Broadcasting and gossiping on de Bruijn, shuffle-exchange and similar networks

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Abstract

We use simple compound properties of de Bruijn related networks to get new bounds about broadcasting and gossiping on such networks. Some asymptotically optimal results on undirected de Bruijn, Kautz, and generalized shuffle-exchange networks are obtained. Our techniques can also be used to derive efficient broadcasting protocols for the undirected graph associated to a line digraph. We give asymptotically optimal broadcast algorithms for some of the generalized de Bruijn and Kautz graphs and an asymptotically optimal gossiping protocol in full duplex model for the shuffle-exchange graph. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Broadcasting (also called One To All) refers to the process of a message dissemination in a communication network. One vertex, called the originator, has to send a given data to the whole set of nodes. The broadcast time of a vertex \( u \), denoted \( b(u) \), is the minimum time to achieve a broadcasting from \( u \). The broadcast time of a graph \( G \) is defined as the maximum of the broadcast time of any vertex in \( G \). Gossiping (also called All To All) can be defined as concurrent broadcastings, that is, each vertex sends a message to all the others. The gossip time of \( G \), \( g(G) \), is the minimum time needed to achieve gossiping in \( G \). The time will depend on the models used. Different models have been proposed in the literature (see [13, 17, 22, 25]). Here, we will suppose that:

1. The routing model is store and forward, that is each message must be completely received before being sent again. As we consider graph behavior facing communications, we shall use a constant time model (that is the time will be counted in

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communication rounds). During a communication round a processor can send all
the data that it knows, which means that during gossiping protocols messages can
be concatenated.

2. A processor will be allowed to use only one of its link during a communication
round, that is it can communicate with only one neighbor. This rule is called
processor bound or whispering or 1-port.

3. If, during one round, two vertices involved in a call can exchange their data, we say
that the mode is full duplex (telephone model). Otherwise, if the message can only
be sent from one of them (the sender) to the other one (the receiver) then the mode is
half duplex (telegraph model).

For broadcasting, half duplex or full duplex modes give the same time, as one can
suppose that two informed vertices will not communicate. For gossiping, following
[13, 25], we will use, \( g_{\text{f}}(G) \) (resp. \( g_{\text{h}}(G) \)) to denote the gossip time of \( G \) under the full
(resp. half) duplex constant time 1-port model. The time depends also on the com-
munication network. Here, we are mainly interested in de Bruijn, Kautz, and shuffle-
exchange networks, and in their generalization.

Several definitions of de Bruijn and Kautz networks as digraphs can be given (see
[8, 9]). Here \( Z_d \) will denote the set of integers modulo \( d \).

The de Bruijn digraph \( \mathcal{B}(d, D) \) of out-degree \( d \) and diameter \( D \) has as vertex set
the set of words of length \( D \) on \( Z_d \), each vertex \( x_1 \ldots x_D \) being joined by the
arc denoted \( \omega_x, x \in Z_d \) to the vertex \( x_2 \ldots x_D x_{D+1} \). The Kautz digraph of
out-degree \( d \) and diameter \( D \), denoted \( \mathcal{K}(d, D) \), has as vertices the words of length
\( D \) on \( Z_{d+1} \), with the additional property \( x_i \neq x_{i+1} \) for \( 1 \leq i \leq D - 1 \). Vertex \( x_1 \ldots x_D \) is joined by the arc denoted \( \omega_x, x \in Z_{d+1} \setminus \{0\}, \) to \( \omega_x(x \ldots x_D) = x_2 \ldots x_D x_{D+1} + x \). These two digraphs can also be defined as iterated line digraphs. Let us recall that
the line digraph \( L(G) \) of a digraph \( G \) is the directed graph whose vertices are the arcs
of \( G \) and whose arcs as defined as follows: there is an arc from a vertex \( e \) to a vertex \( f \in L(G) \) if and only if in \( G \) the initial vertex of \( f \) is the end vertex of \( e \). Let \( K_d^{+} \) denote
the complete symmetric digraph on \( d \) vertices with loops, and \( K_{d+1}^{+} \) the
complete symmetric digraph on \( d + 1 \) vertices; then \( \mathcal{B}(d, D) = L^{D-1}(K_d^{+}) \) and
\( \mathcal{K}(d, D) = L^{D-1}(K_{d+1}^{+}) \).

Generalization of de Bruijn and Kautz digraphs for arbitrary values of \( N \) have been
given by Reddy et al. [24] and Imase and Itoh [18, 19, 20]). The generalized de Bruijn
digraph of order \( N \) with out-degree \( d \) is denoted \( \mathcal{G}(d, N) \). It has as set of vertices the
set of integers \( Z_N \), with arcs from any vertex \( x \) to vertices \( dx + \alpha, 0 \leq \alpha \leq d - 1 \).
The generalized Kautz digraph of order \( N \) with out-degree \( d \) is denoted \( \mathcal{K}(d, N) \). It has
as set of vertices the set of integer \( Z_N \), with arcs from any vertex \( x \) to vertices \( -dx - \alpha \) with \( 1 \leq \alpha \leq d \).

Proposition 1 (Imase et al. [20]). If \( N = d^n n' \) then

\[
\mathcal{G}(d, d^n n') \simeq L^n(\mathcal{G}(d, n')).
\]

\[
\mathcal{K}(d, d^n n') \simeq L^n(\mathcal{K}(d, n')).
\]
As an example if \( N = dD \) we have \( \mathcal{B}(d, N) \cong \mathcal{B}(d, D) \), if \( N = dD^{-1}(d + 1) \) then \( \mathcal{X}(d, N) \cong \mathcal{X}(d, D) \).

