that $\|x\|_{A\mathrm{MN}_B} = \sup \|T_n(x)\|$, where the supremum is over all contractions $T \in \mathrm{B}_A(X, Y)_B$ with $Y \in {}_A\mathrm{NOM}_B$. We conclude that the operator bimodule $\mathrm{MAXN}_A(X)_B$ is characterized by the following: $\mathrm{MAXN}_A(X)_B$ is a normal operator A, B-bimodule and for each $Y \in {}_A\mathrm{NOM}_B$ every map $T \in \mathrm{B}_A(X, Y)_B$ is completely bounded from $\mathrm{MAX}_A(X)_B$ into Y with $\|T\|_{\mathrm{cb}} = \|T\|$. There are similar characterizations for $\mathrm{MAX}_A(X)_B$ (if $X \in {}_A\mathrm{RM}_B$) and $\mathrm{MAXND}_A(X)_B$ (if $X \in {}_A\mathrm{NDRM}_B$). From this and the universal property of the normal part (Proposition 5.7(i)) we deduce:

Corollary 7.4. $\mathrm{MAXN}_A(X)_B$ *is the normal part of* $\mathrm{MAX}_A(X)_B$ *if* $X \in {}_A\mathrm{NRM}_B$.

Example 7.5. In general $\mathrm{MAXN}_C(X)_C \neq \mathrm{MAX}_C(X)_C$ even if C is abelian and X is central. To show this, let $U \subseteq V$ be Banach spaces such that the (completely contractive) inclusion of maximal operator spaces $\mathrm{MAX}(U) \to \mathrm{MAX}(V)$ is not completely isometric. With Δ the spectrum of C and $t_0 \in \Delta$, let

$$X = \{f \in C(\Delta, V) : \ f(t_0) \in U\}.$$

We claim that for each $f \in \mathrm{M}_n(\mathrm{MAX}_C(X)_C)$

$$\|f\|_{C M_C} = \max\{\sup_{t\in\Delta} \|f(t)\|_{\mathrm{M}_n(\mathrm{MAX}(V))}, \|f(t_0)\|_{\mathrm{M}_n(\mathrm{MAX}(U))}\}. \tag{7.4}$$

To show this, it suffices to prove that, when the spaces $\mathrm{M}_n(X)$ $(n = 1, 2. \ldots)$ are equipped with the norms defined by the right-hand side of (7.4), each contraction $T \in \mathrm{B}_C(X, Y)_C$ into $Y \in {}_C\mathrm{OM}_C$, is completely contractive. Replacing Y with the closure of $T(X)$, we may assume that Y is central and therefore has the canonical decomposition $Y \to \oplus_{t\in\Delta} Y(t)$ (Remark 6.3). Since T is a C-module map, T induces for each $t \in \Delta$ a contraction $T_t : X(t) \to Y(t)$. Since the operator space

$$X(t) = \begin{cases} V & \text{if } t \neq t_0, \\ U & \text{if } t = t_0 \end{cases}$$

is maximal, T_t is a complete contraction, hence so is T (since $\|y\| = \sup_t \|y(t)\|$ for each $y \in \mathrm{M}_n(Y)$).

Since the inclusion $\mathrm{MAX}(U) \to \mathrm{MAX}(V)$ is not completely isometric, there exists $u \in \mathrm{M}_n(U)$ with $\|u\|_{\mathrm{M}_n(U)} > \|u\|_{\mathrm{M}_n(V)}$. Hence, if $f \in \mathrm{M}_n(X)$ is the constant function $f(t) = u$, the function $t \mapsto \|f(t)\|$ is not continuous and $\mathrm{MAX}_C(X)_C$ is not normal by Proposition 6.5. On the other hand, $\mathrm{MAXN}_C(X)_C$ is always normal.

To show that $\mathrm{MAXND}_A(\cdot)_B \neq \mathrm{MAXN}_A(\cdot)_B$, we first need to extend [4, 2.8].

