A Note on Words in Braid Monoids

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The purpose of this note is to give a self-contained (apart from simple facts about Coxeter groups) and we hope a bit shorter and more understandable account of some results of [C1, C2] on normal forms of braids which are themselves based on the papers [D1, T]. In particular a motivation was to give a proof of Proposition 5.1 that we use in B-M. Some proofs and results from Section 2 onwards seem to be new. I thank several people for improvements from earlier versions of the manuscript: M. Geck for pointing out some errors, F. Digne for pointing out that some results don’t need the braid group to be of finite type, and J.-Y. Hée for suggesting (and providing) further improvements in that direction. © 1999 Academic Press

Let $W$ be a Coxeter group with presentation

$$\left\langle s \in S | s^2 = 1 (\text{quadratic relations}), \prod_{s,t} m_{s,t} \text{ factors} \right\rangle,$$

where $S$ is a finite set and $M = (m_{s,t})_{s,t \in S}$ is a symmetric matrix of positive integers with $m_{s,s} = 1$ and $m_{s,t} > 1$ if $s \neq t$; it is allowed that $m_{s,t} = \infty$, in which case we impose no braid relation linking $s$ and $t$. The Artin–Tits braid group $B$ associated to $W$ is the group defined by the presentation

$$\left\langle s \in S | \prod_{s,t} m_{s,t} \text{ factors} \right\rangle,$$

where $s$, $t \in S$, $s \neq t$.

We say that $B$ is of finite type if $M$ is chosen so that $W$ is a finite group. We denote by $B^+$ the monoid defined by presentation $(\ast)$; if $B$ is of finite type, we shall see in Corollary 3.2 that $B^+$ can be identified to the submonoid of $B$ generated by $S$. 

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1. THE SECTION \(B_{\text{red}}^+\)

The relations defining \(B^+\) being homogeneous (i.e., both sides have the same length), for \(b \in B^+\) there is a natural length function \(l(b)\) defined as the length of any expression of \(b\) as a product of the elements of \(S\). There is also a natural length function on \(W\), which we denote again by \(l\), defined by the length of a minimal expression as a product of elements of \(S\). It is known (cf., e.g., B. Sect. 1 Ex. 13(b)) that two minimal expressions for an element of \(W\) are equivalent by using only the braid relations. This implies that the natural quotient map \(p: B^+ \to W\) has a canonical section as a map of sets whose image \(B^+_{\text{red}}\) consists of those elements of \(B^+\) which have the same length as their image in \(W\). It also implies:

1.1 Proposition. \(B^+\) has the following presentation:

(i) As a set of generators we take \(B^+_{\text{red}}\) (considered as an abstract set of generators endowed with a bijection \(p: B^+_{\text{red}} \to W\)).

(ii) For relations we take \(ab = c\) whenever \(a, b, c \in B^+_{\text{red}}\) are such that the relation \(p(a)p(b) = p(c)\) holds in \(W\), with \(l(p(a)) + l(p(b)) = l(p(c))\).

More convenient for us will be the following equivalent construction of \(B^+\):

1.2 Proposition. \(B^+\) can be identified with the set of sequences of elements of \(B^+_{\text{red}}\), quotiented by the equivalence relation which is the transitive closure of the following: the sequences \((g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n)\) and \((g_1, \ldots, g_{j-1}, a, b, g_{j+1}, \ldots, g_n)\) are equivalent whenever \(ab = g\) is a relation in Proposition 1.1(ii).

The identification above is by the product \((g_1, \ldots, g_n) \mapsto g_1 \cdots g_n \in B^+\).
We will call \((g_2, \ldots, g_n)\) a decomposition of length \(n\) of the braid element \(g_1 \cdots g_n\). For \(g \in B^+\), we define \(\nu(g)\) as being the minimal \(n\) such that \(g\) has a decomposition of length \(n\) (thus \(\nu(g) = 0\) if and only if \(g = 1\) and \(\nu(g) = 1\) if and only if \(g \in B^+_{\text{red}}\)).

In what follows, properties of \(B^+\) will be proved by first establishing them on \(B^+_{\text{red}}\), where they will in a further step be reduced to properties of \(W\) via the bijection \(p: B^+_{\text{red}} \to W\).