In what follows, \( G \) will always denote a digraph. Sometimes people prefer to consider a network as an undirected graph and to obtain result about graphs. Let \( UG \) be the underlying graph of \( G \), that is the graph obtained from the digraph \( G \) by forgetting the orientation and deleting multiple edges if any. If \( H \) is an undirected graph, we will associate to \( H \) its symmetric digraph \( H^* \) obtained by replacing each edge in \( H \) by two opposite arcs. Finally, we can combine the two operations and associate to a digraph \( G \), the symmetric digraph \( UG^* \) denoted by \( G^* \). The digraph \( G^* \) is obtained from the digraph \( G \) by adding to each arc \((x, y)\) the arc \((y, x)\) if it does not already exist. A protocol for \( G^* \) will clearly induce a protocol on \( UG \).

We define here the shuffle-exchange digraph (see [11]), denoted \( \mathcal{SE}(n) \), as a digraph on the set of the binary words of length \( n \). Vertex \( x_1, \ldots, x_n \) is joined to:

\[
\omega(x_1x_2 \ldots x_n) = x_2 \ldots x_nx_1 \quad \text{by the shuffle arc.}
\]

\[
x_1x_2 \ldots x_{n-1}x_n = x_{n-1}x_n \quad \text{by the exchange arc. (}x_n = x_n + 1 \mod 2\).
\]

Consequently its out(in)-degree is 2, one can also check that it has diameter \( 2n - 1 \). Its associated underlying graph \( U \mathcal{SE}(n) \) is a graph of degree 3 (and not 4) as the pairs of opposite exchange arcs of \( \mathcal{SE}(n) \) gives a single edge in \( U \mathcal{SE}(n) \).

Partial results have been obtained on broadcasting and gossiping in these networks (see 7, 4, 13, 15, 17]. For example, broadcasting protocols have been designed which give the following values and bounds:

\[
b(\mathcal{SE}(n)) = 2n - 1 \quad [17],
\]

\[
b(\mathcal{B}(d, D)) \leq \frac{d + 1}{2} (D + 1) \quad [5],
\]

\[
b(\mathcal{B}(d, D)) \leq \left( \frac{5}{4} \left\lceil \log_2 d \right\rceil + 3 \right) D \quad [15].
\]

Other upper bounds can be found in [5, 15] where all the bounds have been improved. Results on lower bounds can be found in [22, 23]. For gossiping, the problem appears to be open both in full and half duplex models (see [13, 17]).

Here, we propose simple methods, using graph theory tools, to get upper bounds on both broadcast and gossip times. In Section 2, we generalize to higher degrees a well-known result (see [12]) stating that \( \mathcal{SE}(n) \) in a spanning subgraph of \( \mathcal{B}^*(2, n) \), by introducing a generalization of \( \mathcal{SE}(n) \) of in-degree and out-degree \( d + 1 \) that we will denote \( \mathcal{SE}(d, n) \). This result enables us to improve the broadcast time of the

\footnote{Due to our notations, the operation is different for graphs and digraphs, in what follows we will always use the digraph version.}
de Bruijn network in Section 3. In Section 4, we point out an interesting compound property of underlying line digraphs which enables a better understanding of results obtained in Section 3, and generalizes them. Our construction is somewhat related to a previous result [16] and to some similar ideas to appear in [5]. In the case of de Bruijn network the upper bound of [5] is better, but the technique used can be applied only to iterated line digraphs, and does not lead to complete result for generalized de Bruijn and Kautz graphs. Moreover, the broadcast protocols obtained arc used in Section 5 to give the first non-trivial bounds on gossiping time of some compounds of the de Bruijn, and, as consequence, the design of the first quasi-optimal gossiping on the shuffle-exchange network. Our results can be easily extended to other related structures like generalized butterflies or FFT networks. Finally let us note that in [6] we have used ideas of Section 3 to improve some of the bounds.

2. Compound digraphs

Compound graphs have been widely used in the construction of interconnection networks; see, for example, [2, 4, 21]. Here we consider compound digraphs.

A compound digraph of a digraph $G$ by a digraph $H$ is any digraph denoted $G[H]$, obtained by replacing each vertex of $G$ by a copy of $H$ and by joining some of the vertices of a copy associated to $x$ to some vertices of a copy associated to $y$ only if $y$ is a out-neighbor of $x$ in $G$.

Formally, a vertex of a compound digraph will be denoted by $(x, s)$, with $x$ a vertex of $G$ and $s$ a vertex of $H$. For every $x$, the vertex $(x, s)$ is adjacent to the vertex $(x, s')$ if and only if $s$ is adjacent to $s'$ in $H$. For a fixed $x$ in $G$, the sub-digraph induced by the vertices $\{(x, s) | s \in V(H)\}$ is therefore isomorphic to $H$ and will be called the copy $H_s$, associated to $x$. Finally, there exists at least one arc from a copy $H_x$ to a copy $H_y$ if and only if $(x, y)$ is an arc of $G$.

Let us emphasize that, as presented, the compound operation is not unique. To be complete one has to be precise how copies are linked. However, the construction above implies that, if we perform a quotient operation on $G[H]$, by replacing any copy $H_x$ by a vertex $x$, and by placing an arc from $x$ to $y$ if and only if there is at least an arc from $H_x$ to $H_y$, then we obtain $G$.

