THE THREE-PERMUTATIONS PROBLEM

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Given any three permutations on \{1, \ldots, n\}, we want to choose \( f : \{1, \ldots, n\} \to \{-1, 1\} \) so that the maximum absolute partial sum of \( f \) values over the permutations is minimized. The three-permutations problem is to determine the supremum of this minimum taken over all \( n \) and all triples of permutations on \{1, \ldots, n\}. The only thing presently known about the supremum is that it is at least two. This paper establishes a result for a restricted case of the problem in which the maximum absolute partial sum for one of the three permutations equals 1. The supremum in this case is unbounded.

1. Introduction

This note is based on a fundamental and unresolved combinatorial optimization problem from discrepancy theory \([1-4]\) known as the three-permutations problem. We report here a result for a closely-related problem. Beyond this, we hope to stimulate wider interest in what is presently an enticing but intractable problem in the foundations of combinatorial optimization.

Throughout, \( n = \{1, 2, \ldots, n\} \), \( S_n \) is the family of permutations \( \sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) on \( n \), \( F_n \) is the set of all \( f : n \to \{-1, 1\} \), and for each \((f, \sigma) \in F_n \times S_n \),

\[
    f^*(\sigma) = \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} f(\sigma(j)) \right|
\]

Given \( m \) permutations \( \sigma_1, \ldots, \sigma_m \) in \( S_n \), let

\[
    K(\sigma_1, \ldots, \sigma_m) = \min_{f \in F_n} \max \{ f^*(\sigma_1), \ldots, f^*(\sigma_m) \}.
\]

For example, \( K(\sigma_1, \sigma_2, \sigma_3) = 2 \) for \( n = 3 \) and

\[
    \sigma_1 = (1, 2, 3)
\]

\[
    \sigma_2 = (1, 3, 2)
\]

\[
    \sigma_3 = (2, 3, 1)
\]

since if \( f(1) \neq f(2) \) then either \(|f(1) + f(3)| = 2 \) or \(|f(2) + f(3)| = 2 \).
Let
\[ K_m = \sup_{n} \max_{(S_n)^n} K(\sigma_1, \ldots, \sigma_m). \]
The assignment \( f(i) = (-1)^i \) for \( \sigma = (1, 2, 3, \ldots) \) shows that \( K_1 = 1 \), and it has been known for some time that \( K_2 = 1 \).

The three-permutations problem is to determine \( K_3 \). As far as we are aware, nothing is known about \( K_3 \) except that \( K_3 \geq 2 \). It is not even known whether \( K_3 \) is finite.

Our purpose here is to consider a restricted three-permutations problem for which \( f^*(\sigma_i) = 1 \) for some \( i \), so that each neighbor pair in \( \sigma_i \) is assigned 1 and -1, in either order. For \( \sigma_i \in S_n \) let
\[ K^0(\sigma_1, \sigma_2, \sigma_3) = \min_{R_i} \max \{ f^*(\sigma_1), f^*(\sigma_2), f^*(\sigma_3) \} \text{ s.t. } \min f^*(\sigma_i) = 1, \]
and let \( K^3 \) be the supremum of \( K^0(\sigma_1, \sigma_2, \sigma_3) \) over all \( n \) and all \( (\sigma_1, \sigma_2, \sigma_3) \) in \( (S_n)^3 \). We were motivated to consider \( K^3 \) to see if it resolves anything about the basic three-permutations problem. It does not.

**Theorem 1.** \( K^3 = \infty \).

Our proof of Theorem 1 is based on a lemma that we state in an easily-visualized geometric form. Let \( \mathcal{R} \) denote a nonempty finite set of rectangles in \( \mathbb{R}^2 \) with sides parallel to the axes such that no horizontal or vertical line in the plane includes more than one side of all \( R \in \mathcal{R} \). For each real \( x \) let
\[ \mathcal{R}_x = \{ R \in \mathcal{R} : R \text{ is intersected by the vertical through } (x, 0) \} \]
\[ \mathcal{R}^x = \{ R \in \mathcal{R} : R \text{ is intersected by the horizontal through } (0, x) \}. \]
Also let \( G(\mathcal{R}) \) be the set of all \( g : \mathbb{R} \to \{-1, 1\} \), then define \( K(\mathcal{R}) \) by
\[ K(\mathcal{R}) = \min_{g \in G(\mathcal{R})} \left[ \max_{x \in \mathbb{R}} \left| \sum_{R \in \mathcal{R}_x} g(R) \right|, \max_{x \in \mathbb{R}} \left| \sum_{R \in \mathcal{R}^y} g(R) \right| \right]. \]
Thus \( K(\mathcal{R}) \) is the smallest value of the largest absolute \( g \) sum over rectangles in \( \mathcal{R} \) intersected by vertical and horizontal lines.