Proposition 7.6. *If* $X \in {}_A\mathrm{NRM}_B$ *then*: (i) $(\mathrm{MAXN}_A(X)_B)^{\natural\mathrm{p}} = \mathrm{MIN}_{A'}(X^{\natural\mathrm{p}})_{B'}$;
(ii) $(\mathrm{MIN}_A(X)_B)^{\natural\mathrm{p}} = \mathrm{MAXND}_{A'}(X^{\natural\mathrm{p}})_{B'}$.

Proof. (i) Given $\phi = [\phi_{ij}] \in \mathrm{M}_n((\mathrm{MAXN}_A(X)_B)^{\natural\mathrm{p}}) = \mathrm{B}_A(X, \mathrm{M}_n(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B$, its norm is $\|\phi\| = \sup\{\|[\phi_{ij}(x)]\| : \ x \in X, \ \|x\| \leqslant 1\}$. Thus, $(\mathrm{MAXN}_A(X)_B)^{\natural\mathrm{p}}$ is dominated by every operator A', B'-bimodule norm structure Z on $X^{\natural\mathrm{p}}$ since the evaluations $X^{\natural\mathrm{p}} \ni \phi \mapsto \phi(x)$ $(\|x\| \leqslant 1)$ are completely contractive on Z by Theorem 7.2. This proves (i).

(ii) Given $\phi = [\phi_{ij}] \in \mathrm{M}_n((\mathrm{MIN}_A(X)_B)^{\natural\mathrm{p}}) = \mathrm{CB}_A(\mathrm{MIN}_A(X)_B, \mathrm{M}_n(\mathrm{B}(\mathcal{K}, \mathcal{H})))_B$, the norm of ϕ is

$$\|\phi\| = \sup\{\|[\phi_{ij}(x_{kl})]\| : \ [x_{kl}] \in \mathrm{M}_s(X), \ \|[x_{kl}]\|_{A\mathrm{m}_B} \leqslant 1, \ s \in \mathbb{N}\}. \tag{7.5}$$

Since $(\mathrm{MIN}_A(X)_B)^{\natural\mathrm{p}}$ is a normal dual operator A', B'-bimodule, $\|\phi\| \leqslant \|\phi\|_{A\mathrm{MND}_B}$ by maximality of $\|\cdot\|_{A\mathrm{MND}_B}$. For the reverse inequality, it suffices to show that

$$\|[T\phi_{ij}]\| \leqslant \|\phi\| \tag{7.6}$$

for each contraction $T \in \mathrm{N}_{A'}(X^{\natural_{\mathrm{p}}}, \mathrm{M}_m(\mathrm{B}(\mathcal{K}, \mathcal{H})))_{B'}$ ($m \in \mathbb{N}$). Let T_m be the predual of $\mathrm{M}_m(\mathbb{C})$ and put $Y = A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B$. Since for each n the unit ball of $\mathrm{M}_n(A \overset{h}{\otimes} \mathrm{T}_m \overset{h}{\otimes} B)$ is dense in that of $\mathrm{M}_n(Y)$ in the A, B-topology (by a similar argument as that preceding (3.2)), we have that $Y^{\natural_{\mathrm{p}}} = \mathrm{CB}_A(A \overset{h}{\otimes} \mathrm{T}_m \overset{h}{\otimes} B, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B = \mathrm{CB}(\mathrm{T}_m, \mathrm{B}(\mathcal{K}, \mathcal{H}))$, hence

$$Y^{\natural_{\mathrm{p}}} = (A \overset{eh}{\otimes} \mathrm{T}_m \overset{eh}{\otimes} B)^{\natural_{\mathrm{p}}} = \mathrm{M}_m(\mathrm{B}(\mathcal{K}, \mathcal{H})). \tag{7.7}$$