Example 2. Let us define a specific compound of the de Bruijn digraph $\mathcal{B}(2, n - 1)$ by $K_2^\ast$. Vertices are of the form $\langle x_1x_2 \ldots x_{n-1}, \alpha \rangle$ where $\alpha \in \{0, 1\}$ is a vertex of $K_2^\ast$ and $x_1x_2 \ldots x_{n-1}$ is a vertex of $\mathcal{B}(2, n - 1)$. For any $x = x_1x_2 \ldots x_{n-1}$, by putting two opposite arcs between $\langle x_1x_2 \ldots x_{n-1}, 0 \rangle$ and $\langle x_1x_2 \ldots x_{n-1}, 1 \rangle$, we form the copy of $K_2^\ast$ associated to $x$. The arcs between different copies of $K_2^\ast$ are defined as follows: There is an arc from $\langle x_1x_2 \ldots x_{n-1}, \alpha \rangle$ to $\langle x_2 \ldots x_{n-1}, \alpha, 0 \rangle$. Consequently, we have exactly one arc outgoing from the copy representing a node $x_1x_2 \ldots x_{n-1}$ of $\mathcal{B}(2, n - 1)$, to the copies associated to the nodes $x_2x_3 \ldots x_{n-1}$ for $\alpha \in \{0, 1\}$. Hence, the compound definition is satisfied.
It is easy to see that the shuffle-exchange digraph is isomorphic to this compound by associating to each vertex \( \langle x_1 x_2 \ldots x_{n-1}, z \rangle \) the vertex \( x_1 x_2 \ldots x_{n-1} z \) of \( E(n) \). An arc between two copies of \( K_2^* \) corresponds to a shuffle arc in \( E(n) \), and an arc in \( K_2^* \) corresponds to an exchange arc of \( E(n) \).

**Remark 3.** Note that the above result is equivalent to state that the quotient of the shuffle-exchange digraph \( E(n) \) obtained by identifying the vertices \( x_1 x_1 x_{r-1} z \) and \( x_1 x_1 x_{r-1} 0 \) is \( B(2, n-1) \).

**Remark 4.** Similarly, the undirected shuffle exchange can be seen as a compound of \( B(2, n-1) \) by \( K_2 \) (we first build a compound of the de Bruijn digraph by \( K_2^* \) and then we consider the underlying graph of the compound). Other compounds of \( B(2, n-1) \) with a different rule has been defined. For example, in [21], Jerrum and Skyum obtained a compound \( B(2, n-1)[K_2, 2] \) which is cubic graph of diameter \( 1.5 \log_2(N) \) and a compound \( B(3, n-1)[K_1, 3] \) which is also a cubic, but with diameter \( 1.47 \log_2(N) \).

Using the compound technique described above, one can generalize \( E(n) \) to obtain digraphs of higher degrees.

**Definition 5.** The digraph \( E(d, n) \) defined as the compound of \( B(2d, n-1) \) by \( K_2^* \) is built as follows: each vertex \( x \) of \( B(2d, n-1) \) is replaced by the copy \( (K_{2d}^*, x) \) consisting of vertices \( \langle x, s \rangle \) with \( s \in A \cup B = Z_{2d}, |A| = |B| = d \), \( A = \{0, 2, \ldots, 2d-2\} \) and \( B = \{1, 3, \ldots, 2d-1\} \). The sets \( A \) and \( B \) represent the two classes of the bipartition of the digraph \( K_{2d}^* \). Inside a copy \( (K_{2d}^*, x) \), there is an arc from \( \langle x, c_1 \rangle \) to \( \langle x, c_2 \rangle \) if and only if either \( c_1 \in A \) and \( c_2 \in B \), or \( c_1 \in B \) and \( c_2 \in A \). By analogy with \( E(n) \) we will call the corresponding arcs exchange arcs. Finally, to the arc of \( B(2d, n-1) \) from \( x_1 \ldots x_{n-1} \) to \( x_2 \ldots x_{n-1} x \), is associated the shuffle arc from vertex \( \langle x_1 \ldots x_{n-1}, z \rangle \) to \( \langle x_2 \ldots x_{n-1} x, x_1 \rangle \).

Thus, \( E(1, n) \) is nothing else than \( E(n) \), and \( E(d, n) \) has out(in)-degree equal to \( d + 1 \) and \( (2d)^n \) vertices.

In [11] Leighton proposed a different generalization of \( E(n) \); which is equivalent to a compound of the de Bruijn digraph by a cycle of length \( 2d \). However, the underlying graph obtained has degree 3 and is not helpful in our context.

From now on, we will consider the vertices of \( E(d, n) \) as words or elements of \( Z_{2d}^n \); that is vertex \( z = \langle x_1 \ldots x_{n-1} a \rangle \) will be identified with \( x_1 \ldots x_{n-1} a \). Thus, we can also define \( E(d, n) \) as a digraph on the set of words of \( Z_{2d}^n \) with the arcs:

- \( x_1 \ldots x_{n-1} \rightarrow x_1 \ldots x_{n-1}(x_{n} + a + 1), \ a \in 2Z_{2d} = \{0, 2, \ldots, 2d-2\} \), which are called exchange arcs.
- \( x_1 \ldots x_n \rightarrow \omega(x) = x_2 \ldots x_n x_1 \) which is the shuffle arc.

The result of [12] stating that \( E(1, n) = E(n) \) is a spanning subdigraph of \( E(2, n) \) can be generalized as follows:
Proposition 6. The digraph $S(d, n)$ (and consequently $S'(d, n)$) is a spanning subdigraph of $B*(2d, n)$.

Proof. We define the parity $p(x_i)$ of an element $x_i$ of $Z_{2d}$ as $x_i \mod 2$ (thus $p(x_i) \in Z_2$). The parity $p(z)$ of a vertex $z = x_1 \ldots x_{n-1}x_n$ of $S(d, n)$ is the sum in $Z_2$ of the parities of its letters: $\sum_{1 \leq i \leq n} p(x_i)$. Vertices of parity 0 (resp. 1) will be called even (resp. odd) vertices. Let $\omega$ denotes the shuffle action (left-shift) that is $\omega(x_1 \ldots x_n) = x_2 \ldots x_n x_1$; therefore $\omega^{-1}$ is a right shift. Note that, parity being invariant under permutation of the letters, $z, \omega(z)$ and $\omega^{-1}(z)$ have the same parity.