**Lemma 1.** \( K(\mathcal{R}) \) is unbounded over the collection of all nonempty finite \( \mathcal{R} \) as described above.

This is proved next, then is used to prove Theorem 1. We conclude with remarks on potential research.

### 2. Rectangles in the plane

Fig. 1 shows a 12-rectangle \( \mathcal{R} \) configuration. No \( \mathcal{R}_x \) or \( \mathcal{R}^y \) contains more than three rectangles. We claim that \( K(\mathcal{R}) = 3 \). Suppose to the contrary that
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Fig. 1. $K(\mathcal{R}) = 3$. $K(\mathcal{S}) < 3$. Then $\mathcal{R}_x = \{1, 2, 3\}$ forces one of $g(1)$, $g(2)$ and $g(3)$ to differ from the others. If $g(1) = g(2)$ then $K < 3$ and $\mathcal{R}^x$ sets $\{1, 2, 4\}$, $\{1, 2, 5\}$ and $\{1, 2, 6\}$ force $g(4) = g(5) = g(6) = -g(1)$, so $|g(4) + g(5) + g(6)| = 3$. A similar conclusion obtains if $g(1) = g(3)$ or if $g(2) = g(3)$. Hence $K(\mathcal{R}) = 3$.

We extend the construction of Fig. 1 to prove Lemma 1. Call $\mathcal{R}_x$ and $\mathcal{R}^x$ cut sets. Fig. 1 has 13 three-rectangle cut sets, seven for horizontal lines and six for verticals. We add new rectangles so that each of these 13 remains a cut set in the enlarged $\mathcal{R}$. Then, regardless of how $g : \mathcal{R} \to \{-1, 1\}$ is defined on the enlarged $\mathcal{R}$, one of the 13 must have $|g(i) + g(j) + g(k)| = 3$.

The first additions are four new rectangles in each of the 13 strips of positive width whose axis-parallel lines have the same three-rectangle cut set in Fig. 1. There is one strip for each of the 13 cut sets. The four new rectangles in a strip are disjoint, every line through their union that is perpendicular to the strip cuts no rectangle from an earlier step or another added in the present step, and there is a positive-width strip perpendicular to the strip for these four such that every line in this new strip cuts all four.

Fig. 2 pictures the four rectangles $(a, b, c, d)$ added in one of the 13 strips. The vertical strip through these four is the perpendicular strip described at the end of the preceding paragraph. Disregard the rest of Fig. 2 for the moment since it is used in the next step. The four rectangles $(a, b, c, d)$ create five four-rectangle cut sets, four horizontal and one vertical for the configuration shown. This is done for each of the 13 strips from Fig. 1, for a total of 65 four-rectangle cut sets. Since
one of the 13 three-rectangle cut sets from Fig. 1 has $|\Sigma g| = 3$, one of the new four-rectangle cut sets must have $|\Sigma g| = 4$. For example, if $g(10) = g(11) = g(12)$ and none of \{10, 11, 12, a\},..., \{10, 11, 12, d\} has $|\Sigma g| = 4$, then $|g(a) + g(b) + g(c) + g(d)| = 4$. Thus $K(\mathcal{R}) = 4$ at this point.

The procedure continues in the natural way. The next step adds five rectangles within each of the 65 strips for the 65 four-rectangle cut sets. The five-rectangle batches for the strips generated from the addition of a through d are shown on Fig. 2. The placement restrictions for prior additions apply here as well and lead to $K(\mathcal{R}) = 5$ after the placements of the 65 five-rectangle batches.

Continuation forces $K(\mathcal{R})$ as large as we wish.

3. Restricted three-permutations

The bulk of the proof of Theorem 1 is borne by the following lemma. Its proof shows how Lemma 1 applies to permutations.

**Lemma 2.** For every positive integer $N$ there is an even $n$ and $\sigma, \rho \in S_n$ such that
max\{f^*(\sigma), f^*(\rho)\} \geq N \text{ for every } f \in F_n \text{ for which } f(i) + f(i+1) = 0 \text{ for all odd } i < n.

**Proof.** Given \(N \geq 2\) let \(\mathcal{R}\) be a finite set of rectangles as described for Lemma 1 for which \(K(\mathcal{R}) \geq N\). Let \(n = 2|\mathcal{R}|\) and assign the first \(|\mathcal{R}|\) odd positive integers to the lower left corners of the \(R \in \mathcal{R}\), in no particular order. Assign \(i+1\) to the upper right corner of \(R\) when \(i\) is assigned to its lower left corner. Let \(\sigma\) be the left-to-right arrangement of the \(n\) labeled corners projected perpendicularly onto the horizontal axis. Let \(\rho\) be the bottom-to-top arrangement of the \(n\) labeled corners projected perpendicularly onto the vertical axis.

