Decay Rate of Periodic Solutions for a Conservation Law

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1. INTRODUCTION

In this paper we shall estimate the decay rate of periodic solutions of a hyperbolic conservation law

\[ u_t + f(u)_x = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = \psi(x), \quad -\infty < x < \infty, \]

(1.1)

where \( \psi(x) \) is a periodic function with period p and \( f \) is a smooth nonlinear function of \( u \). In general, Eqs. (1.1) do not have a continuous solution for all time; shock waves appear after a finite time. Due to the result on the regularity of solutions [9], we will consider piecewise smooth solutions only. The extension to bounded and measurable \( \psi \)'s follows from the ordering principle of Kruzkov [6] originally due to Douglis [4] (see also Ballou [1] and Keyfitz [6]) and the fact that any \( \psi \) in \( L^1_{\text{loc}}(\infty, \infty) \) can be approximated in \( L^1_{\text{loc}}(\infty, \infty) \) by piecewise smooth data. It is well known that across a line \( x = x(t) \) of discontinuity, the solution satisfies the Rankine–Hugoniot condition (R–H) and the entropy condition (E) [10]:

\[ x'(t) = \sigma(u_-, u_+), \quad \text{(R–H)} \]
\[ \sigma(u_-, u_+) \leq \sigma(u_-, u) \quad \text{for all } u \text{'s between } u_- \text{ and } u_+, \quad \text{(E)} \]

where \( u_x = u(x(t) \pm 0, t) \) and \( \sigma(u_1, u_2) \) is the shock speed defined by

\[ \sigma(u_1, u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}. \]

When \( f(u) \) is convex or concave, the decay rate of solutions of (1.1) was

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obtained by Lax \[8\]. He showed that, for all \( t > 0 \), \( u(x, t) \) satisfies an inequality

\[ |u(x, t) - \bar{u}| \leq A p / t, \]

where \( \bar{u} \), being the average value of \( u \) over a period, is a time invariant and the constant \( A \) depends only on the function \( f \). When the convexity assumption of \( f(u) \) is dropped, Dafermos \[3\] showed that \( u(x, t) \) approaches \( U \) in \( L_1[0, p] \). However, it seems unlikely that his methods will yield the rate of convergence. When \( \bar{u} \) lies at a point of convexity or concavity of \( f(u) \), then from Dafermos' result, \( u(x, t) \) lies on a part of convexity or concavity of \( f(u) \) for sufficiently large \( t \). Thus, we may apply the Lax theory \[8\] to show that \( u(x, t) \to \bar{u} \) uniformly at the rate \((p/t)\). When \( \bar{u} \) lies at a point of inflection of \( f(u) \), no matter how large \( t \) is, we cannot apply the Lax theory. It is this case we wish to deal with in this paper.

Work on this problem was initiated by Greenberg and Tong \[5\]. They gave decay rates for various \( f(u) \)'s but a rigorous proof was given only for a special \( \psi \). Conlon \[2\] also considered the same problem and obtained the decay rate for a special form of the function \( f(u) \), namely, \( f(u) = -\frac{1}{2} u^2 \). The purpose of this paper is to give a rigorous estimation for the decay rates given by Greenberg and Tong \[5\].

We assume without loss of generality that

(i) the set of points of inflection of \( f \) has no accumulation point on the real line,
(ii) \( u = 0 \) is an inflection point of \( f \),
(iii) \( \bar{u} \), the average value of \( u \) over a period, is equal to zero,
(iv) \( f'(0) = 0 \),
(v) there exists an integer \( n \), \( 3 < n < \infty \), such that \( f^{(k)}(0) = 0 \) for \( 2 < k < n \) and \( f^{(n)}(0) \neq 0 \).

We will consider the case when \( n \) is an even integer in Section 2 and the case when \( n \) is an odd integer in Section 3. The reason is that the methods used for estimating the decay rates are quite different for even and odd \( n \).
and

\[(n - 1) \bar{B}u^{n-2} \geq f''(u) \geq (n - 1) \bar{B}u^{n-2}, \quad |u| \leq U_0, \quad \text{if } f^{(n)}(0) > 0.\]

**Proof.** From assumption (v), \(f^{(n)}(0) \neq 0\). Hence we can choose \(U_0\) sufficiently small such that \(f^{(n)}(u) \cdot f^{(n)}(0) > 0\) for all \(|u| \leq U_0\). Thus, if \(f^{(n)}(0) < 0\), we have \(f''(u) < 0\) for all \(0 < |u| \leq U_0\) and if \(f^{(n)}(0) > 0\), we have \(f''(u) > 0\) for all \(0 < |u| \leq U_0\). Consider the function \(G(u) = f''(u)/u^{(n-2)}\), \(G(u)\) is positive if \(f^{(n)}(0) > 0\), negative if \(f^{(n)}(0) < 0\), and is continuous in \(|u| \leq U_0\). Hence there exist \(\bar{B}\) and \(\bar{B}\),

\[(n - 1) \bar{B} = \max_{|u| \leq U_0} |G(u)| \quad \text{and} \quad (n - 1) \bar{B} = \min_{|u| \leq U_0} |G(u)|,\]

such that Lemma 2.1 holds. Q.E.D.