The embedding of $S(d, n)$ into $B*(2d, n)$ is defined as follows:

$$f(z) = \omega^{-p(z)}(z) = \begin{cases} z = x_1 \ldots x_{n-1}a & \text{if } z \text{ is even}, \\ \omega^{-1}(z) = ax_1 \ldots x_{n-1} & \text{if } z \text{ is odd}. \end{cases}$$

The mapping $f$ is a one to one mapping since even (resp. odd) vertices of $S(d, n)$ are sent to even (resp. odd) vertices of $B*(2d, n)$ by the mapping identity (resp. right-shift).

Let us consider a shuffle arc of $S(d, n)$: since $z$ and $\omega(z)$ have the same parity, the arc $(z, \omega(z))$ of $S(d, n)$ is mapped on the arc of $B*(2d, n)$ joining $f(z)$ to $f(\omega(z)) = \omega(f(z))$. An exchange arc of $S(d, n)$ joins the vertices $z_a = x_1 \ldots x_{n-1}a$ and $z_b = x_1 \ldots x_{n-1}b$, which are of different parities. Without loss of generality, suppose that $z_a$ is even, so $z_b$ is odd. Then $f(z_a) = z_a$ and $f(z_b) = \omega^{-1}(z_b) = bx_1 \ldots x_{n-1}$ which is an in-neighbor of $z_a$ in $B*(2d, n)$ (note that it is not true for $B(2d, n)$, because the mapping does not keep the orientation). □

3. Broadcasting in compound digraphs

The knowledge of properties of $G$ and $H$ can give information about a compound digraph $G[H]$. For instance, we have the following easy bound on the diameter of any $G[H]$:

$$D(G[H]) \leq (D(G) + 1) \cdot D(H) + D(G).$$

For undirected graphs $G$ and $H$, one can find in [4] a link between the diameter of $G$, the average broadcast time$^2$ of $H$ and the broadcast time of a compound $G[H]$. Here we give a result on $b(G[H])$, where $G$ and $H$ are digraphs, valid when $G[H]$ has the following additional property:

$^2$During a broadcast protocol a node $x$ in ready at time $r(x)$ if it is informed at a time less than $r(x)$ and is no more involved in the broadcast protocol after time $r(x)$. The average broadcast time is the average of $r(x)$ on the network.
Definition 7. A compound digraph \( G[H] \) has the local matching property if, for any vertex \( x \) of \( G \) having \( d \) out-neighbors \( y_1, y_2, \ldots, y_d \) in \( G \), there exists a set of \( d \) independent\(^3\) arcs \( e_i \) in \( G[H] \), where \( e_i \) is an arc linking \( H_x \) to \( H_{y_i} \).

Example 8. \( SE(d, n) \) has the local matching property. Indeed in \( \mathcal{B}(2d, n - 1) \) the vertex \( x = x_1 \ldots x_{n-1} \) has out-neighbors \( y_i = x_2 \ldots x_{i-1}i \) for \( 0 \leq i \leq 2d - 1 \). In \( SE(d, n) \), the \( 2d \) shuffle arcs \( e_i \) from \( x_1 \ldots x_{n-1}i \) to \( x_2 \ldots x_{n-1}ix_1 \) for \( 0 \leq i \leq 2d - 1 \) are independent.

Theorem 9. Let \( G \) be a digraph of diameter \( D(G) \), and \( H \) be a digraph with broadcast time \( b(H) \); then, if \( G[H] \) satisfies the local matching property:

\[
b(G[H]) \leq (D(G) + 1) \cdot b(H) + D(G)
\]

Proof. The protocol can be decomposed into \( D(G) + 1 \) stages. The first \( D(G) \) stages consist in a local step followed by a forwarding step. The local step is performed in each copy \( H_x \) which has been informed in the preceding stage. More exactly among the vertices informed in the copy \( H_x \), we choose a particular one and let it broadcast the message in \( b(H) \) rounds to all the vertices in \( H_x \). During the forwarding step, for each copy \( H_x \) which have received the message, the end vertices of the arcs \( e_i \) associated to the local matching (see definition 7) send the message along \( e_i \); so due to the local matching property, these arcs are independent and in one round the message reaches one vertex in each copy \( H_y \), where \( y \) is an out-neighbor of \( x \) in \( G \).

Let \( x \) be any originator; after the \( D(G) \) first stages, there is at least one vertex informed in every copy. Then during the last stage in each copy a vertex which has received the message informs all the other vertices of the copy in at most \( b(H) \) rounds. The total number of rounds is therefore at most \( D(G) \cdot (b(H) + 1) + b(H) \).

Remark 10. Note that the formula obtained in very similar to the one concerning diameters. Indeed the formula dealing with diameters can be obtained by considering a broadcast model where a node can send a message to all its neighbors at the same time; in such a model \( b(H) \) is equal to \( D(H) \), and \( D(G[H]) = (D(G) + 1) \cdot D(H) + D(G) \).

Using Proposition 6 and Theorem 9, we obtain:

Corollary 11. \( b(\mathcal{B}^*(2d, n)) \leq b(SE(d, n)) \leq n \cdot b(K_{d,d}^*) + n - 1 = n \cdot \lceil \log_2(2d) \rceil + n - 1 \).

Proof. By Proposition 6, \( SE(d, n) \) is a spanning sub-digraph of \( \mathcal{B}^*(d, n) \), so \( b(\mathcal{B}^*(d, n)) \leq b(SE(d, n)) \). By Definition 5, \( SE(d, n) \) is a compound of \( \mathcal{B}(2d, n - 1) \)

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\(^3\)Arcs are independent when they do not share any common vertex.
by $K_{d,d}^*$, and satisfies the local matching property (Example 8). So we can apply Theorem 9 using the fact that $B(2d, n - 1)$ has diameter $n - 1$ and that $b(K_{d,d}^*) \leq \lceil \log_2(2d) \rceil$.

