Journal of Discrete Algorithms 8 (2010) 36–49

Contents lists available at ScienceDirect

Journal of Discrete Algorithms

www.elsevier.com/locate/jda

Complexity and approximation results for the connected vertex cover problem in graphs and hypergraphs

Bruno Escoffier [a,b], Laurent Gourvès [b,a,*], Jérôme Monnot [b,a]

[a] Université de Paris-Dauphine, LAMSADE, F-75775 Paris, France
[b] CNRS, FRE 3234, F-75775 Paris, France

ARTICLE INFO

Article history:
Received 27 May 2008
Accepted 22 January 2009
Available online 29 May 2009

Keywords:
Connected vertex cover
Chordal graphs
Bipartite graphs
Planar graphs
Hypergraphs
Approximation algorithm

ABSTRACT

We study a variation of the vertex cover problem where it is required that the graph induced by the vertex cover is connected. We prove that this problem is polynomial in chordal graphs, has a PTAS in planar graphs, is APX-hard in bipartite graphs and is 5/3-approximable in any class of graphs where the vertex cover problem is polynomial (in particular in bipartite graphs). Finally, dealing with hypergraphs, we study the complexity and the approximability of two natural generalizations.

1. Introduction

In this paper, we study a variation of the vertex cover problem where the subgraph induced by any feasible solution must be connected. Formally, a vertex cover of a simple graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ which covers all edges, i.e., which satisfies: $\forall e = \{x, y\} \in E$, $x \in S$ or $y \in S$. The vertex cover problem (MinVC in short) consists of finding a vertex cover of minimum size. MinVC is known to be APX-complete in cubic graphs [1] and NP-hard in planar graphs [18]. MinVC is 2-approximable in general graphs [3,23] and admits a polynomial approximation scheme in planar graphs [5]. On the other hand, MinVC is polynomial for several classes of graphs such as bipartite graphs, chordal graphs, graphs with bounded treewidth, etc. [19,7].

The connected vertex cover problem, denoted by MinCVC, is the variation of the vertex cover problem where, given a connected graph $G = (V, E)$, we seek a vertex cover S^* of minimum size such that the subgraph induced by S^* is connected. This problem has been introduced by Garey and Johnson [17], where it is proved to be NP-hard in planar graphs of maximum degree 4. As indicated in [28], this problem has some applications in the domain of wireless network design. In such a model, the vertices of the network are connected by transmission links. We want to place a minimum number of relay stations on vertices such that any pair of relay stations are connected (by a path which uses only relay stations) and every transmission link is incident to a relay station. This is exactly the connected vertex cover problem. Also, notice that MinCVC can be used to solve the Top Right Access point minimum length corridor problem [30], which has many applications in laying optical fibre cables for data communication and electrical wiring in floor plans [20].

* Corresponding author.
E-mail addresses: escoffier@lamsade.dauphine.fr (B. Escoffier), laurent.gourves@lamsade.dauphine.fr (L. Gourvès), monnot@lamsade.dauphine.fr (J. Monnot).

1570-8667/$ – see front matter
doi:10.1016/j.jda.2009.01.005

1.1. Previous related works

The main complexity and approximability results known on this problem are the following: in [34], it is shown that MinCVC is polynomially solvable when the maximum degree of the input graph is at most 3. However, it is NP-hard in planar bipartite graphs of maximum degree 4 [14], in planar biconnected graphs of maximum degree 4 [30], as well as in 3-connected graphs [35]. Concerning the positive and negative results of the approximability of this problem, MinCVC is 2-approximable in general graphs [31,2] but it is NP-hard to approximate within ratio $10\sqrt{5}-21$ [14]. In [11], the authors propose a general technique to derive PTASs for bidimensional problems. In particular, they give a quasi-PTAS (a $(1+\varepsilon)$-approximation algorithm with running time almost polynomial $n^{O(\log\log n)}$ when ε is fixed) for the minimum connected dominating set in a class of graphs that includes planar graphs and claim that an analogous result holds for MinCVC. The minimum connected dominating set problem asks to find a dominating set of $G=(V,E)$, that is a subset $S\subseteq V$ such that $\forall x\in V\setminus S$, $\exists y\in S$ with $\{x,y\}\in E$, of minimum size and where the subgraph of G induced by S is connected. Finally, recently the fixed-parameter tractability of MinCVC with respect to the vertex cover size or to the treewidth of the input graph has been studied in [14,21,26–28]. More precisely, in [14] a parameterized algorithm for MinCVC with complexity $O^*(2.9316^k)$ is presented improving the previous algorithm with complexity $O^*(6^k)$ given in [21] where k is the size of an optimal connected vertex cover. Independently, the authors of [26,27] have also obtained FPT algorithms for MinCVC and they obtain in [27] an algorithm with complexity $O^*(2.7606^k)$. In [28], the author gives a parameterized algorithm for MinCVC with complexity $O^*(2^t\cdot t^{3t+2}n)$ where t is the treewidth of the graph and n the number of vertices.

MinCVC is related to the unweighted version of tree cover. The tree cover problem was introduced in [2] and consists, given a connected graph $G=(V,E)$ with non-negative weights w on the edges, in finding a tree $T=(S,E')$ of G with $S\subseteq V$ and $E'\subseteq E$ satisfying $\forall\{x,y\}\in E\setminus E'$, $\{x,y\}\cap S\neq\emptyset$ and such that $w(T)=\sum_{e\in E'}w(e)$ is minimum. In [2], the authors prove that the tree cover problem is approximable within factor 3.55 and the unweighted version is 2-approximable. Recently, (weighted) tree cover has been shown to be approximable within a factor of 3 in [25], and a 2-approximation algorithm is proposed in [16]. Clearly, the unweighted version of tree cover is (asymptotically) equivalent to the connected vertex cover. Actually, from any connected vertex cover S of G, let $T'=(S,E')$ be any spanning tree of $G[S]$, the subgraph of G induced by S; $T'=(S,E')$ is a tree cover of G with weight $|S|-1$. Conversely, if $T'=(S,E')$ is a tree cover of G, then S is a connected vertex cover of G (of size $|E'|+1$).

1.2. Our contribution

In this article, we mainly deal with complexity and approximability issues for MinCVC in particular classes of graphs. More precisely,

- In Section 2, we first present some structural properties for connected vertex covers.
- Using these properties, we show that MinCVC is polynomial-time solvable in chordal graphs (Section 3).
- Then, in Section 4, we prove that MinCVC is APX-complete in bipartite graphs of maximum degree 4, even if each vertex of one block of the bipartition has a degree at most 3. On the other hand, if each vertex of one block of the bipartition has a degree at most 2 (and the vertices of the other part have an arbitrary degree), then MinCVC is polynomial.
- Section 5 deals with the approximability of MinCVC. We first show that MinCVC is 5/3-approximable in any class of graphs where MinVC is polynomial (in particular in bipartite graphs, or more generally in perfect graphs). Then, we present a polynomial approximation scheme for MinCVC in planar graphs.
- Section 6 concerns two natural generalizations of the connected vertex cover problem in hypergraphs. It is well known that the vertex cover problem in hypergraphs is equivalent to the set cover problem and hence, has many applications. Thus, the study of the connected vertex cover problem in hypergraphs seems quite natural and is equivalent to a connected version of the set cover problem. We mainly prove that the first generalization, called the weak connected vertex cover problem, is polynomial in hypergraphs of maximum degree 3, and is $H(\Delta-1)-1/2$-approximable where Δ is the maximum degree of the hypergraph and $H(k)=\sum_{i=1}^{k}1/i$. Then, we prove that the other generalization, called the strong connected vertex cover problem, is APX-hard, even in 2-regular hypergraphs.
- Finally, in Section 7 we give a conclusion and propose some open problems.

Notations. All graphs considered are undirected, simple and without loops. Unless otherwise stated, n and m will respectively denote the number of vertices and edges of the graph $G=(V,E)$ under consideration. $N_G(v)$ denotes the *neighborhood* of v in G, i.e., $N_G(v)=\{u\in V\colon \{u,v\}\in E\}$ and $d_G(v)$ its *degree*, that is $d_G(v)=|N_G(v)|$. Finally, $G[S]$ denotes the subgraph of G induced by S.

2. Structural properties

We present in this subsection some properties on vertex covers or connected vertex covers. These properties will be useful in the rest of the article to devise polynomial algorithms that solve MinCVC either optimally or approximately.

2.1. Vertex cover and graph contraction

For a subset $A \subseteq V$ of a graph $G = (V, E)$, the *contraction* of G with respect to A is the simple graph $G_A = (V', E')$ where we replace A in V by a new vertex v_A (so, $V' = (V \setminus A) \cup \{v_A\}$) and $\{x, y\} \in E'$ iff either $x, y \notin A$ and $\{x, y\} \in E$ or $x = v_A$, $y \neq v_A$ and there exists $v \in A$ such that $\{v, y\} \in E$. The *connected contraction* of G following $V' \subseteq V$ is the graph $G^c_{V'}$ corresponding to the iterated contractions of G with respect to the vertices in the connected components of the subgraph of G induced by V' (note that contraction is associative and commutative). Formally, $G^c_{V'}$ is constructed in the following way: let $A_1, \dots, A_q$ be the vertices in the connected components of the subgraph induced by V'. Then, we inductively apply the contraction with respect to A_i for $i = 1, \dots, q$. Thus, $G^c_{V'} = G_{A_1}[\dots[G_{A_q}]] = G_{A_1} \circ \dots \circ G_{A_q}$. Finally, let $New(G^c_{V'}) = \{v_{A_1}, \dots, v_{A_q}\}$ be the new vertices of $G^c_{V'}$ (those resulting from the contraction). The following lemma concerns contraction properties that will, in particular, be the basis of the approximation algorithm presented in Section 5.1.

Lemma 1. *Let $G = (V, E)$ be a connected graph and let $S \subseteq V$ be a vertex cover of G. Let $G_0 = (V_0, E_0) = G^c_S$ be the connected contraction of G following S where $A_1, \dots, A_q$ are the connected components of the subgraph induced by S. The following assertions hold*:

(i) *G_0 is connected and bipartite.*
(ii) *If $S = S^*$ is an optimal vertex cover of G, then $New(G_0)$ is an optimal vertex cover of G_0.*
(iii) *If $S = S^*$ is an optimal vertex cover of G and $v \in V \setminus S^*$ with $d_{G^c_{S^*}}(v) \geqslant 2$, then $New(G_1)$ is an optimal vertex cover of $G_1 = G^c_{S^* \cup \{v\}}$.*

Proof. For (i), G_0 is connected since the contraction preserves the connectivity. Let $New(G_0)$ be the new vertices resulting from the connected contraction of G following S. By construction of the connected contraction, $New(G_0)$ is an independent set of G_0. Now, the remaining vertices of G_0 also forms an independent set since S is a vertex cover of G.