Now we have one of our main theorems:

**Theorem 2.2.** Under assumptions (i)–(v), if \(n\) is an even integer, then there exists a finite time \(T_0\) such that the solution of (1.1), \(u(x, t)\), satisfies the inequality

\[|u(x, t)| \leq C t^{- \frac{1}{2}}\]

for all \(t \geq T_0\), where the constants \(T_0\) and \(C\) depend only on \(\|\psi\|_\infty, p\), and the function \(f\).

**Proof.** Dafermos [3] showed that \(u(x, t)\) approaches zero in \(L_1[0, p]\). Hence there exists a finite time \(T\) such that \(|u(x, t)| \leq U_0\) for all \(t \geq T\), where the constant \(T\) depends on \(\|\psi\|_\infty\) and the function \(f\); and the constant \(U_0\) is the one in Lemma 2.1. For \(t \geq T\), let \(U^+(t) = \sup\{|u(x, t)|: 0 < x \leq p\}\) and \(U^-(t) = \inf\{|u(x, t)|: 0 < x \leq p\}\). It is easy to see that \(U^+(t) > 0\) and \(U^-(t) < 0\) for all \(t \geq T\). For definiteness, we consider the case \(f^{(n)}(0) > 0\) only. The other case, \(f^{(n)}(0) < 0\), can be similarly considered. For \(t \geq T\), let \(u(x_1(t) - 0, t) = U^+(t)\) and \(u(x_2(t) + 0, t) = U^-(t)\), where \(0 < x_1(t) - x_2(t) \leq p\). Draw two characteristic lines backward from \((x_1(t) - 0, t)\) and \((x_2(t) + 0, t)\). Since for \(t \geq T\), \(|u(x, t)| \leq U_0\) and \(f''(u)\) is positive for all \(0 < |u| \leq U_0\), the two characteristic lines cannot terminate at some shock curves (no contact discontinuity) before they meet the line \(t = T\). Hence the two characteristic lines meet the line \(t = T\) at two points, say \(x_+\) and \(x_-\), respectively. Now from the construction, we have

\[x_1(t) = x_+ + f''(U^+(t))(t - T)\]

and

\[x_2(t) = x_- + f''(U^-(t))(t - T).\]
Hence
\[ x_1(t) - x_2(t) = (x_+ - x_-) + [f'(U^+(t)) - f'(U^-(t))](t - T) \leq p. \]

Since \( x_+ - x_- \geq 0 \), we have
\[ 0 < |f'(U^+(t)) - f'(U^-(t))|(t - T) \leq p. \]  \hspace{1cm} (2.1)

From Lemma 2.1 and the fact that \( U^+(t) > 0 \) and \( U^-(t) < 0 \), we have
\[ |f''(U^+(t)) - f''(U^-(t))| = |f''(U^+(t)) - f''(0)| - |f''(U^-(t)) - f''(0)| \geq B|U^+(t)|^{n-1} - B|U^-(t)|^{n-1} \geq B|U(t)|^{n-1}, \]  \hspace{1cm} (2.2)

where
\[ U(t) = \max\{U^+(t), -U^-(t)\}. \]

Hence from (2.1) and (2.2) we obtain
\[ 0 < B|U(t)|^{n-1}(t - T) \leq p. \]

If we choose \( T_0 > T \) sufficiently large such that
\[ \left(1 - \frac{T}{T_0}\right) \geq \frac{1}{2^{n-1}}, \]

Then we have
\[ U(t) \leq \left(\frac{p}{B}\right)^{1/(n-1)} \cdot \frac{1}{(t - T)^{1/(n-1)}} \leq \frac{2(p/B)^{1/(n-1)}}{t^{1/(n-1)}}, \]

for all \( t \geq T_0 \). This completes the proof. \hspace{1cm} Q.E.D.

3. The Case When n Is an Odd Integer

For definiteness, we consider the case \( f^{(n)}(0) > 0 \) only. We need some notation. (See Ballou [1].)