Realizing X isometrically as a normal A, B-subbimodule in some $\mathrm{B}(\mathcal{L})$, let $\tilde{X}$ be the smallest strong A, B-bimodule containing X. Note that $(X^{\natural_{\mathrm{p}}})_{\natural_{\mathrm{p}}} = \tilde{X}$ by Theorem 2.17, hence $\tilde{X}^{\natural_{\mathrm{p}}} = X^{\natural_{\mathrm{p}}}$ by Theorem 3.7. Since $\mathrm{MIN}_A(X)_B \subseteq \mathrm{MIN}_A(\tilde{X})_B$ by (7.1), it follows that replacing X by $\tilde{X}$ has no effect on the statement (ii). In other words, we may assume that X is strong. We may regard T as a complete contraction from $\mathrm{MIN}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$ into $\mathrm{MIN}_{A'}(Y^{\natural_{\mathrm{p}}})_{B'}$ (using (7.1) for norms in $\mathrm{M}_n(Y^{\natural_{\mathrm{p}}})$). Since these are normal dual operator bimodules by part (i), we deduce by Propositions 3.11 and 3.12(i) that $T = S^{\natural_{\mathrm{p}}}$ for a contraction $S \in \mathrm{B}_A(Y, X)_B$. Then the norm of $[T\phi_{ij}] \in \mathrm{M}_{mn}(\mathrm{B}(\mathcal{K}, \mathcal{H})) = \mathrm{CB}_A(A \overset{eh}{\otimes} \mathrm{T}_{mn} \overset{eh}{\otimes} B, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ (we have used (7.7)) is equal to

$$\|[T\phi_{ij}]\| = \sup \|[(T\phi_{ij})(v_{kl})]\| = \sup \|[\phi_{ij}(Sv_{kl})]\|, \tag{7.8}$$

where the supremum is over all $[v_{kl}] \in \mathrm{M}_r(A \overset{eh}{\otimes} \mathrm{T}_{mn} \overset{eh}{\otimes} B)$ with $\|[v_{kl}]\| \leqslant 1$ and $r \in \mathbb{N}$. Since S is a complete contraction into $\mathrm{MIN}_A(X)_B$, $\|[Sv_{kl}]\|_{A\mathrm{m}_B} \leqslant 1$. Thus the right-hand side of (7.8) is dominated by $\|\phi\|$ (by (7.5)), which proves (7.6). □

Corollary 7.7. $\mathrm{MAXND}_A(X)_B$ *is a normal dual operator* A, B*-bimodule for each* $X \in {}_A\mathrm{NDRM}_B$.

Proof. Let $X = V^{\sharp}$. If (x_ν) is a net in the unit ball of $\mathrm{M}_n(\mathrm{MIN}_A(X)_B)$ converging to $x \in \mathrm{M}_n(X)$ in the topology induced by $\mathrm{M}_n(V)$, then for each $a \in \mathrm{R}_n(A)$ and $b \in \mathrm{C}_n(B)$ the net $(ax_\nu b)$ converges to axb in the topology induced by V. Since $X \in {}_A\mathrm{NDRM}_B$, it follows that $\|axb\| \leqslant \|a\|\|b\|$ and, using (7.1), we see that the unit ball of $\mathrm{M}_n(\mathrm{MIN}_A(X)_B)$ is closed for each n. By [28, 3.1] this implies that $\mathrm{MIN}_A(X)_B$ is a dual operator space and it follows that $\mathrm{MIN}_A(X)_B \in {}_A\mathrm{NDOM}_B$ (using Theorem 2.9 and Remark 2.10). Then by Theorem 3.7 (applied to $\mathrm{MIN}_A(X)_B$) we have in particular that $X = (X_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}}$ isometrically and weak* homeomorphically. Now Proposition 7.6(ii) applied to $X_{\natural_{\mathrm{p}}}$ shows that $\mathrm{MAXND}_A(X)_B = \mathrm{MAXND}_A((X_{\natural_{\mathrm{p}}})^{\natural_{\mathrm{p}}})_B = (\mathrm{MIN}_{A'}(X_{\natural_{\mathrm{p}}})_{B'})^{\natural_{\mathrm{p}}}$, which is a normal dual operator bimodule by Proposition 2.15. □