For (ii), since the contraction is associative, we only prove the result when $|A_1| = r \geqslant 2$ and $|A_2| = \dots = |A_q| = 1$. By construction, $New(G_0)$ is obviously a vertex cover of G_0; thus $opt(G_0) \leqslant opt(G) - r + 1$. Conversely, Let S^*_0 be an optimal vertex cover of G_0. If $v_{A_1} \notin S^*_0$, then the neighborhood $N_{G_0}(v_{A_1})$ of v_{A_1} in G_0 satisfies $N_{G_0}(v_{A_1}) \subseteq S^*_0$. So, $N_G(A_1) \setminus A_1 \subseteq S^*_0$. Now, let $v \in A_1$. Then $S' = S^*_0 \cup (A_1 \setminus \{v\})$ is a vertex cover of G and hence $opt(G) \leqslant opt(G_0) + r - 1$. Otherwise, $v_{A_1} \in S^*_0$, and $S' = (S^*_0 \setminus \{v_{A_1}\}) \cup A_1$ is a vertex cover of G. Thus, $opt(G) \leqslant opt(G_0) + r - 1$. We conclude that $opt(G) = opt(G_0) + r - 1$ and the result follows.

For (iii), using (ii) and the associativity of the contraction, we only prove the result when S^* is also an independent set of G (in other words, we first apply the connected contraction following S^*); then, the connected components of the subgraph induced by $S^* \cup \{v\}$ satisfy $|A_1| = r \geqslant 3$ and $|A_2| = \dots = |A_q| = 1$. Using the same argument as previously, on the one hand, we get $opt(G_1) \leqslant opt(G) - (r - 1) + 1$ where $G_1 = G^c_{S^* \cup \{v\}}$ since $New(G_1)$ is a vertex cover of G_1; on the other hand, if $v_{A_1} \notin S^*_1$ (where S^*_1 is an optimal vertex cover of G_1) then $S^*_1 \cup \{v\}$ is a vertex cover of G, hence $opt(G) \leqslant opt(G_1) + 1 \leqslant opt(G_1) + (r - 2)$. If $v_{A_1} \in S^*_1$, $(S^*_1 \setminus \{v_{A_1}\}) \cup (A_1 \setminus \{v\})$ is a vertex cover of G and then $opt(G) \leqslant opt(G_1) + r - 2$. The proof is now complete. □

2.2. Connected vertex covers and biconnectivity

Now, we deal with connected vertex covers. It is easy to see that if the removal of a vertex v disconnects the input graph (v is called a *cut-vertex*, or an *articulation point*), then v has to be in any connected vertex cover. In this section we show that solving MinCVC in a graph is equivalent to solving it on the biconnected components of the graph, under the constraint of including all cut vertices.

Formally, a connected graph $G = (V, E)$ with $|V| \geqslant 3$ is *biconnected* if for any two vertices x, y there exists a simple cycle in G containing both x and y. A *biconnected component* (also called *block*) $G_i = (V_i, E_i)$ is a maximal connected subgraph of G that is biconnected if $|V_i| \geqslant 3$ or $V_i = \{x, y\}$ and $E_i = \{\{x, y\}\}$. Note that the biconnected components of G partition the edges of G. For a connected graph $G = (V, E)$, V_c denotes the set of cut-vertices of G and $V_{i,c}$ its restriction to V_i, that is $V_{i,c} = V_c \cap V_i$.

Lemma 2. *Let $G = (V, E)$ be a connected graph. $S \subseteq V$ is a connected vertex cover of G iff for each biconnected component $G_i = (V_i, E_i)$, $i = 1, \dots, p$, $S_i = S \cap V_i$ is a connected vertex cover of G_i containing $V_{i,c}$.*

Proof. Let $S \subseteq V$ be a connected vertex cover of a connected graph G. Obviously, $V_c \subseteq S$ since on the one hand, each biconnected component contains at least one edge, and on the other hand, the only vertices linking two distinct biconnected components are the cut-vertices. Moreover, trivially the restriction of S to V_i (i.e., S_i) is a vertex cover of G_i containing $V_{i,c}$. Finally, if S_i is not connected in G_i, then there are two connected components $S_{i,1}$ and $S_{i,2}$ in the subgraph of G_i induced by S_i. By construction, there is a path which connects a vertex of $S_{i,1}$ to a vertex of $S_{i,2}$ and which only contains vertices of S (since S is connected in G). Among all such paths that link a vertex from $S_{i,1}$ to a vertex of $S_{i,2}$, choose a shortest path μ. Thus, all vertices of μ (except its endpoints) are outside G_i. In this case, the subgraph $G_i + \mu$ would be biconnected, a contradiction since G_i is assumed to be maximal.

Conversely, let S_i be a connected vertex cover of $G_i = (V_i, E_i)$ containing $V_{c,i}$ for $i = 1, \dots, p$. Let us prove that $S = \bigcup_{i=1}^{p} S_i$ is a connected vertex cover of G. Obviously, S is a vertex cover of G since $E_1, \dots, E_p$ is a partition of E. Moreover, since $S = \bigcup_{i=1}^{p} S_i$ contains V_c, the solution is connected. □

Lemma 2 allows us to characterize the optimal connected vertex covers of G.

Corollary 3. *Let $G = (V, E)$ be a connected graph. $S^* \subseteq V$ is an optimal connected vertex cover of G iff for each biconnected component $G_i = (V_i, E_i)$, $i = 1, \dots, p$, $S_i^* = S^* \cap V_i$ is an optimal connected vertex cover of G_i among the connected vertex covers of G_i containing $V_{i,c}$.*

Proof. Let $S^* \subseteq V$ be an optimal connected vertex cover of G. If for some $i_0 \in \{1, \dots, p\}$, $S^* \cap V_{i_0}$ is not an optimal connected vertex cover of G_{i_0} among the connected vertex covers of G_{i_0} containing $V_{i_0,c}$, then we deduce that there exists a vertex cover $S_{i_0}^*$ of G_{i_0} with $V_{i_0,c} \subseteq S_{i_0}^*$ and $|S_{i_0}^*| < |S^* \cap V_{i_0}|$ (since from Lemma 2, we know that $V_{i_0,c}$ is included in $S^* \cap V_{i_0}$). In this case, using one more time Lemma 2, we obtain that $S = (\bigcup_{j \neq i_0} S^* \cap V_j) \cup S_{i_0}^*$ is also a connected vertex cover of G with $|S| < |S^*|$, contradiction.

Conversely, let S_i^* be an optimal connected vertex cover of $G_i = (V_i, E_i)$ among the connected vertex covers of G_i containing $V_{i,c}$ for any $i = 1, \dots, p$. if $S = \bigcup_{i=1}^{p} S_i^*$ is not an optimal connected vertex cover of G, then there exists another connected vertex cover S^* of G with $|S^*| < |S|$. Thus, we deduce that there exists at least one index $i_0 \in \{1, \dots, p\}$, such that $|S^* \cap V_{i_0}| < |S_i^*|$. However, using Lemma 2, we know that $S^* \cap V_{i_0}$ is a connected vertex cover of G_{i_0} containing $V_{i_0,c}$, contradiction. □

For instance, we deduce from Corollary 3 that MinCVC is polynomial in *trees* and *split graphs*. A split graph is a graph where the vertices can be partitioned into an independent set and a clique. More generally, we will see in Section 3 that this result holds in chordal graphs. Let MinPrExtCVC (by analogy with the well known PreExtension Coloring problem) be the variation of MinCVC where given $G = (V, E)$ and $V_0 \subseteq V$, we seek a connected vertex cover S of G containing V_0 and of minimal size. We obtain the following result:

Lemma 4. *Let $\mathcal{G}$ be a class of connected graphs defined by a hereditary property (i.e. a property which holds for induced subgraphs). Solving* MinCVC *in $\mathcal{G}$ polynomially reduces to solving* MinPrExtCVC *in the biconnected graphs of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is closed by pendant addition (i.e., is closed under addition of a new vertex v and a new edge $\{u, v\}$ where $u \in V$), then they are polynomially equivalent.*

Proof. Let $G = (V, E) \in \mathcal{G}$ be a biconnected graph and $V_0 \subseteq V$, be an instance of MinPrExtCVC. By adding a new pendent edge for each vertex $v \in V_0$ (i.e., a new vertex $v' \notin V$ and an edge $\{v, v'\}$), we obtain a new graph G' such that any connected vertex cover S' of G' contains V_0. Since $\mathcal{G}$ is assumed to be closed by pendent addition, then $G' \in \mathcal{G}$ and MinCVC is NP-hard in $\mathcal{G}$ if MinPrExtCVC is NP-hard in the subclass of biconnected graphs of $\mathcal{G}$.

Conversely, given a graph $G \in \mathcal{G}$, we can compute the biconnected components G_i and the cut-vertices V_c of G in $O(n + m)$ time, see [32] for instance. Since the graph property is hereditary, we deduce $G_i \in \mathcal{G}$. Using Corollary 3, we deduce that if we had a polynomial algorithm which solves MinPrExtCVC in the subclass of biconnected graphs of $\mathcal{G}$, then we could solve MinCVC in $\mathcal{G}$ in polynomial time. □

3. Chordal graphs

The class of *chordal graphs* is a very well known class of graphs which arises in many practical situations. A graph G is chordal if any cycle of G of size at least 4 has a chord (i.e., an edge linking two non-consecutive vertices of the cycle). There are many characterizations of chordal graphs, see for instance [7]. It is well known that the vertex cover problem is polynomial in this class [19].

In this section, we devise a polynomial time algorithm to compute an optimal connected vertex cover in chordal graphs. To achieve this, we need the following lemma.

Lemma 5. *Let $G = (V, E)$ be a connected chordal graph and let S be a vertex cover of G. The following properties hold:*

(i) *The connected contraction $G_0 = (V_0, E_0) = G_S^c$ of G following S is a tree.*
(ii) *If G is biconnected, then S is a connected vertex cover of G.*

Proof. Let S be a vertex cover of G.