**Remark 12.** In the models used, $b(K_{d,d}^*)$ is $\lceil \log_2(2d) \rceil$; but the relation above can also be applied with other models and therefore other values for $b(K_{d,d}^*)$, for example if we allow to send information to $k$ neighbors at the time.

Note that the obvious lower bound is

$$b(B^*(2d, n)) \geq \lceil \log_2((2d)^n) \rceil = \lceil n \log_2(2d) \rceil.$$  

So, when $d$ tends to infinity, we have obtained an asymptotically optimal upper bound. Clearly, our result is not optimal for de Bruijn graphs of small degree. For example, for $d = 1$ we obtain $b(B^*(2, n)) \leq 2n - 1$ although it has been proved that $b(B(2, n)) \leq \frac{3}{2}(n + 1)$. The previous results [7, 15] were

$$b(B^*(2d, n)) \leq \frac{2d + 1}{2} (n + 1),$$

$$b(B^*(2d, n)) \leq \left( \frac{3}{2} \lceil \log_2(2d) \rceil + 3 \right) n.$$  

The bound obtained is always better than the previous ones as soon as $d \geq 4$. We must point out that, when $d = 2^k$ and $k < 4$, a better bound was obtained in [15]. Note that, in our result, the ceiling is performed after multiplying by $n$ and this round estimate hides a non-constant over-cost; in fact, $b(B^*(2b, n)) - \log_2(2d^n) \leq 2n - 1$.

Recently, we have found a more sophisticated protocol for undirected de Bruijn graphs using some ideas of this paper. The new bound obtained in [6] improves all the results of the literature and can be summarized as: $b(B^*(d, n)) \leq \left\lfloor n(\log_2(d) + \frac{1}{2}) \right\rfloor + \log_2(d)$. However, the protocol given in this article is useful, since it was the first to give the asymptotically optimal bounds. Moreover we will generalize the protocol in Section 4 to undirected line digraphs, and use the high symmetry of the protocol in Section 5 to derive gossiping schemes.

4. Line digraphs and compound graphs

Let us recall proposition 6 stating that, $L(d, n)$ is a subdigraph of $B^*(2d, n)$, and that $B(2d, n) = L(B(2d, n - 1))$. So, in fact, we have

$$L(d, n) = B(2d, n - 1)[K_{d,d}^*] \subset B^*(2d, n)$$

that is also

$$B(2d, n - 1)[K_{d,d}^*] \subset L^*(B(2d, n - 1)).$$

This fact is more general and it is easy to prove when a specific coloring of the digraph, called *fair coloring*, is given.
Definition 13. A regular digraph of out-degree $2d$ (and so in-degree $2d$) admits a fair coloring if its arcs can be colored in blue or red so that each vertex has as many blue incoming (resp. outgoing) arcs as red incoming (resp. outgoing) arcs.

Proposition 14. Any regular digraph $G$ of even out-degree admits a fair coloring.

Proof. A proper coloring of a regular digraph $G$ is a coloring of the arcs using a number of colors equal to the out-degree of $G$ such that the incoming (resp. outgoing) arcs of any vertex have different colors. In the case of an even out-degree $2d$, one can consider the color of a proper coloring as the set of integers $\{0, 1, \ldots, 2d - 1\}$ and obtain a fair coloring by considering the colors modulo 2.

Proposition 15. If $G$ is a regular digraph of out-degree $2d$, then $L^*(G)$ is a compound $G[K_{*\times d}]$, satisfying the local matching property.

Proof. Let $A \cup B$ be the bipartition of the digraph $K_{d, *d}^*$. $A = \{a_0, a_1, \ldots, a_{d-1}\}$ and $B = \{b_0, b_1, \ldots, b_{d-1}\}$. Let us define a one to one mapping of the vertices of $L^*(G)$ onto vertices of $G[K_{d, *d}^*]$, as follows:

to the vertex $e$ of $L(G)$, representing the arc $(x, y)$ of $G$ is associated the vertex:
- $\langle y, a \rangle$ with $a \in A$ if $e$ is red,
- $\langle x, b \rangle$ with $b \in B$ if $e$ is blue.

$a$ and $b$ are chosen in such a way that the labeling is a one-to-one mapping. That is possible since there are $d$ red arcs entering (resp. $d$ blue arcs leaving) a given vertex $x$ of $G$, and $d$ elements in $A$ (resp. $B$).

Now we prove that according to this labeling $L^*(G)$ contains a compound $G[K_{d, *d}^*]$ which satisfies the local matching property. Let us first check that each copy $H_x$ is isomorphic to $K_{d, *d}^*$. A vertex $\langle x, a \rangle$ represents a red arc $e$ of the kind $(z, x)$ and is consequently adjacent in $L^*(G)$ to all the vertices $f$ representing arcs of kind $(x, y)$. The only vertices representing an arc $(x, y)$ which are in the copy $H_x$ represent a blue arc and are labeled $\langle x, b \rangle$. Consequently the neighbors of $\langle x, a \rangle$ in $H_x$ are exactly all the vertices $\langle x, b \rangle$ for $b \in B$. Hence $H_x$ is isomorphic to $K_{d, *d}^*$.