Define the bijection \(t: G(\mathcal{R}) \rightarrow \{f \in F_n: f(i) + f(i+1) = 0 \text{ for each odd } i < n\}\) between the function sets as follows: when the lower left and upper right corners of \(R\) are labeled \(i\) and \(i+1\) respectively,

\[g(R) = 1 \iff (f(i), f(i+1)) = (1, -1)\]
\[g(R) = -1 \iff (f(i), f(i+1)) = (-1, 1)\]

for each \(R \in \mathcal{R}\). By the definition of \(K(\mathcal{R})\), for every \(g \in G(\mathcal{R})\) there is an \(x \in \mathbb{R}\) such that

\[\max\left(\left|\sum_{R \in \mathcal{R}_i} g(R)\right|, \left|\sum_{R \in \mathcal{R}^*} g(R)\right|\right) \geq N.\]

Then \(f = t(g)\) has \(\max\{f^*(\sigma), f^*(\rho)\} \geq N\).

To see this, suppose for definiteness that \(g \in G(\mathcal{R})\) has \(\left|\sum_{R \in \mathcal{R}} g(R)\right| \geq N\). Nothing is lost by assuming that the vertical through \((x, 0)\) includes no side of any \(R \in \mathcal{R}\). Let \(\sigma(1), \sigma(2), \ldots, \sigma(k)\) be the part of \(\sigma\) on the horizontal axis to the left of \((x, 0)\). Then for each odd \(i\), \(\{i, i+1\}\) makes a net nonzero contribution to \(\left|\sum_{j<i} f(\sigma(j))\right|\), in the amount \(f(i) = g(R)\) when \(R\) is the rectangle for \(\{i, i+1\}\), if and only if \(R \in \mathcal{R}_i\). Hence \(f^*(\sigma) \geq \left|\sum_{j<i} f(\sigma(j))\right| = \left|\sum_{R \in \mathcal{R}_i} g(R)\right| \geq N.\]

To complete the proof of Theorem 1, let \(N\) be a large positive integer with \(n, \sigma\) and \(\rho\) as in the conclusion of Lemma 2, and let \(\iota = (1, 2, \ldots, n)\). Also let \((t^*, \sigma^*, \rho^*)\) and \((t^{**}, \sigma^{**}, \rho^{**})\) be translated copies of \((\iota, \sigma, \rho)\) to \(\{n+1, \ldots, 2n\}\) and \(\{2n+1, \ldots, 3n\}\) respectively, and define \(\sigma_i\) in \(S_{3n}\) by juxtapositions:

\[\sigma_1 = t^* \sigma^*\]
\[\sigma_2 = \sigma^* \rho^{**}\]
\[\sigma_3 = \rho \sigma^{*} t^{**}.\]

Then, regardless of which \(\sigma_i\) has \(f^*(\sigma_i) = 1\), we conclude that \(f^*(\sigma_i) \geq N\) for one of the others. For example, if \(f \in F_n\) has \(f^*(\sigma_3) = 1\), then a partial sum within the \(\sigma^{**}\) part of \(\sigma_1\) or the \(\rho^{**}\) part of \(\sigma_2\) has absolute value at least \(N\), and it follows that \(\max\{f^*(\sigma_1), f^*(\sigma_2)\} \geq N\).

Since \(N\) can be as large as we wish, \(K^3_\mathcal{R} = \infty.\)
4. Discussion

Since the three-permutations problem has resisted a variety of attacks, we hesitate to suggest specific approaches other than the obvious one of trying to construct an example to show that $K_3 \geq 3$.

A potential avenue of progress that has not been widely explored is to focus on $K_m$ for fixed $m \geq 4$ to see if one can get $K_m = \infty$. Here is a specific question related to $K_4$: Does Lemma 1 hold when $\mathcal{R}$ is further restricted so that every rectangle in $\mathcal{R}$ contains the origin? If the answer is "yes" then $K_4 = \infty$, and if it is "no" then $K_4$ must be finite, as must $K_3$.

The relationship between the four-permutations problem and origin-containing rectangles comes about by positioning the $\sigma_1(j)$ in order on the negative abscissa, the $\sigma_2(j)$ in order on the positive abscissa, the $\sigma_3(j)$ in order on the negative ordinate, and the $\sigma_4(j)$ in order on the positive ordinate, and then constructing the rectangle for $i$ by intersections of perpendicular extensions for its four corners.

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References