For (i): from Lemma 1, we know that $G_0 = (V_0, E_0) = G_S^c$ is bipartite and connected. Assume that G_0 is not a tree, and let Γ be a cycle of G_0 with a minimal size. By construction, Γ is chordless, has a size at least 4 and alternates between

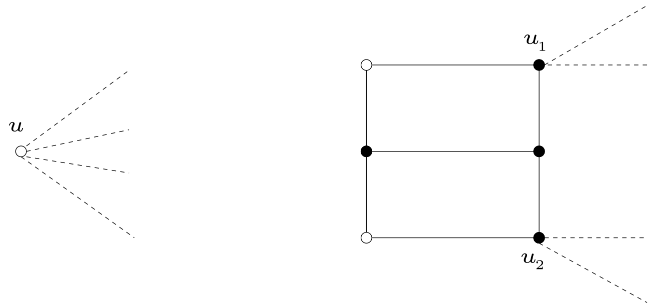

Fig. 1. Local replacement of a vertex $u \in V_0$ by gadget $H(u)$.

vertices of $New(G_0)$ and vertices of $V \setminus S$. From Γ, we can build a cycle Γ' of G using the following rule: if $\{x, v_{A_i}\} \in \Gamma$ and $\{v_{A_i}, y\} \in \Gamma$ where $x, y \notin S$ and $v_{A_i} \in New(G_0)$ (where we recall that A_i is some connected component of $G[S]$), then we replace these two edges by a shortest path $\mu_{x,y}$ from x to y in G among the paths from x to y in G which only use vertices of A_i (such a path exists since A_i is connected and is linked to x and y); by repeating this operation, we obtain a cycle Γ' of G with $|\Gamma'| \geqslant |\Gamma| \geqslant 4$. Let us prove that Γ' is chordless which will lead to a contradiction since G is assumed to be chordal. Let v_1, v_2 be two non-consecutive vertices of Γ'. If $v_1 \notin S$ and $v_2 \notin S$, then $\{v_1, v_2\} \notin E$ since otherwise Γ would have a chord in G_0. So, we can assume that $v_1 \in (\mu_{x,y} \setminus \{x, y\})$ and $v_2 \in \mu_{x,y}$ (since there is no edge linking two vertices of disjoint paths $\mu_{x,y}$ and $\mu_{x',y'}$) where x and y are two vertices of Γ with $x \in S$ and $y \in S$; in this case, using edge $\{v_1, v_2\}$, we could obtain a path which uses strictly less edges than $\mu_{x,y}$.

For (ii): Suppose that S is not connected. Then, from (i) we deduce that there are two edges $\{v_{A_i}, x\}$ and $\{x, v_{A_j}\}$ in G_0 where A_i and A_j are two distinct connected components of S. We deduce from (i) that x would be a cut-vertex of G, a contradiction since G is assumed to be biconnected. □

In particular, using (ii) of Lemma 5, we deduce that any optimal vertex cover S^* of a biconnected chordal graph G is also an optimal connected vertex cover.

Now, we give a simple linear algorithm for computing an optimal connected vertex cover of a chordal graph.

Theorem 6. MinCVC *is polynomial in chordal graphs. Moreover, an optimal solution can be found in linear time.*

Proof. Following Lemma 4, solving MinCVC in a chordal graph $G = (V, E)$ can be done by solving MinPrextCVC in each of the biconnected components $G_i = (V_i, E_i)$ of G. Since G_i is both biconnected and chordal, by Lemma 5, MinPrextCVC is the same problem as MinPrextVC (in G_i). But, by adding a pendent edge to vertices required to be taken in the vertex cover, we can easily reduce MinPrextVC to MinVC (note that the graph remains chordal). Since computing the biconnected components and solving MinVC in a chordal graph can be done in linear time (see [7]), the result follows. □

4. Bipartite graphs

A bipartite graph $G = (V, E)$ is a graph where the vertex set is partitioned into two independent sets L and R. For chordal graphs, we saw that biconnected components are important to solve MinCVC within polynomial-time. One may ask whether this result remains true for bipartite graphs. Unfortunately, MinCVC is NP-hard in biconnected bipartite graphs.

In this section, we first show, using Lemma 4, a preliminary result (Lemma 7) which extends the results of [14] and [30] (i.e. NP-hardness in planar bipartite graphs of maximum degree 4, and, respectively, in planar biconnected graphs as of maximum degree 4). Next we show (Theorem 8) that MinCVC is not only NP-hard but also APX-hard in bipartite graphs (even if the maximum degree is 4). Finally, we close this section by exhibiting a polynomial subcase of bipartite graphs (Lemma 9) that will be very useful in Section 5 to devise an approximation algorithm.

Lemma 7. MinCVC *is* NP-*hard in biconnected planar bipartite graphs of maximum degree* 4.

Proof. Using the NP-hardness of MinCVC in planar bipartite graphs of maximum degree 4, given in [14], we only prove that MinPrextCVC in the subclass of biconnected planar bipartite graphs of maximum degree 4 can be polynomially reduced to MinCVC in the subclass of biconnected planar bipartite graphs of maximum degree 4. Note that the simple reduction given in Lemma 4 does not preserve the biconnectivity.

Let $G = (V, E)$ be a planar biconnected bipartite graph of maximum degree 4 and let V_0 an instance of MinPrextCVC. We replace each vertex $u \in V_0$ by the gadget $H(u)$ depicted in Fig. 1. Actually, if the neighborhood of u is $N = \{v_1, \ldots, v_p\}$ with $2 \leqslant p \leqslant 4$ (since G is biconnected of maximum degree 4), then we link u_1 to some vertices of $v_1, \ldots, v_p$ and u_2 to the remaining vertices in such a way that on the one hand u_1 and u_2 have at least one neighbor in N and at most 2 neighbors

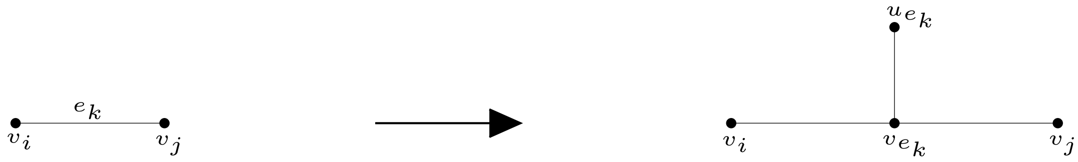

Fig. 2. Local replacement of edge $e_k = \{v_i, v_j\}$ using gadget $H(e_k)$.

Fig. 3. The graph H''.

in N, and on the other hand, the new graph remains planar. Let G' be the new graph. It is easy to see that G' is planar, bipartite, biconnected and of maximum degree 4.

Let S^* containing V_0 be an optimal connected vertex cover of G. Then, by deleting V_0 and by adding the vertices drawn in black for each gadget $H(u)$ (see Fig. 1), we obtain a connected vertex cover of G'. Thus,

$$opt(G') \leqslant opt(G) + 3|V_0|. \tag{1}$$

Conversely, let S' be a connected vertex cover of G'. It is easy to see that S' takes at least 4 vertices for each gadget $H(u)$. Thus, without loss of generality, we can assume that S' only takes the black vertices for each gadget $H(u)$. By deleting these black vertices and by adding V_0, we obtain a solution S of G satisfying

$$|S| = |S'| - 3|V_0|. \tag{2}$$

Using inequalities (1) and (2), the expected result follows. □

Now, one can show that MinCVC has no PTAS in bipartite graphs of maximum degree 4.

Theorem 8. MinCVC *is not* 1.001031*-approximable in connected bipartite graphs* $G = (L, R; E)$ *where* $\forall l \in L$, $d_G(l) \leqslant 4$ *and* $\forall r \in R$, $d_G(r) \leqslant 3$, *unless* P = NP.

Proof. We give a reduction from the vertex cover problem in cubic graphs. In [10] it is proved that, given a connected cubic graph $G = (V, E)$ of n vertices, it is NP-hard to decide whether $opt(G) \leqslant 0.5103305n$ or $opt(G) \geqslant 0.5154986n$ where $opt(G)$ is the value of an optimal vertex cover of G.

Given a cubic connected graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ instance of MinVC, we build an instance $H = (V', E')$ of MinCVC in the following way.

- We start from G and each edge $e_k = \{v_i, v_j\}$ is replaced by the gadget $H(e_k)$ described in Fig. 2. Let H' be this graph.
- We add the graph H'' depicted in Fig. 3.
- Finally, we connect the graph H' to the graph H''. For each $i = 1, \ldots, n$, we link v_i to v'_i by using a gadget isomorphic to $H(e_k)$ (we denote by w_i the vertex of degree 3 in the gadget, i.e., the vertex v_{e_k} in Fig. 2).

Clearly H is of maximum degree 4 and bipartite. Finally, we can easily observe that any vertex of this graph has degree at most 4 for one part of the bipartition and at most 3 for the other part.

Let S^* be an optimal vertex cover of G with value $opt(G)$. Clearly, $S^* \cup \{v_{e_k}\colon k = 1, \ldots, m\} \cup \{v'_i, v''_i, w_i\colon i = 1, \ldots, n\}$ is a connected vertex cover of H. Conversely, let S^* be an optimal connected vertex cover of H with value $opt(H)$. Without loss of generality, we can assume that S^* contains $\{v_{e_k}\colon k = 1, \ldots, m\} \cup \{v'_i, v''_i, w_i\colon i = 1, \ldots, n\}$ since these vertices are cut vertices of H. Thus, $S = S^* \setminus (\{v_{e_k}\colon k = 1, \ldots, m\} \cup \{v'_i, v''_i, w_i\colon i = 1, \ldots, n\})$ is a vertex cover of G. Indeed, if an edge $e_k = \{v_i, v_j\}$ is not covered by S, then the vertex v_{e_k} will not be connected to the other vertices of S^*, which is impossible. Thus, we deduce:

$$opt(H) = opt(G) + m + 3n. \tag{3}$$

Using the NP-hard gap of [10], the fact that G is cubic and equality (3), we deduce that it is NP-hard to decide whether $opt(H) \leqslant 5.0103305n$ or $opt(H) \geqslant 5.0154986n$. □

In Theorem 8, we proved in particular that MinCVC is NP-hard when all the vertices of one part of the bipartition have a degree at most 3. It turns out that if all the vertices of one part of this bipartition have a degree at most 2, the problem becomes easy. This property will be very useful to devise our approximation algorithm in Section 5.1.