**Definition 3.1.** Given \( \eta < 0 \), define \( \eta^* = \eta^*(\eta) \) by
\[ \eta^* = \sup\{u > \eta: \sigma(v, \eta) > \sigma(u, \eta) \ \forall v \in (\eta, u)\}. \]  \hspace{1cm} (3.1)
Given \( \eta > 0 \), define \( \eta_\ast = \eta_\ast(\eta) \) by

\[
\eta_\ast = \inf\{u < \eta: \sigma(v, \eta) > \sigma(u, \eta) \ \forall v \in (u, \eta)\}. \tag{3.2}
\]

We shall need certain properties of the function \( f \).

**Lemma 3.2.** (a) There are constants \( U_0, 0 < B \leq \bar{B} < \infty \), such that

\[
\bar{B}(n - 1) u^{n-2} \leq f''(u) \leq B(n - 1) u^{n-2} \quad \text{for} \quad -U_0 \leq u \leq 0,
\]

\[
\bar{B}(n - 1) u^{n-2} \leq f''(u) \leq \bar{B}(n - 1) u^{n-2} \quad \text{for} \quad 0 \leq u \leq U_0.
\]

(b) The functions \( \eta^\ast \) and \( \eta_\ast \) defined in (3.1) and (3.2), may be written as \( \eta^\ast(\eta) = -\gamma^\ast(\eta) \eta \) and \( \eta_\ast(\eta) = -\gamma_\ast(\eta) \eta \), where \( \gamma^\ast \) and \( \gamma_\ast \) are positive and continuous and satisfy

\[
0 < \gamma^\ast \equiv \inf_{-U_0 < \eta < 0} \gamma^\ast(\eta) \leq \bar{\gamma}^\ast \equiv \sup_{-U_0 < \eta < 0} \gamma^\ast(\eta) < 1, \tag{3.3}
\]

\[
0 < \gamma_\ast \equiv \inf_{0 < \eta < U_0} \gamma_\ast(\eta) \leq \bar{\gamma}_\ast \equiv \sup_{0 < \eta < U_0} \gamma_\ast(\eta) < 1. \tag{3.4}
\]

**Proof.** From assumption (v) and the specification that \( f^{(n)}(0) > 0 \), we can choose \( U_0 \) sufficiently small such that \( f^{(n)}(u) > 0 \) for all \( |u| \leq U_0 \). Hence \( f''(u) < 0 \) for \( -U_0 \leq u < 0 \) and \( f''(u) > 0 \) for \( 0 < u \leq U_0 \). Consider the function \( F(u) = f''(u)/u^{(n-2)} \). \( F(u) \) is positive and continuous in \( |u| \leq U_0 \). Hence we can choose \( (n - 1) \bar{B} = \max_{|u| \leq U_0} F(u) \) and \( (n - 1) B = \min_{|u| \leq U_0} F(u) \) to complete the proof for part (a). From (3.1), it is easy to see that (see Ballou [1] or Greenberg and Tong [5])

\[
f'(\eta^\ast(\eta)) = \frac{f(\eta^\ast(\eta)) - f(\eta)}{\eta^\ast(\eta) - \eta}. \tag{3.5}
\]

If we choose \( U_0 \) sufficiently small, then from the assumptions about the function \( f \), we have

\[
f(u) = f'(0) u + \frac{f^{(n)}(0)}{n!} u^n + O(u^{n+1}), \quad |u| \leq U_0.
\]

Hence from (3.5), we have

\[
f''(0) + \frac{f^{(n)}(0)}{(n - 1)!} (\eta^\ast)^{n-1}
\]

\[
= f'(0) + \frac{f^{(n)}(0)}{n!} (\eta^{n-1} + \eta^{n-2} + \cdots + \eta^{n-1}) + O(\eta^n). \tag{3.6}
\]
If we let \( q^* = -\gamma^*(\eta) \eta \), then we have from (3.6)

\[
(n-1) \gamma^{*n-1} + \gamma^{*n-2} - \gamma^{*n-3} + \cdots - 1 = O(\eta). \tag{3.7}
\]

Thus, if \( U_0 \) is sufficiently small, we have from (3.7) that \( 0 < \gamma^*(\eta) < 1 \) for all \( -U_0 \leq \eta < 0 \) and \( 0 < \lim_{\eta \to 0} \gamma^*(\eta) < 1 \). Similarly we can prove that if \( U_0 \) is sufficiently small, then we have \( 0 < \gamma^*(\eta) < 1 \) for all \( 0 < \eta \leq U_0 \) and \( 0 < \lim_{\eta \to 0} \gamma^*(\eta) < 1 \). This completes the proof. Q.E.D.