We denote by \(<\) divisibility on the left in \(B^+\) (that is, for \(a, b \in B^+\), we have \(a < b\) if and only if there exists \(c \in B^+\) such that \(b = ac\)), and we denote similarly by \(>\) divisibility on the right. From the inequality \(l(a) + l(b) \geq l(ab)\) which holds in \(W\), it follows that if \(a < b, b \in B^+_{\text{red}}\) then also \(a \in B^+_{\text{red}}\). We also note

1.3 Remark. Elements of \(B^+_{\text{red}}\) are cancellable, that is, if \(a \in B^+, b, c \in B^+_{\text{red}},\) and \(ab = ac \in B^+_{\text{red}}\) or \(ba = ca \in B^+_{\text{red}}\) then \(b = c\).
Proof. This is obvious, since then \( b \) and \( c \) have same image in \( W \).

For \( s, t \in S \) such that \( m_{s,t} < \infty \) we denote by \( \Delta_{\{s,t\}} \) the element of \( B^+ \) which appears on both sides of the braid relation involving \( s \) and \( t \).

The following lemma is adapted from [D1, 1.14].

1.4 Lemma. Let \( M \) be a finite set of elements of \( B^+ \) such that:

(i) If \( b \in M, a \in B^+, a < b \) then \( a \in M \).

(ii) If \( a \in B^+, s, t \in S, as, at \in M \) then \( m_{s,t} < \infty \) and \( a \Delta_{\{s,t\}} \in M \).

Then there exists \( g \in M \) such that \( M = \{ a \in B^+ | a < g \} \).

Proof. Let \( g \) be an element of maximal length in \( M \). We will prove by contradiction that every element of \( M \) divides \( g \). If it were not the case, we can find some \( a < g, s \in S \) such that \( as \in M, as \not< g \) (take an element \( x \in M, x \not< g \), and take for \( a \) any maximal (for \( < \) ) divisor of \( x \) which divides \( g \)). We consider such an \( a \) of maximal length and derive a contradiction by constructing a longer one. Since \( a < g \) and \( l(a) < l(g) \) (since \( l(a) < l(as) \leq l(g) \) by maximality of \( l(g) \)), there exists \( t \in S \) such that \( at < g \) (and \( t \neq s \)). Thus \( at \in M \) and by (ii) we have \( a \Delta_{\{s,t\}} \in M \). Since \( as < a \Delta_{\{s,t\}} \), we cannot have \( a \Delta_{\{s,t\}} < g \). So there exists some \( v < \Delta_{\{s,t\}} \) such that \( at < av, av < g \), and there exists \( r \in S \) equal to either \( s \) or \( t \) such that \( avr < a \Delta_{\{s,t\}}, avr \not< g \), whence the contradiction.

For \( a \in B^+ \) we put \( \mathcal{L}(a) = \{ s \in S | s < a \} \) and \( \mathcal{R}(a) = \{ s \in S | a > s \} \). For \( w \in W \) we also define \( \mathcal{L}(w) = \{ s \in S | l(sw) < l(w) \} \) and \( \mathcal{R}(w) = \{ s \in S | l(ws) < l(w) \} \) (the definitions agree on \( B^+_{\text{red}} \)).

1.5 Proposition. Let \( a \in B^+_{\text{red}} \), and \( s, t \in \mathcal{L}(a) \). Then \( \Delta_{\{s,t\}} < a \).

Proof. We only have to check this property in \( W \) where it is well known: write \( p(a) = uv' \) where \( v \) is an element of the parabolic subgroup \( W_{\{s,t\}} \) of \( W \) generated by \( s \) and \( t \) and \( v' \) is \( \{ s, t \} \)-reduced—i.e., \( \mathcal{L}(v') \cap \{ s, t \} = \emptyset \). Then \( v \) is an element of \( W_{\{s,t\}} \) such that \( \mathcal{L}(v) = \{ s, t \} \); there exists such an element if and only if the group \( W_{\{s,t\}} \) is finite (equivalently \( m_{s,t} < \infty \)) and then the only element is \( p(\Delta_{\{s,t\}}) \), the longest element of \( W_{\{s,t\}} \).

1.6 Proposition. Let \( a, b \in B^+_{\text{red}} \). There exists a unique maximal (for the partial order \( < \) ) \( c \in B^+_{\text{red}} \) such that \( c < a \) and \( c < b \) (we will call \( c \) the g.c.d. of \( a \) and \( b \) and denote \( c = \gcd(a, b) \)).