Lemma 9. MinCVC *is polynomial in bipartite graphs* $G=(L,R;E)$ *such that* $\forall r\in R$, $d_G(r)\leqslant 2$. *Moreover, if* $L_2=\{l\in L\colon\ d_G(l)\geqslant 2\}$, *then* $opt(G)=|L|+|L_2|-1$.

Proof. Let $G=(L,R;E)$ be a bipartite graph such that $\forall r\in R$, $d_G(r)\leqslant 2$ and assume that $|L|\geqslant 3$ and G is connected. Let $L_1=L\setminus L_2$ and let $R_1=N_G(L_1)$ be the neighbors of L_1. Let $G'=(L_2,R\setminus R_1;E')$ be the bipartite subgraph of G induced by $L_2\cup(R\setminus R_1)$ and let $G_{L_2}=(L_2,E_{L_2})$ where $e_r=\{l,l'\}\in E_{L_2}$ iff $\exists r\in R\setminus R_1$ with $\{l,r\}\in E'$ and $\{r,l'\}\in E'$. Finally, let T be a spanning tree of G_{L_2}.

We claim that $S_T=L_2\cup R_1\cup\{r\in R\setminus R_1\colon\ e_r\in T\}$ is an optimal connected vertex cover of G.

Let S^* be an optimal connected vertex cover of G and let $L'_2=N_G(R_1)\cap L_2$ be the neighbors of R_1 in G not in L_1. Clearly $R_1\subseteq S^*$, since $|L|\geqslant 3$ and each vertex of L_1 has degree 1. Moreover, since each vertex of R has a maximum degree 2, then $L'_2\subseteq S^*$. Now, let us prove that $L_2\subseteq S^*$. Assume the reverse and let $l_0\in L_2\setminus S^*$. Using the previous remark, we know that $l_0\in L_2\setminus L'_2$. Let $r_1,\dots,r_q$ be the neighbors of l_0 in G. By construction, $q\geqslant 2$ and $r_i\in S^*$ since S^* is a vertex cover. Moreover, $\forall i=1,\dots,q$, $d_G(r_i)=2$ since S^* must induce a connected subgraph and if l_i is the other neighbor of r_i, then $l_i\in S^*$. Let us prove that $(S^*\setminus\{r_1\})\cup\{l_0\}$ is a connected vertex cover of G. First, $S^*\setminus\{r_1\}$ is a connected vertex cover in the subgraph $(L,R;E\setminus\{l_0,r_1\})$ since $S^*\setminus\{r_1\}$ is connected (r_1 is a leaf of the subgraph induced by S^*) and r_1 only covers edges $\{l_0,r_1\},\{r_1,l_1\}$, but the edge $\{r_1,l_1\}$ is also covered by $l_1\in S^*$. Then, by adding l_0, we now cover the missing edge $\{l_0,r_1\}$ and since $q\geqslant 2$, l_0 is linked to r_2 in $(S^*\setminus\{r_1\})\cup\{l_0\}$. By repeating this operation, we obtain another optimal solution with $L_2\subseteq S^*$. Thus, in S^*, we need to connect together the vertices of L_2 by using some vertices of R. Since the vertices of R_1 cannot link together vertices of L_2 (we recall that the degree of each vertex of R is at most 2), the vertices of $(S^*\setminus L_2)\setminus R_1$ correspond to a set of edges $E^*_{L_2}$ in G_{L_2} such that the subgraph $(L_2,E^*_{L_2})$ of G_{L_2} is connected. Hence $|E^*_{L_2}|\geqslant|T|$ or equivalently $|(S^*\setminus L_2)\setminus R_1|\geqslant|(S_T\setminus L_2)\setminus R_1|$. In conclusion, S_T is an optimal connected vertex cover of G with value $opt(G)=|L_2|+|T|+|R_1|=2|L_2|-1+|R_1|$ since T is a spanning tree of G_{L_2}. Now, observe that $|R_1|=|L_1|$ since otherwise G would not be connected, and the proof is complete. □

5. Approximation results

MinCVC is trivially APX-complete in k-connected graphs for any $k\geqslant 2$ since starting from a graph $G=(V,E)$, instance of MinVC, we can add a clique K_k of size k and link each vertex of G to each vertex of K_k. This new graph G' is obviously k-connected and $S\neq V$ is a vertex cover of G iff S together with the k vertices of K_k (if $S=V$, then S together with $k-1$ vertices of K_k is a connected vertex cover) is a connected vertex cover of G'. Thus, using the negative result of [24] it is quite improbable that one can improve the approximation ratio of 2 for MinCVC, even in k-connected graphs. Thus, in this subsection we deal with the approximability of MinCVC in particular classes of graphs.

In Section 5.1, we devise a 5/3-approximation algorithm for any class of graphs where the classical vertex cover problem is polynomial. In Section 5.2, we show that MinCVC admits a PTAS in planar graphs.

5.1. *When* MinVC *is polynomial*

Let $\mathcal{G}$ be a class of connected graphs where MinVC is polynomial (for instance, the connected bipartite graphs). The underlying idea of the algorithm is simple: we first compute an optimal vertex cover, and then try to connect it by adding vertices (either using high degree vertices or Lemma 9). The analysis leading to the ratio 5/3 is based on Lemma 1 which deals with graph contraction.

Now, let us formally describe the algorithm. Recall that given a vertex set V', $G^c_{V'}$ denotes the connected contraction of V following V', and $New(G^c_{V'})$ denotes the set of new vertices (one for each connected component of $G[V']$).

---

algo$_{CVC}$ input: A graph $G=(V,E)$ of $\mathcal{G}$ with at least 3 vertices.

1 Find an optimal vertex cover S^* of G such that in $G^c_{S^*}$, $\forall v\in New(G^c_{S^*})$, $d_{G^c_{S^*}}(v)\geqslant 2$;
2 Set $G_1=G^c_{S^*}$, $N_1=New(G^c_{S^*})$, $S=S^*$ and $i=1$;
3 While $|N_i|\geqslant 2$ and there exists $v\notin N_i$ such that v is linked in G_i to at least 3 vertices of N_i do
  3.1 Set $S:=S\cup\{v\}$ and $i:=i+1$;
  3.2 Set $G_i:=G^c_S$ and $N_i=New(G^c_S)$;
4 If $|N_i|\geqslant 2$, apply the polynomial algorithm of Lemma 9 on G_i (let S' be the produced solution) and set $S:=S\cup(V\cap S')$;
5 Output S;

Now, we show that algo$_{CVC}$ outputs a connected vertex cover of G in polynomial time. First of all, given an optimal vertex cover S^* of a graph G (assumed here to be computable in polynomial time), we can always transform it in such a way that $\forall v\in New(G^c_{S^*})$, $d_{G^c_{S^*}}(v)\geqslant 2$. Indeed, if a vertex of $G^c_{S^*}$ corresponding to a connected component of S^* has only one neighbor in $G^c_{S^*}$, then we can take this neighbor in S^* and remove one vertex on this connected component (and the number of such 'leaf' connected components decreases, as long as $G^c_{S^*}$ has at least 3 vertices). Now, using (ii) of

Lemma 1, we know that $New(G^c_{S^*})$ is an optimal vertex cover of $G^c_{S^*}$. Then, from $New(G^c_{S^*})$, we can find such a solution within polynomial time.

Moreover, using (i) of Lemma 1 with S^*, we deduce that the graph G_i is bipartite, for each possible value of i. Assume that $G_i = (N_i; R_i, E_i)$ for iteration i where N_i is the left set corresponding to the contracted vertices and R_i is the right set corresponding to the remaining vertices and let p be the last iteration. Clearly, if $|N_p| = 1$, the output solution S is connected. Otherwise, the algorithm uses step 4; we know that G_p is bipartite and by construction $\forall r \in R_p$, $d_{G_p}(r) \leqslant 2$. Thus, we can apply Lemma 9 on G_p. Moreover, a simple proof also gives that $\forall l \in N_p$, $d_{G_p}(l) \geqslant 2$. Indeed, otherwise there exists $l \in N_p$ such that l has a unique neighbor $r_0 \in R_p$. Let $\{x_1, \ldots, x_j\} \subseteq N_{p-1}$ with $j \geqslant 3$ and r_1 be the vertices contracted in G_{p-1} in order to obtain G_p. We conclude that the neighborhood of $\{x_1, \ldots, x_j\}$ is $\{r_0, r_1\}$ in G_{p-1} which is impossible since on the one hand, N_{p-1} is an optimal vertex cover of G_{p-1} (using (iii) of Lemma 1), and on the other hand, by flipping $\{x_1, \ldots, x_j\}$ with $\{r_0, r_1\}$, we obtain another vertex cover of G_{p-1} with smaller size than N_{p-1}! Finally, using Lemma 9, an optimal connected vertex cover of G_p consists of taking N_p and $|N_p| - 1$ of R_p. In conclusion, S is a connected vertex cover of G.

We now prove that this algorithm improves the ratio 2.