Now we can state another main theorem:

**Theorem 3.3.** Under assumptions (i)-(v), if \( n \) is an odd integer, then there exists a finite time \( T_0 \) such that the solution of (1.1), \( u(x, t) \), satisfies the inequality

\[
|u(x, t)| \leq Ct^{-1/(n-1)}
\]

for all \( t \geq T_0 \), where the constants \( T_0 \) and \( C \) depend only on \( \|\psi\|_{\infty} \) \( p \) and the function \( f \).

**Proof.** We shall decompose the proof into several steps.

**Step 1.** Dafermos [3] showed that \( u(x, t) \) approaches zero in \( L_1[0, p] \). Hence there exists a finite time \( T \) such that \( |u(x, t)| \leq U_0 \) for all \( t \geq T \), where the constant \( T \) depends on \( \|\psi\|_{\infty} \) \( p \) and \( \) the constant \( U_0 \) is the one in Lemma 3.2.

**Step 2.** We can choose the \( T \) in Step 1 sufficiently large such that all shock waves for \( t \geq T \) are type II shock. (See Greenberg and Tong [5] and Conlon [2].) This means that if \( x = x(t) \) is a curve of discontinuity and \( t \geq T \), then (1) \( x''(t) < 0 \) for all \( t > T \) and (2) \( u^*(x(t) - 0, t) = u(x(t) + 0, t) \) or \( u^*(x(t) - 0, t) = u(x(t) + 0, t) \). (See Ballou [1].) Let \( U(t) = \sup\{|u(x, t)|: 0 < x \leq p\} \). It is easy to see that \( U(t) \) is continuous and monotone decreasing for \( t \geq T \).

**Step 3.** Choose a shock curve \( x = x_0(t) \) with \( u(x_0(t) - 0, t) < 0 \) and a constant \( \delta \) such that \( 1 > \delta \geq \frac{1}{2}[1 + \max\{\tilde{\gamma}^*, \tilde{\gamma}^*\}] \). Following Greenberg and Tong [5], we draw a straight line from the point \( (x_0(t), t) \) with \( dx/dt = \min\{f'(\delta U(t)), f'(-\delta U(t))\} \). This straight line intersects the shock curve \( x = x_0(t) + p \) at some time \( \tau(t) \equiv t + \tau^*(t) \). Hence \( \tau(t) \) and \( \tau^*(t) \) satisfy

\[
x_0(t) + \min\{f'(\delta U(t)), f'(-\delta U(t))\} \tau^*(t) = p + x_0(\tau(t)). \tag{3.8}
\]

From the properties of \( x_0(t)(x_0^*(t) < 0) \) and \( U(t) \) (continuous and monotone decreasing), we see that \( \tau(t) \) is continuous and strictly increasing. Hence the inverse function of \( \tau(t) \) exists. From (3.8) we have

\[
p = \min\{f'(\delta U(t)), f'(-\delta U(t))\} \tau^*(t) - \int_{\tau}^{\tau^*(t)} \frac{dx_0(t + s)}{ds} ds. \tag{3.9}
\]
But
\[ \frac{dx_0(t + s)}{ds} = \frac{f(u^*(x(t + s) - 0, t)) - f(u(x(t + s) + 0, t))}{u^*(x(t + s) - 0, t) - u(x(t + s) + 0, t)}. \tag{3.10} \]

It is easy to see that (Greenberg and Tong [5])
\[ f'(0) \leq \frac{dx_0(t + s)}{ds} \leq \frac{f(-U(t))}{-U(t)}, \quad 0 \leq s \leq \tau^*(t). \tag{3.11} \]

From (3.9) and (3.11) we get
\[ \left[ \min\{f'[-\delta U(t)], f'[-\delta U(t)]\} - f'(0) \right] \tau^*(t) \]
\[ \geq p \geq \left[ \min\{f'[-\delta U(t)], f'[-\delta U(t)]\} - \frac{f(-U(t))}{-U(t)} \right] \tau^*(t). \tag{3.12} \]

But from Lemma 3.2(a), we have
\[ \left[ \min\{f'[-\delta U(t)], f'[-\delta U(t)]\} - f'(0) \right] \leq -\delta U(t)^{n-1} \tag{3.13} \]

and
\[ \left[ \min\{f'[-\delta U(t)], f'[-\delta U(t)]\} - \frac{f(-U(t))}{-U(t)} \right] \geq \left( \frac{-\delta^{n-1} - \bar{B}}{n} \right) U^n. \tag{3.14} \]

We can choose \( \delta \) and \( U_0 \) such that \( (\delta^{n-1} - \bar{B}/n) > 0 \). (This is because \( \lim_{\nu \to 0} (\bar{B} - \bar{B}) = 0 \).) From (3.12) and (3.14), we have
\[ \frac{C}{U^{n-1}(t)} \leq \tau^*(t) \leq \frac{\bar{C}}{U^n(t)}, \tag{3.15} \]

where \( C = p/\delta^{n-1} - \bar{B}/n \) and \( \bar{C} = (\delta^{n-1} - \bar{B}/n)^{-1} \).