Proof. It is enough to show that the set \( c \subseteq B^+ \), \( c < a \), and \( c < b \) satisfies the hypotheses of Lemma 1.4: (i) is obvious; to see (ii), since \( B^+_{\text{red}} \) has the left cancellation property (1.3), it is enough to show that if \( a \in B^+_{\text{red}}, s, t \in S, s < a, t < a \) then \( \Delta_{\{s,t\}} < a \). But this is just the statement of Proposition 1.5.
Note that the above construction of \( a \wedge b \) shows that the operation \( \wedge \) is associative.

1.7 Proposition / Definition. Let \( a, b \in B_{\text{red}}^+ \). There exists a unique maximal \( c < b \) such that \( ac \in B_{\text{red}}^+ \). In this situation, we define two elements of \( B_{\text{red}}^+ \): we put \( \alpha_2(a, b) = ac \) and \( \omega_2(a, b) = \) the unique \( d \) such that \( b = cd \) (thus we have \( ab = \alpha_2(a, b) \omega_2(a, b) \)).

Proof. In order to check that the set of \( c \) such that \( c < b \) and \( ac \in B_{\text{red}}^+ \) satisfies the assumptions of Lemma 1.4, the only new thing we have to check is that if \( a \in B^+, s, t \in S \), as, at \( \in B_{\text{red}}^+ \), and \( m_{s,t} < \infty \) then \( a \Delta_{\{s,t\}} \in B_{\text{red}}^+ \) (we know that \( m_{s,t} < \infty \) since we are in a case where \( s \) and \( t \) divide some tail of \( c \)). Again this is a verification in \( W \): \( a \) must be \( (s,t) \)-reduced, thus it adds to \( \Delta_{\{s,t\}} \).

Note also that it follows from Remark 1.3 that \( \omega_2(a, b) \) is unique.

Before extending them to the whole of \( B^+ \), we check some properties of \( \alpha_2 \) and \( \omega_2 \).

1.8 Proposition. Let \( a, b, c, ab \in B_{\text{red}}^+ \). Then \( \alpha_2(ab, c) = \alpha_2(a, \alpha_2(b, c)) \).

Proof. By definition \( \alpha_2(b, c) \) is of the form \( by \) with \( y < c \), so \( \alpha_2(a, \alpha_2(b, c)) \) is of the form \( aby' \in B_{\text{red}}^+ \) with \( y' < y < c \), so \( aby' < \alpha_2(ab, c) \). Let \( \alpha_2(ab, c) = abz \). We thus have \( y' < z < c \). But this implies \( y' = z \) since \( bz \in B_{\text{red}}^+ \), \( bz < bc \) so \( bz < by = \alpha_2(b, c) \) and since \( abz \in B_{\text{red}}^+ \), \( abz \) cannot be longer than \( aby' = \alpha_2(a, by) \).

1.9 Proposition. Let \( a, b, c, ab \in B_{\text{red}}^+ \). Then \( \omega_2(ab, c) = \omega_2(a, \alpha_2(b, c)) \omega_2(b, c) \).

Proof. We first notice that, using Propositions 1.8 and 1.7, both sides have the same value when multiplied on the left by \( \alpha_2(ab, c) \). This implies that the equality holds when projected to \( W \), and by Remark 1.3 it will hold in \( B^+ \) if we show that the right-hand side is in \( B_{\text{red}}^+ \). But this results from the fact that the equality holds in \( W \) and that (since it holds when multiplied by \( \alpha_2(ab, c) \)) we have \( l(\omega_2(ab, c)) = l(\omega_2(a, \alpha_2(b, c))) + l(\omega_2(b, c)) \).

2. Cancellability in \( B^+ \)

We first define a function \( \alpha \) on \( B^+ \), which extends \( \alpha_2 \), considered as a function defined on those \( g \in B^+ \) such that \( \nu(g) = 2 \). The following proof of the existence of \( \alpha \) is inspired by [T].
2.1 Proposition. There is a unique function \( \alpha: B^+ \to B_{\text{red}}^+ \) which induces the identity on \( B_{\text{red}}^+ \), satisfies \( \alpha(ab) = \alpha_2(a, b) \) for \( a, b \in B_{\text{red}}^+ \) and such that \( \alpha(g) = \alpha(g\alpha(h)) \) for all \( g, h \in B^+ \). Further, \( \alpha(g) \) is the unique maximal (for \( \prec \)) element in the set \( \{ c \in B_{\text{red}}^+ | c < g \} \).