Theorem 10. *Let $\mathcal{G}$ be a class of connected graphs where* MinVC *is polynomial. Then,* $\mathtt{algo}_{CVC}$ *is a* 5/3*-approximation for* MinCVC *in $\mathcal{G}$.*

Proof. Let $G = (V, E) \in \mathcal{G}$. Let S be the approximate solution produced by $\mathtt{algo}_{CVC}$ on G. Using the previous notations and Lemma 9, the solution S has a value $apx(G)$ satisfying:

$$apx(G) = |S^*| + p - 1 + |N_p| - 1, \tag{4}$$

where $p - 1$ is the number of iterations of step 3. Obviously, we have:

$$opt(G) \geqslant |S^*|. \tag{5}$$

Now let us prove that for any $i = 1, \ldots, p - 1$, we also have $opt(G_i) \geqslant opt(G_{i+1}) + 1$. Let S^*_i be an optimal connected vertex cover of G_i. Let $r_i \in R_i$ be the vertex added to S during iteration i and let $N_{G_i}(r_i)$ be the neighbors of r_i in G_i. The graph G_{i+1} is obtained from the contraction of G_i with respect to the subset $S_i = \{r_i\} \cup N_{G_i}(r_i)$. Thus, if v_{S_i} denotes the new vertex resulting from the contraction of S_i, then $(S^*_i \setminus S_i) \cup \{v_{S_i}\}$ is a connected vertex cover of G_{i+1}. Moreover, $|S^*_i \cap S_i| \geqslant 2$ since either $r_i \in S^*_i$ and at least one of these neighbors must belong to S^*_i (S^*_i is connected and $i < p$) or $N_{G_i}(r_i) \subseteq S^*_i$ since S^*_i is a vertex cover. Thus $opt(G_{i+1}) \leqslant |S^*_i \setminus S_i| + 1 = opt(G_i) - |S^*_i \cap S_i| + 1 \leqslant opt(G_i) - 1$. Summing up these inequalities for $i = 1$ to $p - 1$, and using that $opt(G) \geqslant opt(G_1)$, we obtain:

$$opt(G) \geqslant opt(G_p) + p - 1. \tag{6}$$

Moreover, thanks to Lemma 9, we know that $opt(G_p) = 2|N_p| - 1$. Together with Eq. (6), we get:

$$opt(G) \geqslant 2|N_p| - 1 + p - 1. \tag{7}$$

Finally, since each vertex chosen in step 3 has degree at least 3, we get $|N_{i+1}| \leqslant |N_i| - 2$. This immediately leads to $|N_1| \geqslant |N_p| + 2(p - 1)$. Since $|S^*| \geqslant |N_1|$, we get:

$$|S^*| \geqslant |N_p| + 2(p - 1). \tag{8}$$

Combination of Eqs. (5), (7) and (8) with coefficients 4, 1 and 1 (respectively) gives:

$$5opt(G) \geqslant 3|S^*| + 3|N_p| - 1 + 3(p - 1). \tag{9}$$

Then, Eq. (4) allows to conclude. □

5.2. Planar graphs

Given a planar embedding of a planar graph $G = (V, E)$, the level of a vertex is defined inductively as follows: the set L_1 of vertices at level 1 is constituted by vertices on the exterior face of G. Then, the set L_i of vertices at level i is the set of vertices on the exterior face of the subgraph of G induced by $V \setminus (L_1 \cup \cdots \cup L_{i-1})$. A planar graph is said to be k-outerplanar if every vertex is at level at most k (for at least one planar embedding).

Baker gave in [4] a polynomial time approximation scheme for several problems including vertex cover in planar graphs. The underlying idea is to consider k-outerplanar subgraphs of G constituted by k consecutive layers. The polynomiality of vertex cover in k-outerplanar graphs (for a fixed k) allows to achieve a $k/(k-1)$ approximation ratio. More precisely, consider for a given $l \leqslant k - 1$ the following subsets of vertices: W_0 is the set of vertices of level at most l, and W_i for $1 \leqslant i \leqslant t$ is the set of vertices of levels between $l + (i-1)(k-1)$ and $l + i(k-1)$ (for t such that each vertex is in at least one W_i). The graphs induced by the W_i's are k-outerplanar. Since W_{i-1} and W_i overlap, the union S^l of optimal vertex

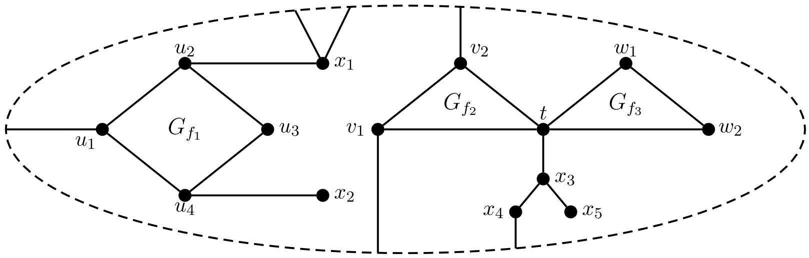

Fig. 4. Level of a planar graph.

covers in the subgraphs $G[W_i]$'s induced by the W_i's is a vertex cover. Given an optimal vertex cover S^* on the whole graph, since $S^* \cap W_i$ is a vertex cover on $G[W_i]$, we have $|S^l| \leqslant |S^*| + \sum_{i=0}^{t} |S^* \cap L_{l+i(k-1)}|$ (each subset $S^* \cup L_{l+i(k-1)}$ is counted twice since Layer $L_{l+i(k-1)}$ is both in W_i and in W_{i+1}).

Doing this for all values of $l \leqslant k-1$, we get $k-1$ solutions S^l such that:

$$\sum_{l=1}^{k-1} |S^l| \leqslant (k-1)|S^*| + \sum_{l=1}^{k-1} \sum_{i=0}^{t} |S^* \cap L_{l+i(k-1)}| = k|S^*|.$$

Taking the best of the $k-1$ solutions computed gives the claimed ratio.

We adapt this technique in order to achieve an approximation scheme for MinCVC (MinCVC is NP-hard in planar graphs, see [17]). First of all, note that k-outerplanar graphs have treewidth bounded above by $3k-1$ [6]. Since MinCVC is polynomially solvable for graphs with bounded treewidth [28], MinCVC is polynomial for k-outerplanar graphs.

Theorem 11. MinCVC *admits an approximation scheme in planar graphs.*

Proof. Given an embedding of a planar (connected) graph G, we define, as previously, the layers $L_1, \ldots, L_q$ of G. For each layer L_i, we define F_i as the set of vertices of L_i that are in an interior face of L_i. For instance, in Fig. 4, the dashed ellipse represents an interior face on level $i-1$. Then, depicted vertices are at level i. There are 3 interior faces (constituted respectively by the u_i's, by $\{v_1, v_2, t\}$ and $\{t, w_1, w_2\}$). All vertices but the x_i's are in F_i.

Following the principle of the approximation scheme for vertex cover, we define an algorithm for any integer $k > 0$. Let $V_i = F_i \cup L_{i+1} \cup L_{i+2} \cup \cdots \cup L_{i+k}$, and G_i be the graph induced by V_i. Note that G_i is not necessarily connected since for example there can be several disjoint faces in F_i (there are two connected components in Fig. 4).

Let S^* be an optimum connected vertex cover on G with value $opt(G)$, and $S_i^* = S^* \cap V_i$. Then of course S_i^* is a vertex cover of G_i. However, even restricted to a connected component of G_i, it is not necessarily connected. Indeed, S^* is connected but the path(s) connecting two vertices of S^* in a connected component of G_i may use vertices out of this connected component. To overcome this problem, notice that only vertices in F_i or in F_{i+k} connect V_i to $V \setminus V_i$. Hence, $S_i^* \cup F_i \cup F_{i+k}$ can be partitioned into a set of connected vertex covers on each of the connected components of G_i (since F_i and F_{i+k} are made of cycles). Now, take an optimum connected vertex cover on each of these connected components, and define S_i as the union of these optimum solutions. Then, we have:

$$|S_i^* \cup F_i \cup F_{i+k}| \geqslant |S_i|. \tag{10}$$

Now, let $p \in \{1, \ldots, k\}$. Let $V_0 = L_1 \cup L_2 \cup \cdots \cup L_p$, G_0 be the subgraph of G induced by V_0, $S_0^* = S^* \cap V_0$, and S_0 be an optimum vertex cover on G_0. With similar arguments as previously, we have:

$$|S_0^* \cup F_p| \geqslant |S_0|. \tag{11}$$

We build a solution S^p on the whole graph G as follows. S^p is the union of S_0 and of all S_i's for $i = p \bmod k$. Of course, S^p is a vertex cover of G, since any edge of G appears in at least one G_i (or G_0). Moreover, it is connected since:

- S_0 is connected, and each S_i is made of connected vertex covers on the connected components of G_i;
- each of these connected vertex covers in S_i is connected to S_{i-k} (or to S_0 if $i = p$): this is due to the fact that F_i belongs to V_i and to V_{i-k} (or V_0). Hence, a level i interior face f is common to S_{i-k} (or S_0) and to the connected vertex cover of S_i we are dealing with. Both partial solutions cover all the edges of this face f. Since f is a cycle, the two solutions are necessarily connected. In other words, each connected component of S_i is connected to S_{i-k} (or S_0) and, by recurrence, to S_0. Consequently, the whole solution S^p is connected.

Summing up Eq. (10) for each $i = p \bmod k$ and Eq. (11), we get:

$$|S_0^* \cup F_p| + \sum_{i=p \bmod k} |S_i^* \cup F_i \cup F_{i+k}| \geqslant |S_0| + \sum_{i=p \bmod k} |S_i|. \tag{12}$$

By definition of S^p, we have $|S^p| \leqslant |S_0| + \sum_{i=p \bmod k} |S_i|$. On the other hand, since only vertices in F_i ($i = p \bmod k$) appear in two different V_i's ($i = 0$ or $i = p \bmod k$), we get that $|S_0^* \cup F_p| + \sum_{i=p \bmod k} |S_i^* \cup F_i \cup F_{i+k}| \leqslant |S^*| + 2\sum_{i=p \bmod k} |F_i|$. This leads to:

$$opt(G) + 2 \sum_{i=p \bmod k} |F_i| \geqslant |S^p|. \tag{13}$$

If we consider the best solution S with value $apx(G)$ among the S^p's ($p \in \{1, \ldots, k\}$), we get:

$$opt(G) + \frac{2}{k} \sum_{i=1}^{q} |F_i| \geqslant apx(G). \tag{14}$$

To conclude, we observe that the following property holds:

Property 12. *S^* takes at least one fourth of the vertices of each F_i.*

To see this property of $S^* \cap F_i$, consider F_i and the set E_i of edges of G that belong to one and only one interior face of F_i (for example, in Fig. 4, if there were edges $\{u_2, u_4\}$ and $\{u_3, v_1\}$, they would not be in E_i). Let n_i be the number of vertices in F_i, and m_i the number of edges in E_i. This graph is a collection of edge-disjoint (but not vertex-disjoint, as one can see in Fig. 4) interior faces (cycles). Of course, $S^* \cap F_i$ is a vertex cover of this graph. Since this graph is a collection of interior faces (cycles), on each of these faces f $S^* \cap F_i$ cannot reject more than $|f|/2$ vertices. In all,

$$|S^* \cap F_i| \geqslant n_i - \sum_{f \in F_i} \frac{|f|}{2}.$$

But since faces are edge-disjoint, $\sum_{f \in F_i} |f| = m_i$. On the other hand, if N_f denotes the number of interior faces in F_i, since each face contains at least 3 vertices, $m_i = \sum_{f \in F_i} |f| \geqslant 3N_f$. Since the graph is planar, using Euler formula we get $1 + m_i = n_i + N_f \leqslant n_i + m_i/3$. Hence, $m_i \leqslant 3n_i/2$. Finally, $|S^* \cap F_i| \geqslant n_i - m_i/2 \geqslant n_i/4$. Based on this property, we get:

$$opt(G)\left(1 + \frac{8}{k}\right) \geqslant apx(G). \tag{15}$$

Taking k sufficiently large leads to a $1 + \varepsilon$ approximation. The polynomiality of this algorithm follows from the fact that each subgraph we deal with is (at most) $(k+1)$-outerplanar, hence for a fixed k we can find an optimum solution in polynomial time. □

6. Connected vertex cover in hypergraphs

Here, we extend the notions of vertex cover and connected vertex cover to hypergraphs. Whereas the generalization of the vertex cover problem to hypergraphs is quite natural, it turns out that the generalization of the connected vertex cover problem is a task much harder due to the notion of connected hypergraphs. Actually, we will give two generalizations: the *weak connected vertex cover* problem and the *strong connected vertex cover* problem.