Example 7.8. In general $\mathrm{MIN}_A(X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}})_B \neq (\mathrm{MIN}_A(X)_B)^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$, hence by Proposition 7.6 we have that $\mathrm{MAXND}_{A'}(X^{\natural_{\mathrm{p}}})_{B'} = (\mathrm{MIN}_A(X)_B)^{\natural_{\mathrm{p}}} \neq \mathrm{MAXN}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$.

We sketch a counterexample. Let A be the injective II_1 factor represented normally on a Hilbert space $\mathcal{L}$ such that $\mathcal{L}$ is not locally cyclic for A. Let $X = A \check{\otimes} A' \subseteq \mathrm{B}(\mathcal{L} \otimes \mathcal{L})$.

By [41, 3.4] A (identified with $A \otimes 1$) is a norming subalgebra of X, which by (7.1) means that X carries the minimal operator A-bimodule structure. By Corollary 3.5 $X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}} = (X^{A\sharp A})^{\sharp} \subseteq X^{\sharp\sharp} = \tilde{X}$. Let $\mathcal{G}$ be the Hilbert space of the universal representation Φ of X (hence $\tilde{X}$ is the weak* closure of $\Phi(X)$). Since A is a C*-subalgebra of X, $\tilde{A} = A^{\sharp\sharp}$ can be regarded as a von Neumann subalgebra of $\tilde{X}$. Let P be the central projection in $\tilde{A}$ such that the weak* continuous extension α of $\Phi^{-1}|\Phi(A)$ to $\tilde{A}$ has kernel $P^{\perp}\tilde{A}$, so that α maps $P\tilde{A}$ isomorphically onto A. Since A is a factor, $\mathrm{C}^*(A \cup A')$ is weak* dense in $\mathrm{B}(\mathcal{L})$, hence the representation $a \otimes a' \mapsto aa'$ of X (bounded since A is injective [18]) is cyclic, therefore it can be regarded as a direct summand in Φ. So, we may regard $\mathcal{L}$ as a subspace in $\mathcal{G}$ and denote by $e \in \tilde{X}'$ the projection onto $\mathcal{L}$. Then $\Phi(X)e \cong \mathrm{C}^*(A \cup A')$. If C_e is the central carrier of e in $\tilde{X}$, the map

$$\tilde{X}C_e \to \tilde{X}e, \quad x \mapsto xe$$

is an isomorphism of von Neumann algebras [27, p. 335], hence normal, and maps the C*-subalgebra $\Phi(A \otimes 1)C_e$ of $\tilde{X}C_e$ onto $\Phi(A \otimes 1)e \cong A$. Since the representation $a \mapsto \Phi(a \otimes 1)|e\mathcal{G}$ of A is just the identity, it is normal, hence the representation $A \ni a \mapsto \Phi(a \otimes 1)|C_e\mathcal{G}$ is also normal. Using [27, 10.1.18] this implies that $C_e \leqslant P$, hence $\tilde{X}C_e \subseteq P\tilde{X}P = PX^{\sharp\sharp}P = X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$ by Theorem 5.8.

If the operator A-bimodule structure on $X^{\natural_{\mathrm{p}}\natural_{\mathrm{p}}}$ were minimal, the same would hold for the subbimodule $\tilde{X}C_e$, hence also for the completely isometric A-bimodule $\tilde{X}e$. But $\tilde{X}e \cong \mathrm{B}(\mathcal{L})$, thus $\mathrm{B}(\mathcal{L})$ carries the minimal operator A-bimodule structure, hence by (7.1) A is a norming subalgebra of $\mathrm{B}(\mathcal{L})$. But this is a contradiction since by [41, 2.7] A is norming for $\mathrm{B}(\mathcal{L})$ only if $\mathcal{L}$ is locally cyclic for A.