Proof. We define recursively (by induction on \( n \)) a function \( \alpha \) on decompositions of length \( n \) of elements of \( B^+ \), considered just as sequences. We first define it on decompositions of length \( \leq 1 \) by \( \alpha(()) = 1 \), \( \alpha((a)) = a \), and for longer decompositions we set \( \alpha((g_1, \ldots, g_n)) = \alpha_2(g_1, \alpha((g_2, \ldots, g_n))) \). We claim that \( \alpha \) induces a well-defined function on \( B^+ \). Indeed, in view of the equivalence relation of Proposition 1.2 which we put on decompositions and the inductive definition of \( \alpha \), it is enough to check \( \alpha_2(ab, c) = \alpha_2(a, \alpha_2(b, c)) \), which is the statement of Proposition 1.8.

It is clear that the \( \alpha(g) \) we have thus defined for \( g \in B^+ \) is maximal among divisors in \( B_{\text{red}}^+ \) of \( g \): if \( a \prec g \), \( a \in B_{\text{red}}^+ \) then there exists a decomposition \( g = ag_2 \cdots g_n \) with \( g_j \in B_{\text{red}}^+ \), and \( \alpha(g) = \alpha((a, g_2, \ldots, g_n)) \) is by construction a multiple of \( a \).

Also, if \( g, h \in B^+ \) and \( \nu(g) = 1 \) there is a decomposition of \( gh \) which starts with \( g \) so by construction we have \( \alpha(gh) = \alpha(g\alpha(h)) \). We show by induction on \( \nu(g) \) that this equality remains true for all \( g \): if \( g = g'g'' \) with \( g' \in B_{\text{red}}^+ \), \( \nu(g'') = \nu(g) - 1 \) then \( \alpha(g'g''h) = \alpha(g'\alpha(g''h)) = \alpha(g'\alpha(g'\alpha(h))) \) where the first equality holds by the case \( \nu = 1 \) and the second case by induction, and on the other hand \( \alpha(g'g''h) = \alpha(g'\alpha(g''\alpha(h))) \) by using the case \( \nu = 1 \).

The unicity of \( \alpha \) is clear by construction.

We would like now to define \( \omega(g) \) as the unique \( h \) such that \( g = \alpha(g)h \), but until we know that \( B^+ \) has the left cancellation property we will not know the unicity of \( h \), so we proceed differently.

2.2 Proposition. There is a unique function \( \omega: B^+ \to B^+ \) which maps \( B_{\text{red}}^+ \) to 1, such that \( \omega(ab) = \omega_2(a, b) \) for \( a, b \in B_{\text{red}}^+ \) and such that \( \omega(g) = \omega(g\alpha(h))\omega(h) \) for all \( g, h \in B^+ \).

Proof. As for \( \alpha \), we define \( \omega \) recursively on decompositions by setting \( \omega((g_1, \ldots, g_n)) = \omega_2(g_1, \alpha(g_2 \cdots g_n))\omega(g_2, \ldots, g_n) \). This time, to check that this definition is compatible with the equivalence relation we put on decompositions, we must check that \( \omega_2(ab, c) = \omega_2(a, \alpha(bc))\omega_2(b, c) \), which is the statement of Proposition 1.9.

Again we have by definition \( \omega(g) = \omega(g\alpha(h))\omega(h) \) when \( \nu(g) = 1 \) and again by an easy induction on \( \nu(g) \) this extends to all elements \( g \).
We can now prove:

2.3. Proposition. Let $g \in B^+$. Then there is a unique $y$ such that $g = \alpha(g)y$, and this $y$ equals $\omega(g)$.

Proof. We show by induction on the length of $g$ that there is a unique $y$ such that $g = \alpha(g)y$, and this $y$ equals $\omega(g)$.