Before establishing a definition of these two problems, we recall some definitions on hypergraphs. A simple hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $\mathcal{E} = \{e_1, \ldots, e_m\} \subseteq 2^V$ is the hyperedge set. Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, $d_{\mathcal{H}}(v)$, $N_{\mathcal{H}}(v)$ and $s_{\mathcal{H}}(e)$ denote respectively the degree, the neighborhood of a vertex $v \in V$ and the size of an hyperedge $e \in \mathcal{E}$, that is $d_{\mathcal{H}}(v) = |\{e \in \mathcal{E}\colon\ v \in e\}|$, $N_{\mathcal{H}}(v) = \{u \in V \setminus \{v\}\colon\ \exists e \in \mathcal{E}$ containing vertices $u, v\}$ and $s_{\mathcal{H}}(e) = |\{v\colon\ v \in e\}|$. $\Delta(\mathcal{H})$ and $s(\mathcal{H})$ denote respectively the *maximum degree* of a vertex and the *maximum size* of a hyperedge in $\mathcal{H}$. The following definitions are introduced in [7]: $\mathcal{H}' = (V', \mathcal{E}')$ is a *partial hypergraph* of $\mathcal{H} = (V, \mathcal{E})$ if $\mathcal{E}' \subseteq \mathcal{E}$ and V' is the union of the hyperedges in $\mathcal{E}'$. The *restriction* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ to $V' \subseteq V$ is the partial hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ (that is satisfying $\mathcal{E}' = \{e \in \mathcal{E}\colon\ e \cap V' = e\}$). The subhypergraph of $\mathcal{H} = (V, \mathcal{E})$ *induced by* V' is the hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ where $\mathcal{E}' = \{e \cap V'\colon\ e \in \mathcal{E}\}$. A hypergraph is *simple* if no hyperedge is a subset of any other hyperedge. A hypergraph is *r-uniform* if each hyperedge has a size r and *r-regular* if each vertex has a degree r. A *path* of length k from v_1 to v_k in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a sequence $(v_1, e_1, v_2, \ldots, e_k, v_{k+1})$ with $k \geqslant 1$ such that $e_1, \ldots, e_k$ and $v_1, \ldots, v_{k+1}$ are sets of distinct hyperedges and vertices respectively, and $\forall i = 2, \ldots, k$, $v_i \in e_{i-1} \cap e_i$ and $v_1 \in e_1$, $v_{k+1} \in e_k$. A hypergraph $\mathcal{H}$ is *connected* if between every pair (v_i, v_j) of disjoint vertices, there is path from v_i to v_j.

The dual hypergraph of an hypergraph $\mathcal{H} = (V, \mathcal{E})$ is the hypergraph $\mathcal{H}^* = (V_{\mathcal{E}}, E^*)$ such that the vertices of $V_{\mathcal{E}} = \{v_e\colon\ e \in \mathcal{E}\}$ correspond to the hyperedges of $\mathcal{E}$ and the hyperedge set is $E^* = \{\mathcal{E}_v, v \in V\}$ where $\mathcal{E}_v = \{v_e\colon\ v \in e\}$.

The generalization of MinVC and MinCVC in graphs to hypergraphs can be defined as follows. Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a vertex cover of $\mathcal{H}$ is a subset of vertices $S \subseteq V$ such that for any hyperedge $e \in \mathcal{E}$, we have $S \cap e \neq \emptyset$; the vertex cover problem in hypergraphs is the problem of determining a vertex cover S^* of $\mathcal{H}$ minimizing $|S^*|$. It is well known that this problem is equivalent to the *set cover problem* (in short MinSC) by considering the dual hypergraphs (see for instance [9]). Thus, the vertex cover problem in hypergraphs is not approximable within performance ratio $(1-\varepsilon)\ln m$ for all $\varepsilon > 0$ unless NP $\subset$ DTIME$(m^{\log\log m})$ [13]. Moreover, it is not $\ln(\Delta) - c\ln(\ln(\Delta))$-approximable (for some constant c), for any constant Δ, in hypergraphs of degree Δ [33]. Recently, new inapproximability results have been given. In [12], the authors prove that the vertex cover problem in k-uniform hypergraphs is not $(k-1-\varepsilon)$-approximable unless P = NP for any $k \geqslant 3$ and $\varepsilon > 0$. At the same time, based on the so-called *unique games conjecture*, it is shown that $(k-\varepsilon)$ is a lower bound of the approximation of vertex cover in k-uniform hypergraphs for any $k \geqslant 2$ and $\varepsilon > 0$ [24].

We consider two versions of the connected vertex cover problem in hypergraphs, namely a weak and strong one. Given a connected hypergraph $\mathcal{H} = (V, \mathcal{E})$, the weak (resp., strong) connected vertex cover problem, denoted by MinWCVC (resp., MinSCVC) consists of finding a minimum size vertex cover S^* of $\mathcal{H}$ such that the subhypergraph induced by S^* (resp., the restriction of $\mathcal{H}$ to S^*) is connected. Obviously, when we restrict these problems to graphs, we again find the connected vertex cover problem.

6.1. The weak connected vertex cover problem

The weak connected vertex cover problem is as hard as the vertex cover problem in hypergraphs since starting from any hypergraph $\mathcal{H} = (V, \mathcal{E})$ and by adding a new hyperedge e containing the entire vertex set (i.e., $e = V$), any vertex cover of $\mathcal{H}$ is a weak connected vertex cover of the new hypergraph and conversely. Thus, we deduce that on the one hand MinWCVC is NP-hard in connected hypergraphs of maximum degree 4 and is not $c\ln m$ approximable, for some constant c, unless P = NP [9]. Moreover, using another simple reduction, the negative approximation results established in [12,24] also hold for MinWCVC. Actually, starting with a k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ where $V = \{v_1, \ldots, v_n\}$ and $\mathcal{E} = \{e_1, \ldots, e_m\}$, we first add a new vertex v_0 connected to each vertex v_i by edges e'_i for any $i = 1, \ldots, n$. Then, we replace each edge e'_i by a hyperedge by introducing $k-2$ new vertices. Obviously, this new hypergraph $\mathcal{H}'$ is connected and k-uniform, and it is easy to see that S is a vertex cover of $\mathcal{H}$ iff $S \cup \{v_0\}$ is a weak connected vertex cover of $\mathcal{H}'$. In conclusion, for k-uniform hypergraphs, MinWCVC is not $(k-\varepsilon)$ (or $(k-1-\varepsilon)$)-approximable under the same hypothesis as those given in [12,24]. We now present a simple approximation algorithm which shows that the previous bound is sharp.

For a connected hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a hyperedge $e \in \mathcal{E}$, we set $N_{\mathcal{H}}(e) = \bigcup_{v\in e} N_{\mathcal{H}}(v)$; remark that $e \subseteq N_{\mathcal{H}}(e)$ (assuming without loss of generality that there is no edge of size 1). The following greedy algorithm is a generalization of the classical 2-approximation algorithm for the vertex cover problem.

`Greedy2HCVC` input: A connected hypergraph $\mathcal{H} = (V, \mathcal{E})$.

1 Set $S = \emptyset$ and *Label* $= \{v\}$ where v is a vertex of $\mathcal{H}$;
2 While there exists a hyperedge $e \in \mathcal{E}$ with $e \cap \mathit{Label} \neq \emptyset$ do
  2.1 $S := S \cup e$ and $\mathit{Label} := \mathit{Label} \cup N_{\mathcal{H}}(e)$;
  2.2 Delete from $\mathcal{H}$ all the hyperedges adjacent to e and all the vertices in e. Let $\mathcal{H}$ be the resulting hypergraph;
3 Output S;

Let us first prove that S is a vertex cover of the initial hypergraph $\mathcal{H}$. Otherwise, we have $\mathit{Label} \neq V$ and let $\mathcal{H}'' = (V \setminus \mathit{Label}, \mathcal{E}'')$ be the subhypergraph of $\mathcal{H}$ induced by $V \setminus \mathit{Label}$. By assumption, $\mathcal{H}''$ contains some hyperedges of $\mathcal{E}$; actually, it is easy to prove that each vertex $v \notin \mathit{Label}$ is not isolated in $\mathcal{H}''$ and each hyperedge of $\mathcal{H}''$ is a hyperedge of $\mathcal{H}$ with the same size (thus, $\mathcal{H}''$ is also the restriction of $\mathcal{H}$ to $V \setminus \mathit{Label}$). Since $\mathcal{H}$ is connected, there is a hyperedge $e \in \mathcal{E} \setminus \mathcal{E}''$ such that $v \in e \cap \mathit{Label}$ and $w \in e \cap (V \setminus \mathit{Label})$. This hyperedge e has been deleted by `Greedy2HCVC` because either e has been added to S or e is adjacent to a hyperedge $e' \notin \mathcal{E}''$ with $e' \subset S$. In any case, w would have been added to *Label*, contradiction. Finally, we can easily prove that at each iteration of `Greedy2HCVC`, the current set S induces a connected subhypergraph and then, the solution output by this algorithm is a weak connected vertex cover.

The following result is an obvious generalization of the analysis of the classical matching algorithm for MinVC.