Now we would like to establish that
\[ U(t) = U(t + \tau^*(t)) \leq \delta U(t). \tag{3.16} \]

We first observe that no characteristic
\[ x = x + f'(u(x, t)) s, \quad x \in [x_0(t), x_0(t) + p] \]
carrying a value \( u(x, t) \) and satisfying \( |u(x, t)| > \delta U(t) \) can intersect the line \( \tau(t) \) inside the interval \( [x_0(\tau(t)), x_0(\tau(t)) + p] \). Thus, \( \{u: u = u(x, \tau(t)) \} \) for some \( x \in [x_0(\tau(t)), x_0(\tau(t)) + p] \) is contained in the union of the following sets:
\[ \{u: u = u(x, t), |u| \leq \delta U(t)\} \]
and
\[ |u: u = v^* \text{ or } u = v_* \text{ and } |v| \leq U(t)|. \]

Values from the first set are propagated along characteristics issuing from the interval \([x_0(t), x_0(t) + p]\), while values from the second are obtained from characteristics issuing from the right side of shocks across which \(u\) changes sign. Since \(v^* = -\gamma^*(v) v\) and \(v_* = -\gamma_*(v) v\), we have \(|v^*| \leq \delta |v|\) and \(|v_*| \leq \delta |v|\). This proves (3.16).

Step 4. Now for a given \(t \geq \tau(T)\), we define \(t(1), t(2), \ldots, t(m)\) by
\[
\begin{align*}
\tau(t(1)) &= t \\
\tau(t(2)) &= t(1) \\
\tau(t(m)) &= t(m - 1) \text{ or } t(m) = \tau^{-1}(t(m - 1)),
\end{align*}
\]
with \(T \leq t(m) < \tau(T)\). Hence
\[
\begin{align*}
\tau^*(t(1)) &= \tau(t(1)) - t(1) = t - t(1), \\
\tau^*(t(2)) &= \tau(t(2)) - t(2) = t(1) - t(2), \\
\vdots \\
\tau^*(t(m)) &= \tau(t(m)) - t(m) = t(m - 1) - t(m).
\end{align*}
\]

From (3.15), it is easy to see that for all \(t \geq \tau(T)\), there exists a finite integer \(m\) such that (3.17) and (3.18) hold. Now repeatedly using (3.15) and (3.16), we have
\[
\begin{align*}
\tau^*(t(1)) U^n(t(1)) &\leq \bar{C}, \\
\vdots \\
\tau^*(t(m)) U^n(t(m)) &\leq \bar{C}
\end{align*}
\]
and
\[
\begin{align*}
U(t) &\leq \delta U(t(1)), \\
U(t(1)) &\leq \delta U(t(2)), \\
\vdots \\
U(t(m - 1)) &\leq \delta U(t(m)).
\end{align*}
\]
Hence
\[
\begin{align*}
\tau^*(t(1)) U^n(t(1)) &\leq \tau^*(t(1)) \delta^p U^n(t(1)) \leq \bar{C} \delta^{p-1}, \\
\tau^*(t(m)) U^n(t(m)) &\leq \tau^*(t(m)) (\delta^{p-1})^m U^n(t(m)) \leq \bar{C} (\delta^{p-1})^m.
\end{align*}
\]
From (3.21) and (3.18), we finally have

\[(t - t(m)) \leq U^{-1}(t) \leq \frac{\overline{C} \delta^{(n-1)}(1 + \delta^{(n-1)} + \cdots + \delta^{(n-1)(m-1)})}{1 - \delta^{(n-1)}} = \overline{C}.\]

Hence

\[U^{-1}(t) \leq \frac{\overline{C}}{(t - t(m))} \leq \frac{\overline{C}}{(t - \tau(T))}.\]  \hspace{1cm} (3.22)

Choose \(T_0\) such that

\[1 - \frac{\tau(T)}{T_0} \geq \frac{1}{2^{(n-1)}}\]  \hspace{1cm} (3.23)

Then from (3.22) and (3.23), we get

\[U(t) \leq \frac{C}{t^{1/(n-1)}} \quad \text{for all} \quad t \geq T_0,\]

where \(C \equiv 2(\overline{C})^{1/(n-1)}\). This completes the proof. \hspace{1cm} Q.E.D.

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