We now prove the second part of the statement by showing that in each copy $H_x$ there is a set of $2d$ independent arcs linking it to the $2d$ copies $H_y$, where $y$ is a out-neighbor of $x$ in $G$. Let us consider an out-neighbor $y$ of $x$ in $G$, then two cases might appear:

- if the arc $(x, y)$ is red, then it is associated to a vertex $\langle y, a_i \rangle$. Let us consider the vertex labeled $\langle x, a_i \rangle$ it is associated to a red arc $(z, x)$ and is consequently adjacent to $(x, y)$ in $L^*(G)$. Hence, if $y$ is a neighbor of $x$ along a red arc in $G$, we find in $L^*(G)$ an arc $\langle x, a_i \rangle, \langle y, a_j \rangle$ between the copies $H_x$ and $H_z$;
- if the arc $(x, y)$ is blue, it is associated to a vertex $\langle y, b_i \rangle$. Let us consider the vertex labeled $\langle y, b_i \rangle$ it represents a blue arc $(y, z)$, and is consequently adjacent to $\langle x, b_i \rangle$ in $L^*(G)$. Hence, if $y$ is a neighbor of $x$ along a blue arc in $G$, we find in $L^*(G)$ an arc $\langle x, b_i \rangle, \langle y, b_j \rangle$ between the copies $H_x$ and $H_y$ (see Fig. 1).
By construction the arcs are clearly independent and therefore the local matching property is insured. □

An immediate consequence of Proposition 15 and Theorem 9 is:

**Corollary 16.** If \( G \) is 2d regular, then

\[
b(L^*(G)) \leq (D(G) + 1) \cdot b(K^*_d,d) + D(G).
\]

According to the preceding results we can reconsider Proposition 6 and see that the mapping from \( \mathcal{B}(d, n) \) to \( \mathcal{B}^*(2d, n) \) is induced by a fair coloring of the de Bruijn digraph. Moreover, it gives also a result about the symmetric Kautz graph.

**Corollary 17.** \( b(\mathcal{K}(2d, n)) \leq n \cdot \lceil \log_2(2d) \rceil + n - 1. \)

**Proof.** \( \mathcal{K}(2d, n - 1) \) admit a fair coloring, hence as \( \mathcal{K}(2d, n) = L(\mathcal{K}(2d, n - 1)) \) we know that \( \mathcal{K}^*(2d, n) \) contains a compound graph \( \mathcal{K}(2d, n - 1) [K^*_d,d] \) with the local matching property, thus the result hold. □

**Corollary 18.** If 2d is a divisor of \( N \), then

\[
b(\mathcal{B}^*(2d, N)) \leq \lceil \log_{2d}(N) \rceil \cdot \lceil \log_2(2d) \rceil + \lceil \log_{2d}(N) \rceil - 1.
\]

\[
b(\mathcal{K}^*(2d, N)) \leq \lceil \log_{2d}(N) \rceil \cdot \lceil \log_2(2d) \rceil + \lceil \log_{2d}(N) \rceil - 1.
\]
Proof. It can be proved that if \( d \) is relatively prime with \( N \) then \( B(2d, N) \) and \( K(2d, N) \) admit a fair coloring. Consequently, by line graph operation \( B(2d, N) \) and \( K(2d, N) \) admit a fair coloring too. The result is then obtained according to the fact that when \( 2d \) divides \( N \), \( B(2d, N) \) (resp. \( K(2d, N) \)) is isomorphic to \( L(B(2d, N/2d)) \) (resp. \( L(K(2d, N/2d)) \)), as the diameter of \( B(2d, N) \) and \( K(2d, N) \) is \( \lceil \log_2(N) \rceil \).

Remark 19. This result can be generalized to many line digraphs like \( d \) consecutive digraphs (see the survey [1] or the article [10]).

5. Gossiping on compounds of the de Bruijn graph

One might think of using the same protocol as the one described in the proof of Theorem 9, by replacing the broadcast schemes in the local steps, by local gossip schemes (so, in fact by replacing \( h(H) \) by \( g(H) \) in the formula). However, this works well in the local steps, where copies exchange information independently, but there is a problem to exchange information between copies during the forwarding step.

Example 20. In \( S(d, n) \), during the forwarding step, the vertex \( x \) has to send all its information to \( \omega(x) \) (we will note that by \( x \to \omega(x) \)). But, as we perform a gossip, \( \omega(x) \) is itself involved in \( \omega(x) \to \omega^2(x) \), and so, there are conflicts. However, if all the cycles \( (x, \omega(x), \omega^2(x), \ldots) \) generated by the action \( \omega \) are of even length, we can perform this shuffle or forwarding step in two rounds. That is the case if and only if no odd number greater than 1 divides \( n \), so \( n = 2^p \), giving \( g(S(n)) \sim 3n \) when \( n = 2^p \) (this idea was known to B. Monien and D. Sotteau - private communications).

Here, we give a different protocol which avoids this difficulty and enables us to perform quasi-optimal gossiping. We first give a protocol for \( S(n) \), and then, a generalized protocol for \( S(d, n) \).

For simplicity, we will use regular language expressions: \( * \) will denote any letter of the alphabet \( (Z_2 \) and later \( Z_{2d} \)), the set of words of length \( p \) on the alphabet will be denoted \( *^p \). For example \( *^401 * \) denotes the set of word of length 7 whose fifth (resp. sixth) latter is a zero (resp. a one).

5.1. Gossiping on \( S(n) \)

Theorem 21. \( g_{fr}(S(n)) \leq 2n + 5 \quad g_{fr}(S(n)) \leq 3n + 3 \)

Proof. Recall that \( S(n) \) is a compound of \( B(2, n - 1) \) by \( H = K_2^p \). We use the following protocol:

Phase 1: We concentrate the information is such a way that the information of any vertex of the graph will be on at least one vertex \( w \) of some copy \( H_x \), where \( x \) belongs to \( *^{n-3}01 \).
**Phase 2:** We now send to whole information to all the vertices \( \{ w \in H_y | y \in 01^n \} \).

In fact, we do independent broadcasts in parallel from each vertex belonging to a copy \( H_x \) where \( x \) belongs to \( 0^n-301 \), to all the vertices of all the copies \( H_y \) where \( y \) belongs to \( 01^n-3 \). That is called in [14] a set to set broadcasting. Our first set is \( \{ w \in H_x | x \in 0^n-301 \} \), the second is \( \{ w \in H_y | y \in 01^n-3 \} \).

**Phase 3:** We do the "reverse operation" of phase 1, by informing any vertex of the graph from some vertex \( w \) of \( \{ w \in H_y | y \in 01^n \} \).