Theorem 13. *`Greedy2HCVC` is a $s(\mathcal{H})$-approximation of* MinWCVC *where $s(\mathcal{H})$ is the maximum size of the hyperedges of $\mathcal{H}$.*

We now establish a connection between the weak connected vertex cover problem in hypergraphs and the *minimum labeled spanning tree problem* (MinLST in short) in multigraphs. In this problem, we are given a connected and undirected multigraph $G = (V, E)$ on n vertices. Each edge e in E is colored (or labeled) with the color $\mathcal{L}(e) \in \{c_1, c_2, \ldots, c_q\}$ and for $E' \subseteq E$, we denote by $\mathcal{L}(E') = \bigcup_{e\in E'} \mathcal{L}(e)$ the set of colors used by E'. Given $I = (G, \mathcal{L})$, the goal is to find a spanning tree T that uses the minimum number of colors, that is minimizing $|\mathcal{L}(T)|$. If $\mathcal{L}^{-1}(\mathcal{C}) \subseteq E$ denotes the set of edges with color $c_i \in \mathcal{C}$ for any set $\mathcal{C} \subseteq \{c_1, c_2, \ldots, c_q\}$, then another formulation of MinLST is to find a minimum cardinality subset $\mathcal{C} \subseteq \{c_1, c_2, \ldots, c_q\}$ such that the subgraph induced by the edge set $\mathcal{L}^{-1}(\mathcal{C})$ is connected and touches all vertices in V. The minimum labeled spanning tree problem has been studied in the context of simple graphs (see [8]) but it is easy to see that all the obtained results also hold in multigraphs [22]. The *color frequency* of $I = (G, \mathcal{L})$ denoted by r, is the maximum number of times that a color appears, that is $r = \max\{|\mathcal{L}^{-1}(c_i)|:\ i = 1, \ldots, q\}$.

Theorem 14. *A $\rho(r)$-approximation of* MinLST *can be polynomially converted into a $\rho(r)$-approximation of* MinWCVC *in hypergraph of maximum degree $r+1$.*

Proof. Let $\mathcal{H}=(V,\mathcal{E})$ be a connected hypergraph with maximum degree Δ, instance of MinWCVC. We build the multigraph $G=(V_{\mathcal{E}},E)$ where the vertex set is given by $V_{\mathcal{E}}=\{v_e\colon e\in\mathcal{E}\}$; the edge set is $E=\bigcup_{v\in V}T_v$ where T_v is an arbitrary spanning tree on the subset of vertices $\{v_e\in V_{\mathcal{E}}\colon v\in e\}$. Finally, the color set is $\{c_v\colon v\in V\}$ and if $e\in T_v$, then the edge e is colored with color c_v, that is $\mathcal{L}(e)=c_v$. It is easy to observe that color c_v appears exactly $d_{\mathcal{H}}(v)-1$ times. In conclusion $I=(G,\mathcal{L})$ is an instance of MinLST with color frequency $r=\Delta-1$.

We claim that $S\subseteq V$ is a weak connected vertex cover of $\mathcal{H}$ iff the subgraph $G'=(V_{\mathcal{E}},E')$ where $E'=\bigcup_{v\in S}T_v$ is connected.

Assume that $G'=(V_{\mathcal{E}},\bigcup_{v\in S}T_v)$ is a connected subgraph of $G=(V_{\mathcal{E}},\bigcup_{v\in V}T_v)$. Let $e\in\mathcal{E}$; since G' spans all the vertices of $V_{\mathcal{E}}$, there exists $v\in S$ such that $v_e\in T_v$ (formally, v_e is adjacent to e' with $e'\in T_v$). Thus, v covers the hyperedge e in H and more generally S is a vertex cover of $\mathcal{H}$. Let us prove that the subhypergraph induced by S is a connected hypergraph. Let $s,t\in S$; since G' is connected, there is a shortest path μ in G' linking a vertex of T_s to a vertex of T_t. Assume that this path μ uses edges colored with colors $c_{v_1},\dots,c_{v_p}$. By construction, $\{v_1,\dots,v_p\}\subseteq S$ and since μ is a shortest path, we can assume, without loss of generality, that the colors met in μ are $c_{v_1},\dots,c_{v_p}$ in this order. Let v_{e_j} for $j=1,\dots,p-1$ be the vertex adjacent to colors c_{v_j} and $c_{v_{j+1}}$ in μ. By construction, $\{v_j,v_{j+1}\}\subseteq e_j$ in hypergraph $\mathcal{H}$. Moreover, for the same reasons, there is also two hyperedges e_0 and e_p such that $\{s,v_1\}\subseteq e_0$ and $\{v_p,t\}\subseteq e_p$. In conclusion, $(s,e_0,v_1,e_1,v_2,\dots,e_p,t)$ is a path from s to t in $\mathcal{H}'$ and $\mathcal{H}'$ is connected.

Conversely, let $S\subseteq V$ be a weak vertex cover of $\mathcal{H}$. Obviously, $\bigcup_{v\in S}T_v$ spans all the vertices of $V_{\mathcal{E}}$ since S is a vertex cover of $\mathcal{H}$. Besides, it turns out that any path $(v_1,e_1,v_2,\dots,e_{p-1},v_p)$ from v_1 to v_p in the restriction $\mathcal{H}'$ of $\mathcal{H}$ to S can be transformed into a path going through edges from $\bigcup_{i=1}^{p}T_{v_i}$. In conclusion, $G'=(V_{\mathcal{E}},\bigcup_{v\in S}T_v)$ is a connected subgraph.

Now, since the number of colors used by $G'=(V_{\mathcal{E}},E')$ where $E'=\bigcup_{v\in S}T_v$ is exactly $|S|$, the result follows. In particular, any ρ-approximation for MinLST can be polynomially converted into a ρ-approximation for MinWCVC. If ρ depends on parameter r, the final performance ratio is valid in hypergraphs of degree Δ. □

In [8], it is proved that the restriction of MinLST to the instances $I=(G,\mathcal{L})$ where each color appears at most twice (i.e., $r\leqslant 2$) is polynomial, even if G is a multigraph. Thus, using Theorem 14, we strengthen the result of [34], establishing that the connected vertex cover problem is polynomial in simple graphs with maximum degree 3.

Corollary 15. MinWCVC *is polynomial in hypergraphs with maximum degree* 3.

On the other hand, using the $(H(r)-1/6)$-approximation for MinLST where $H(r)=\sum_{i=1}^{r}\frac{1}{i}$ is the rth harmonic number given in [22], we deduce:

Corollary 16. MinWCVC *is* $(H(\Delta-1)-1/6)$*-approximable in hypergraphs of maximum degree* Δ.

Note that this result is very close to the lower bound of $\ln(\Delta)-c\ln(\ln(\Delta))$ already mentioned [33].

6.2. The strong connected vertex cover problem

It turns out that the complexity of the strong connected vertex cover problem is much harder than the one of the weak connected vertex cover problem. Actually, in contrast to Corollary 15, we now prove that MinSCVC has no approximation scheme in 2-regular hypergraphs.

Theorem 17. MinSCVC *is* APX*-complete in connected* 2*-regular hypergraphs.*

Proof. We give an approximation preserving L-reduction [29] from the vertex cover problem in cubic graphs. This restriction has been proved APX-complete in [1]. An approximation preserving L-reduction is a mapping f (built within polynomial time) from any instance I of an NPO problem π to an instance $f(I)$ of a NPO problem π' such that (i) $opt_{\pi'}(f(I))\leqslant\alpha opt_{\pi}(I)$ and (ii) $|apx_{\pi}(I)-opt_{\pi}(I)|\leqslant\beta|apx_{\pi'}(f(I))-opt_{\pi'}(f(I))|$ for some positive constants α,β.

Let $G'=(V',E')$ be a cubic graph with $V'=\{v'_1,\dots,v'_n\}$ and $E'=\{e'_1,\dots,e'_m\}$, instance of MinVC. We build the connected 2-regular hypergraph $\mathcal{H}=(V,\mathcal{E})$ containing vertices $v_{i,j}$ for $i=1,\dots,n$, $j=1,\dots,4$ and u_j for $j=1,2,3$. Moreover,

- Each vertex v'_i of G' with $i=1,\dots,n$, is split into $d_{G'}(v'_i)+1$ ($=4$ since G' is cubic) vertices $v_{i,1},\dots,v_{i,4}$ such that the edges of G' become a matching in the hypergraph $\mathcal{H}$ saturating vertices $v_{i,j}$ for $i=1,\dots,n$, $j=1,\dots,3$. Moreover, we add the hyperedge $e_i=\{v_{i,1},\dots,v_{i,4}\}$. This gadget $\mathcal{H}(v'_i)$ is described in Fig. 5.
- We add the path μ of length 2 where $\mu=\{\{u_1,u_2\},\{u_2,u_3\}\}$ and we add the hyperedge $e_0=\{v_{i,4}\colon i=1,\dots,n\}\cup\{u_1,u_3\}$.

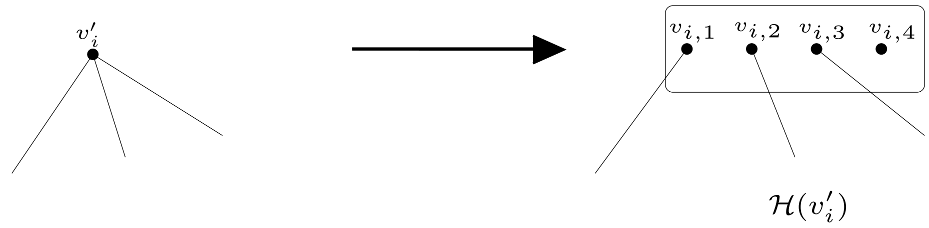

Fig. 5. The gadget $\mathcal{H}(v'_i)$.

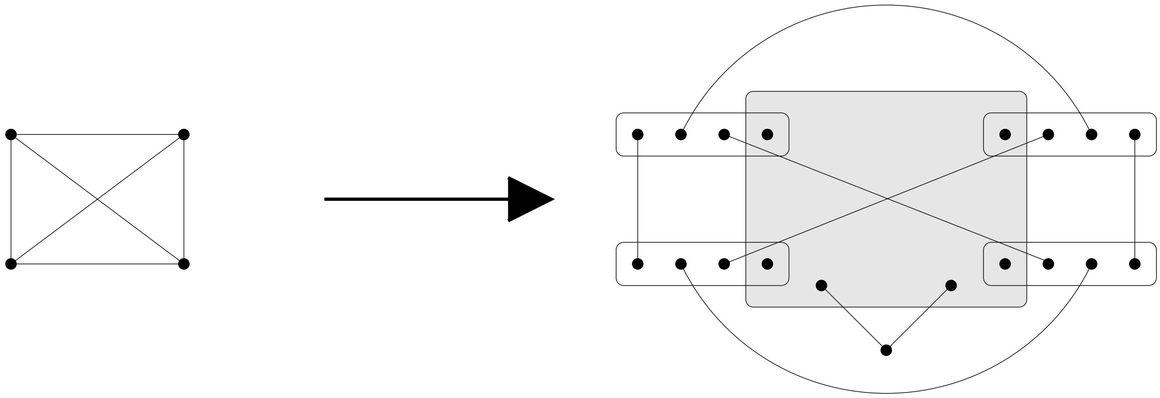

Fig. 6. An example of the construction from a K_4.