Remark 7.9. By Proposition 7.6(i) and Corollary 3.5(i) $\mathrm{MIN}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$ is a dual operator space, hence $X^{\natural_{\mathrm{p}}}$ is the dual of a Banach space V. If there is an operator space on V such that $Y := \mathrm{MAXN}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$ is the operator space dual of V, then Y is a normal *dual* operator A', B'-bimodule (Theorem 2.9 and Remark 2.10), hence $Y = \mathrm{MAXND}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$ by maximality. But, with X as in Example 7.8, $Y \neq \mathrm{MAXND}_{A'}(X^{\natural_{\mathrm{p}}})_{B'}$, hence there is no operator space on V predual to Y.

Acknowledgment

Supported by the Ministry of Science and Education of Slovenia.

References

[1] C. Akemann, The dual space of an operator algebra, Trans. Amer. Math. Soc. 126 (1967) 286–302.
[2] W.B. Arveson, Subalgebras of C^*-algebras, Acta Math. 123 (1969) 141–224.
[3] C. Anantharaman, C. Pop, Relative tensor products and infinite C^*-algebras, J. Operator Theory 47 (2002) 389–412.
[4] D.P. Blecher, The standard dual of an operator space, Pacific J. Math. 153 (1992) 15–30.

[5] D.P. Blecher, Tensor products of operator spaces II, Can. J. Math. 44 (1992) 75–90.
[6] D.P. Blecher, On selfdual Hilbert modules, Operator algebras and their applications, 65–80, Fields Institute Communications, Vol. 13, American Mathematical Society, Providence, RI, 1997.
[7] D.P. Blecher, The Shilov boundary of an operator space and the characterization theorems, J. Funct. Anal. 182 (2001) 280–343.
[8] D.P. Blecher, Multipliers and dual operator algebras, J. Funct. Anal. 183 (2001) 498–525.
[9] D.P. Blecher, E.G. Effros, V. Zarikian, One-sided M-ideals and multipliers in operator spaces, I, Pacific J. Math. 206 (2002) 287–319.
[10] D.P. Blecher, C. Le Merdy, Operator algebras and their Modules—an Operator Space Approach, LMS Monographs, Oxford University Press, Oxford, 2004.
[11] D.P. Blecher, B. Magajna, Duality and operator algebras, Preprint, July 2004.
[12] D.P. Blecher, R.R. Smith, The dual of the Haagerup tensor product, J. London Math. Soc. 45 (1992) 126–144.
[13] E. Christensen, E.G. Effros, A. Sinclair, Completely bounded multilinear maps and C^*-algebraic cohomology, Invent. Math. 90 (1987) 279–296.
[14] A. Defant, K. Floret, Tensor norms and operator ideals, North-Holland Mathematics Studies, Vol. 176, North-Holland Publishing Co., Amsterdam, 1993.
[15] M.J. Dupré, R.M. Gillette, Banach bundles, Banach modules and automorphisms of C^*-algebras, Research Notes in Mathematics, Vol. 92, Pitman, Boston, MA, 1983.
[16] E.G. Effros, R. Exel, On multilinear double commutant theorems, Operator algebras and applications, Vol. 1, London Mathematical Society, Lecture Note Series, Vol. 135, Cambridge University Press, Cambridge, 1988, pp. 81–94.
[17] E.G. Effros, A. Kishimoto, Module maps and Hochschild–Johnson cohomology, Indiana Univ. Math. J. 36 (1987) 257–276.
[18] E.G. Effros, E.C. Lance, Tensor products of operator algebras, Adv. in Math. 25 (1977) 1–34.
[19] E.G. Effros, N. Ozawa, Z.-J. Ruan, On injectivity and nuclearity for operator spaces, Duke Math. J. 110 (2001) 489–521.
[20] E.G. Effros, Z.-J. Ruan, Representation of operator bimodules and their applications, J. Operator Theory 19 (1988) 137–157.
[21] E.G. Effros, Z.-J. Ruan, Self-duality for the Haagerup tensor product and Hilbert space factorizations, J. Funct. Anal. 100 (1991) 257–284.
[22] E.G. Effros, Z.-J. Ruan, Operator space tensor products and Hopf convolution algebras, J. Operator Theory 50 (2003) 131–156.
[23] E.G. Effros, Z.-J. Ruan, Operator spaces, London Mathematical Society Monographs, New Series, Vol. 23, Oxford University Press, Oxford, 2000.
[24] E.G. Effros, S. Winkler, Matrix convexity: operator analogues of the bipolar and Hahn–Banach theorems, J. Funct. Anal. 144 (1997) 117–152.
[25] J. Glimm, A Stone–Weierstrass theorem for C^*-algebras, Ann. of Math. 72 (1960) 216–244.
[26] U. Haagerup, C. Winslow, The Effros—Maréchal topology in the space of von Neumann algebras II, J. Funct. Anal. 171 (2000) 401–431.
[27] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, vols. 1, 2, Academic Press, London, 1983, 1986.
[28] C. Le Merdy On the duality of operator spaces, Canad. Math. Bull. 38 (1995) 334–346.
[29] B. Magajna, The Haagerup norm on the tensor product of operator modules, J. Funct. Anal. 129 (1995) 325–348.
[30] B. Magajna, Strong operator modules and the Haagerup tensor product, Proc. London Math. Soc. 74 (1997) 201–240.
[31] B. Magajna, A topology for operator modules over W^*-algebras, J. Funct. Anal. 154 (1998) 17–41.
[32] B. Magajna, The minimal operator module of a Banach module, Proc. Edinburgh Math. Soc. 42 (1999) 191–208.
[33] B. Magajna, C^*-convex sets and completely bounded bimodule homomorphisms, Proc. Roy. Soc. Edinburgh 130A (2000) 375–387.