We will prove that phase 1 (and therefore phase 3 which is similar) can be performed in five rounds. Then, we will prove that phase 2 can be done in \((n - 2)g(K^*_2) + n - 3\) rounds; that follows from the more general Lemma 1. So, all together, as \( g_F(K^*_2) = 1 \) (resp. \( g_H(K^*_2) = 2 \)), we obtain that the whole protocol can be realized in:

- \( 2n + 5 \) rounds in case of a full duplex mode.
- \( 3n + 3 \) rounds in case of a half duplex mode.

**Time of phase 1.** We send the information of vertex \( z = \langle x_1x_2 \ldots x_{n-3}0\beta, \gamma \rangle \) belonging to any copy of \( K^*_2 \) to some vertex belonging to a copy \( H_x \) of \( K^*_2 \) associated to a vertex \( x \) belonging to \( 0^n-301 \). We will use five rounds consisting of disjoint calls \( x \to y \) (\( x \to y \) means that \( x \) sends all its information to \( y \), or \( x \) calls \( y \); a call is directed).

We only have to check that any vertex is involved in at most one call in each round. Vertices with \( \alpha\beta = 01 \) have already their own information. We look at the six other possible values for \((\alpha, \beta, \gamma)\) (patterns), namely \( \{ (00, 0), (00, 1), (10, 0), (10, 1), (11, 0), (11, 1) \} \):

**Round 1**

\[
\begin{align*}
\langle x_1 \ldots x_{n-3}00, 0 \rangle &\to \langle x_1 \ldots x_{n-3}00, 1 \rangle \\
\langle x_1 \ldots x_{n-3}10, 0 \rangle &\to \langle x_1 \ldots x_{n-3}10, 1 \rangle \\
\langle x_1 \ldots x_{n-3}11, 1 \rangle &\to \langle x_1 \ldots x_{n-3}11, 0 \rangle
\end{align*}
\]

All these calls are done without conflict since the six patterns of the three last digits differ. And there are also no conflicts between vertices \( \langle x, \gamma \rangle \) and \( \langle x', \gamma' \rangle \) when \( x \neq x' \):

**Round 2**

\[
\begin{align*}
\langle x_1 \ldots x_{n-3}00, 1 \rangle &\to \langle x_2 \ldots x_{n-3}001, x_1 \rangle \\
\langle x_1 \ldots x_{n-3}11, 0 \rangle &\to \langle x_2 \ldots x_{n-3}110, x_1 \rangle
\end{align*}
\]

Here again, the patterns of the three last digits are distinct. Note that we are done with pattern \((00, 1)\). Note that we could not do concurrently \( y_1 \ldots y_{n-3}10, 1 \to y_2 \ldots y_{n-3}101, y_1 \), as we would find \( y_1 \ldots y_{n-3}10, 1 \) identical to \( x_2 \ldots x_{n-3}110, x_1 \) if \( y_1 = x_{i+1}, y_{n-3} = 1, \) and \( x_1 = 1 \) creating a conflict. After this round, only the information of vertices with patterns \( \{(110, x_1), (10, 1)\} \) still have to be sent on vertices of some \( H_x \) where \( x \in 0^n-301 \):
Round 3
\[
\langle x_1 \ldots x_{n-3}10, 1 \rangle \rightarrow \langle x_2 \ldots x_{n-3}101, x_1 \rangle
\]

Pattern (10, 1) is now ok; the last patterns to consider are \{(110, x_1)\}:

Round 4
\[
\langle x_1 \ldots x_{n-3}110, 0 \rangle \rightarrow \langle x_2 \ldots x_{n-3}110, 1 \rangle
\]

The last pattern is (110, 1):

Round 5
\[
\langle x_1 \ldots x_{n-3}110, 1 \rangle \rightarrow \langle x_3 \ldots x_{n-3}1101, x_2 \rangle
\]

The process is therefore completed after five rounds. Note that such a process could not cost less than four rounds, as there are vertices at distance four of the target subgraph. 

Time of phase 2. Here, we state the basic lemma which enables us to perform a quasi-total gossiping, using concurrent broadcasting schemes defined in Section 3, Theorem 9. We are only interested in compounds with the one operator property:

Definition 22. We say that a compound digraph was the one operator property if on each vertex of any copy, there is only one incoming arc from another copy and one outgoing arc to a neighbor copy. In other words, there is a unique operator \( \omega_x \) associated to a vertex, and one converse operator.

This is one of the properties of some compounds studied by Bermond et al. [4]. It implies the local matching property. Note that \( \mathcal{D}(d, n) \) satisfies this property.

Lemma 1. Let \( G[H] \) be a compound of the de Bruijn digraph \( \mathcal{B}(2, n-1) \) by \( H \) satisfying the one operator property; then, one can perform a set to set broadcasting from each vertex of the set \( \{w \in H_x | x \in \ast^{n-3}01\} \) to each vertex of the set \( \{w \in H_y | y \in 01\ast^{n-3}\} \) in \((n - 3)(g(H) + 1) + g(H)\) rounds.

Proof. It is similar to that of Theorem 9. In the first local stage, we perform gossiping on each copy \( H_x \) where \( x \in \ast^{n-3}01 \). Then, in each copy, each vertex use the one operator property to send its information to the unique vertex of an out-neighbor copy. A vertex can be at most involved in two calls (one as a sender, one as a receiver). But, the receivers are in copies \( H_y \), where \( y \) belongs to \( \ast^{n-4}01\ast \), and are therefore distinct from senders, which are in copies of \( H_x \) with \( x \) in \( \ast^{n-3}01 \) (the \((n - 2)\)th letter being, respectively, 1 and 0). At stage \( i \), with \( 1 \leq i \leq n - 3 \), we perform a gossiping in each copy \( H_x \), where \( x = \ast^{n-2-i}01\ast^{i-1} \) (active copy) and then, each vertex lying on an active copy sends its information along its unique outgoing arc. Here again, the sending can be done independently as the \((n - 1 - i)\)th letter of the active copies is...
and the \((n - 1 - i)\)th letter of the receiving copies is 1. After \(n - 3\) stages, the active copies (i.e. \(\{H_y \mid y = 01^{n-3}\}\)) still have to perform a gossiping so that all the vertices of the copy \(H_y\) know all the original information. So, although, we have \((n - 3)(g(H) + 1) + g(H)\) rounds. Note that this proof just states that we can perform the first \(n - 3\) stages of a concurrent broadcasting involving senders in copies \(H_x\) with \(x\) in \(*^{n-3}01\) without conflict.