Clearly, $\mathcal{H} = (V, \mathcal{E})$ is a connected hypergraph where each vertex has a degree 2. Fig. 6 gives a simple illustration of this construction when G' is a K_4.

If S^* is an optimal vertex cover of G' with value $opt(G')$, then by taking $\{e_i\colon v'_i \in S^*\} \cup e_0$, we obtain a strong connected vertex cover of $\mathcal{H}$. Thus,

$$opt(\mathcal{H}) \leqslant 3opt(G') + n + 2. \tag{16}$$

Conversely, let V_0 be a strong connected vertex cover of $\mathcal{H}$ with value $apx(\mathcal{H})$. By construction, V_0 contains e_0 (i.e., the vertices of this hyperedge) since it is the only way to connect the edges of the path μ to the rest of the solution. Moreover, for each edge $e'_k = \{v_{i,i_1}, v_{j,j_1}\}$ where $i_1, j_1 \in \{1, 2, 3\}$ of $\mathcal{H}$ we have $e_i \subseteq V_0$ or $e_j \subseteq V_0$ since on the one hand, V_0 is a vertex cover of $\mathcal{H}$ and on the other hand, as previously the only way to connect the hyperedge e_0 to v_{i,i_1} or v_{j,j_1} consists of taking the whole hyperedge e_i or e_j. Finally, without loss of generality we may assume that $e_i \cap V_0 = e_i$ or $e_i \cap V_0 = \{v_{i,4}\}$ for $i = 1, \ldots, n$. Thus, $\{v'_i\colon e_i \in V_0\}$ is a vertex cover of G', with value:

$$apx(G') \leqslant \frac{apx(\mathcal{H}) - n - 2}{3}. \tag{17}$$

Using inequalities (16) and (17), we obtain $3opt(G') = opt(\mathcal{H}) - n - 2$. Thus, on the one hand we have $apx(G') - opt(G') \leqslant (apx(\mathcal{H}) - opt(\mathcal{H}))/3$ and on the other hand, $opt(\mathcal{H}) = 3opt(G') + n + 2 \leqslant 6opt(G')$ since G' is an instance of MinVC and cubic. Actually, we get $2opt(G) \geqslant n$ and $opt(G) \geqslant 2$. Thus, by setting $\alpha = 6$ and $\beta = 1$ we obtain the expected result. □

7. Conclusion and open questions

In this article, we proposed some complexity and approximation results for the connected vertex cover problem. We proved that MinCVC is polynomial in chordal graphs and APX-complete in bipartite graphs. We also gave a 5/3-approximation for any class of graphs for which the vertex cover problem is polynomial and a PTAS for planar graphs. Finally, we extended the notion of connected vertex cover to hypergraphs.

This work leads to several open questions. In particular, one could further study the links with the usual vertex cover problems by (a) studying the complexity in other classes of graphs where MinVC is polynomial (such as AT-free graphs, distance-hereditary graph,...), (b) finding, generalizing Theorem 10, an approximation ratio in classes of graphs where MinVC is approximable (better than 2), (c) improving the ratio 5/3, either in general or on some particular classes of graphs.

Alternatively, it would be worth studying the generalization of MinCVC to weighted graphs. As a first step, one can easily see that the problem remains solvable in linear time in chordal graphs (using the fact that MinVC is linear in weighted chordal graphs [15]). But, for instance, could we get similar results as the ones in Section 5?

Acknowledgements

We are very grateful to an anonymous referee for its insightful comments and suggestions.

References

[1] P. Alimonti, V. Kann, Some APX-completeness results for cubic graphs, Theoret. Comput. Sci. 237 (1–2) (2000) 123–134.
[2] E.M. Arkin, M.M. Halldórsson, R. Hassin, Approximating the tree tour covers of a graph, Inform. Process. Lett. 47 (6) (1993) 275–282.
[3] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, Complexity and Approximation (Combinatorial Optimization Problems and Their Approximability Properties), Springer, Berlin, 1999.
[4] B.S. Baker, Approximation algorithms for NP-complete problems on planar graphs, J. ACM 41 (1) (1994) 153–180.
[5] R. Bar-Yehuda, S. Even, On approximating a vertex cover for planar graphs, in: Proceedings of the Annual ACM Symposium on Theory of Computing, STOC'82, 1982, pp. 303–309.
[6] H.L. Bodlaender, A partial k-arboretum of graphs with bounded treewidth, Theoret. Comput. Sci. 209 (1–2) (1998) 1–45.
[7] A. Brandstadt, V.B. Le, J. Spinrad, Graph Classes: A Survey, Society for Industrial and Applied Mathematic, Philadelphia, 1999.
[8] T. Brüggemann, J. Monnot, G.J. Woeginger, Local search for the minimum label spanning tree problem with bounded color classes, Oper. Res. Lett. 31 (3) (2003) 195–201.
[9] M. Chlebík, J. Chlebíková, Approximation hardness of dominating set problems, in: Proceedings of the Annual European Symposium, ESA'04, in: Lecture Notes in Computer Science, vol. 3221, Springer, 2004, pp. 192–203.
[10] M. Chlebík, J. Chlebíková, Complexity of approximating bounded variants of optimization problems, Theoret. Comput. Sci. 354 (3) (2006) 320–338.
[11] E.D. Demaine, M.T. Hajiaghayi, Bidimensionality: new connections between FPT algorithms and PTASs, in: Proceedings of the Annual ACM–SIAM Symposium on Discrete Algorithms, SODA'05, 2005, pp. 590–601.
[12] I. Dinur, V. Guruswami, S. Khot, O. Regev, A new multilayered PCP and the hardness of hypergraph vertex cover, in: Proceedings of the Annual ACM Symposium on Theory of Computing, STOC'03, 2003, pp, 595–601.
[13] U. Feige, A threshold of $\ln n$ for approximating set cover, J. ACM 45 (1998) 634–652.
[14] H. Fernau, D. Manlove, Vertex and edge covers with clustering properties: Complexity and algorithms, in: Algorithms and Complexity in Durham, 2006, pp. 69–84.
[15] A. Frank, Some polynomial algorithms for certain graphs and hypergraphs, in: Proceedings of the Fifth British Combinatorial Conference, 1975, pp. 211–226.
[16] T. Fujito, How to trim an MST: A 2-approximation algorithm for minimum cost tree cover, in: Proceedings of the 33rd International Colloquium on Automata, Languages and Programming, ICALP'06, in: Lecture Notes in Computer Science, vol. 4051, Springer, 2006, pp. 431–442.
[17] M.R. Garey, D.S. Johnson, The rectilinear Steiner tree problem is NP complete, SIAM J. Appl. Math. 32 (1977) 826–834.
[18] M.R. Garey, D.S. Johnson, Computer and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, 1979.
[19] F. Gavril, Algorithms for minimum coloring maximum clique, minimum covering by cliques, and maximum independent set of a chordal graph, SIAM J. Comput. 1 (2) (1972) 180–187.
[20] A. Gonzalez-Gutierrez, T.F. Gonzalez, Complexity of the minimum-length corridor problem, Comput. Geometry 37 (2) (2007) 72–103.
[21] J. Guo, R. Niedermeier, S. Wernicke, Parameterized complexity of generalized vertex cover problems, in: Proceedings of the International Workshop on Algorithms and Data Structures, WADS'05, in: Lecture Notes in Computer Science, vol. 3608, Springer, 2005, pp. 36–48.
[22] R. Hassin, J. Monnot, D. Segev, Approximation algorithms hardness results for labeled connectivity problems., J. Combin. Opt. 14 (4) (2007) 437–453.
[23] D.S. Hochbaum, Approximation Algorithms for NP-Hard Problems, PWS Publishing Company, Boston, MA, 1996.
[24] S. Khot, O. Regev, Vertex cover might be hard to approximate to within $2 - \varepsilon$, in: Proceedings of the IEEE Conference on Computational Complexity, CCC'03, 2003, pp. 379–386.
[25] J. Könemann, G. Konjevod, O. Parekh, A. Sinha, Improved approximations for tour tree covers, Algorithmica 38 (3) (2003) 441–449.
[26] D. Mölle, S. Richter, P. Rossmanith, Enumerate and expand: Improved algorithms for connected vertex cover and tree cover, in: Proceedings of the International Computer Science Symposium in Russia, CSR'06, in: Lecture Notes in Computer Science, vol. 3967, Springer, 2006, pp. 270–280.
[27] D. Mölle, S. Richter, P. Rossmanith, Enumerate and expand: New runtime bounds for vertex cover variants, in: Proceedings of the 12th Annual International Conference on Computing and Combinatorics, COCOON'06, in: Lecture Notes in Computer Science, vol. 4112, Springer, 2006, pp. 265–273.
[28] H. Moser, Exact algorithms for generalizations of vertex cover, Master's thesis, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, 2005.
[29] C.H. Papadimitriou, M. Yannakakis, Optimization, approximation, and complexity classes, J. Comput. System Sci. 43 (3) (1991) 425–440.
[30] P.K. Priyadarsini, T. Hemalatha, Connected vertex cover in 2-connected planar graph with maximum degree 4 is NP-complete, Internat. J. Math. Phys. Engrg. Sci. 2 (1) (2008) 51–54.
[31] C.D. Savage, Depth-first search the vertex cover problem, Inform. Process. Lett. 14 (5) (1982) 233–237.
[32] A. Schrijver, Combinatorial Optimization – Polyhedra and Efficiency, Springer-Verlag, Berlin, 2003.
[33] L. Trevisan, Non-approximability results for optimization problems on bounded degree instances, in: Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, STOC'01, 2001, pp. 453–461.
[34] S. Ueno, Y. Kajitani, S. Gotoh, On the nonseparating independent set problem feedback set problem for graphs with no vertex degree exceeding three, Discrete Math. 72 (1988) 355–360.
[35] T. Wanatabe, S. Kajita, K. Onaga, Vertex covers and connected vertex covers in 3-connected graphs, in: Proceedings of the IEEE International Symposium on Circuits and Systems, 1991, pp. 1017–1020.