[34] B. Magajna, Normal operator modules and tensor products, Functional analysis vii (Dubrovnik, 2001), Various Publications Series, Vol. 46, University of Aarhus, 2002, pp. 131–142.
[35] B. Magajna, On completely bounded bimodule maps over W^*-algebras, Studia Mathematica 154 (2003) 137–164.
[36] P.S. Muhly, B. Solel, Duality of W^*-correspondences and applications, Preprint, arXiv:math.OA/0404316 v1, 18 April 2004.
[37] Q. Na, Standard duals of operator bimodules, J. Funct. Anal. 135 (1996) 132–156.
[38] V.I. Paulsen, Completely bounded maps and operator algebras, Cambridge Studies in Advanced Mathematics, Vol. 78, Cambridge University Press, Cambridge, 2002.
[39] G. Pisier, Introduction to the theory of operator spaces, LMS Lecture Note Series, Vol. 294, Cambridge Univ. Press, Cambridge, 2003.
[40] C. Pop, Bimodules normés représentables sur des espaces hilbertiens, Ph.D. Thesis, Université d'Orléans, January 1999.
[41] F. Pop, A.M. Sinclair, R.R. Smith, Norming C^*-algebras by C^*-subalgebras, J. Funct. Anal. 175 (2000) 168–196.
[42] M.A. Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, J. Pure Appl. Algebra 5 (1974) 51–96.
[43] R.R. Smith, Completely bounded module maps and the Haagerup tensor product, J. Funct. Anal. 102 (1991) 156–175.
[44] G. Wittstock, Extension of completely bounded C^*-module homomorphisms, Operator algebras and group representations, Vol. II (Neptun, 1980), 238–250, Monogr. Studies in Mathematics, Vol. 18, Pitman, Boston, MA, 1984.