5.2. Generalization to higher degree

For gossip in \(\mathcal{S}_d(d, n)\), which is a compound of \(\mathcal{B}(2d, n - 1)\) by \(K^{\times}_{d, d}\), we use exactly the same protocol as in the case of degree 2. Phase 1 is still valid, but one has to concentrate information on the subgraph consisting of the copies \(H_x\) where \(x\) belongs to \(*^{n-3}ab\), \(a \in A\), \(b \in B\), where \(A\) (resp. \(B\)) is the set of even (resp. odd) letters. For phase 2, we use the Lemmas 2 and 3, instead of Lemma 1.

**Lemma 2.** Let \(G[H]\) be a compound of the de Bruijn \(\mathcal{B}(2, n - 1)\) by \(H\), satisfying the one operator property; let \(B_{ab}\) denote the set to set broadcasting consisting in sending information from any vertex of the set \(S_{ab} = \{w \in H_x \mid x \in *^{n-3}ab\}\) to any vertex of the set \(F(S_{ab}) = \{w \in H_y \mid y \in ab*^{n-3}\}\); then, one can perform concurrently all the \(B_{ab}\) for couples \((a, b)\) such that \(a \in A\) and \(b \in B\) in \((n - 3)(g(H) + 1) + g(H)\) rounds.

**Proof.** The proof is the same as the one of Lemma 1. Just recall that, at stage \(i\), the \((n - 1 - i)\)th letter of active copies is in \(A\) and the \((n - 1 - i)\)th letter of receiving copies is in \(B\).

In the same way, we also have the following result:

**Lemma 3.** Let \(G[H]\) be a compound of the de Bruijn digraph \(\mathcal{B}(2d, n - 1)\) by \(H\), satisfying the one operator property; the partial gossiping in the subdigraph \(S = \{w \in H_x \mid x \in *^{n-3}ab, a \in A, b \in B\}\) can be performed in \((n - 1)(g(H) + 1) + g(H) = ng(H) + (n - 1)\) rounds.

**Proof.** At the end of phase 2, any vertex in a copy \(H_x\) with \(x\) in \(ab*^{n-3}\) knows all the information of the vertices of the copies \(*^{n-3}ab\) with the same couple \((a, b)\) (Lemma 2). Now, in one round, the appropriate vertices of copies \(H_x\) where \(x\) belongs to \(*^{n-3}ab\) send their information to the out-neighbor copies \(H_y\) where \(y\) belongs to \(b*^{n-3}a'\), using the one operator property (it can be considered as a shuffle). Then, we perform a gossiping in \(g(H)\) rounds in the copies of the form \(b*^{n-3}a'\). After that, appropriate vertices of copies \(H_x\) with \(x\) in \(b*^{n-3}a'\) forward their information to copies \(H_y\) with \(y\) in \(*^{n-3}a'b'\). At this time, it remains to perform a gossiping in all the copies \(*^{n-3}a'b'\) to complete the process. Altogether we have \(2(1 + g(H))\) rounds. Here again, all the calls are done independently.
Note that the subgraph $S$ represents one-quarter of the whole graph, and to obtain a more general bound, we just need to compute the time of phase 1; its complexity should be in most cases $\Theta(1)$. That can be done for $\mathcal{SE}(d, n)$:

**Theorem 23.** $g(\mathcal{MA}(2d, n)) \leq g(\mathcal{SE}(d, n)) \leq n(g(K^*_d, d) + 1) + 9$.

**Proof.** Phases 1 and 3 can be done in five rounds exactly and phase 2 requires $ng(K^*_d, d) + n - 1$ rounds (by Lemma 3). \(\square\)

Depending on the model, $g(K^*_d, d)$ is either $\lceil \log_2(2d) \rceil$ in full duplex mode, or related to Fibonacci parameter $\lceil fib^{-1}(2d) \rceil$, defined as the first integer $u$ such that $fib(u) \geq 2d$, in half duplex mode. In this case, $g_F(K^*_d, d) \leq \lceil fib^{-1}(d) \rceil + 1$ (see [22]).

**Corollary 24.** $g_F(\mathcal{MA}(2d, n)) \leq g_F(\mathcal{SE}(d, n)) \leq n(\lceil \log_2 2d \rceil + 1) + 9$.

6. Conclusion

In this article we have given some new bounds asymptotically optimal to broadcast in de Bruijn, Kautz undirected graphs, or on some of their generalization. We also gave the first efficient gossiping protocols on the shuffle-exchange and on undirected de Bruijn graphs. These results are corollaries of more general ones concerning compounds of smaller de Bruijn digraphs with complete bipartite graphs. Our proofs can be easily applied to compounds with other graphs, and also be extended to similar networks. Note that we always use underlying graphs of digraphs of even degrees. But one can also consider the case of odd degrees, fair coloring being replaced by a quasi-fair coloring which leads to a compound by $K_{(d-1)/2,(d+1)/2}$, and allows similar bounds.

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References


\(fib\) is in fact the golden ratio $(1 + \sqrt{5})/2$, and $fib(u)$ is the $u$th number of the Fibonacci sequence, roughly equal to $fib^u$. 