2.4. Proposition. The monoid $B^+$ has the left and right cancellation property.

Proof. By symmetry, it is enough to check, e.g., left cancellability. First by induction on $n$ we notice that it is enough to prove cancellability for $X^Y$. By the case $n = 1$ we get $ab = ac$, from which we deduce by induction $b = c$. So assume now $a \in B^*$. Then there is $x \in B^*$, such that $\alpha(ab) = ax$: indeed, by the formula $\alpha(ab) = \alpha(axb)$ we have $\alpha(ab) = ax$ where $x < a(b) < b$. Similarly $x < c$. So there exists $b', c'$ such that $ab = (ax)b' = (ax)c'$. We then have $b' = c' = \omega(ab)$ by Proposition 2.3 whence $b = xb' = xc' = c$.

We are now in a position to extend Propositions 1.5 and 1.6 to the whole of $B^+$.

2.5. Proposition. Let $a \in B^+$, and $s, t \in \mathcal{D}(a)$. Then $\Delta_{(s, t)} < a$.

Proof. If $s, t \in \mathcal{D}(a)$ then $s < \alpha(a)$ and $t < \alpha(a)$ so by Proposition 1.5 we have $\Delta_{(s, t)} < \alpha(a) < a$.

2.6. Proposition. Let $a, b \in B^+$. There exists a unique maximal (for the partial order $\prec$) $c \in B^+$ such that $c \prec a$ and $c \prec b$ (we still denote $c = a \wedge b$).

Proof. Since $B^+$ has the left cancellation property, we can just follow the proof of Proposition 1.6 using Proposition 2.5 instead of Proposition 1.5.

3. The Center and Monoids of Finite Type

If $B$ is of finite type, $W$ is finite and thus has a unique longest element. We define $\Delta$ as the element of $B^*$ such that $p(\Delta)$ is the longest element of $W$. We can in this case redefine $\alpha$ on $B^+$ by $\alpha(g) = \Delta \wedge g$. 
3.1. Proposition. Assume \( B \) is of finite type. There is an automorphism \( g \mapsto \bar{g} \) of order 2 of \( B^+ \) defined by \( \Delta g = \bar{g} \Delta \). This automorphism preserves \( S \).

Proof. It is sufficient to prove that there is an involution \( s \mapsto \bar{s} \) of \( S \) such that \( \Delta s = s \Delta \). It is well known (cf. e.g., [B1, Sect. 1 Ex. 22]) that the same property holds in \( B \), i.e., that for \( s \in S \) there exists \( \bar{s} \in S \) such that \( p(s)p(\Delta) = p(\Delta)p(\bar{s}) \) and \( \bar{s} = s \). But \( p(s) \) is of order 2, so if \( w' \in B_{\text{red}}^+ \) is defined by \( p(w') = p(s)p(\Delta) \), we have \( \Delta = sw' = w'\bar{s} \), thus \( s\Delta = sw'\bar{s} = \Delta \bar{s} \).

3.2. Corollary. Assume \( B \) is of finite type. The element \( \pi = \Delta^2 \) is central in \( B^+ \). The monoid \( B^+ \) injects in \( B \), and for any \( b \in B \), there exists \( i > 0 \) such that \( \pi^i b \in B^+ \). Any element \( g \in B \) can be written uniquely as \( x^{-1}y \), where \( x, y \in B^+ \) have no common divisor.

Proof. The first assertion is clear from Proposition 3.1. To see the second one, notice that any \( x \in B_{\text{red}}^+ \) divides \( \Delta \), thus in particular divides \( \pi \). It results by induction on \( n \) that if we write \( x \in B^+ \) as \( x_1 \cdots x_n \), with \( x_i \in B_{\text{red}}^+ \), then \( x < \pi^n \); indeed, \( x_2 \cdots x_n < \pi^{n-1} \) so \( x < x_1 \pi^{n-1} = \pi^{n-1}x_1 < \pi^{n-2}\pi = \pi^n \). It follows that every two elements of \( B^+ \) have a common multiple (some power of \( \pi \)), thus by Proposition 2.4, \( B^+ \) satisfies the Ore condition and thus injects in \( B \). The group \( B \) can be identified with the group of fractions of the monoid \( B^+ \); in particular every element of \( B \) is of the form \( x^{-1}y \) with \( x, y \in B^+ \). We have \( \pi^i x^{-1}y \in B^+ \) for large enough \( i \) (such that \( x < \pi^i \)); also the existence of the g.c.d. proves the unicity of such a decomposition with no common divisor.

Note that the existence of a g.c.d. implies that of an l.c.m.: given \( a, b \in B^+ \) they have some common multiple (e.g., some power of \( \pi \)), so the set of elements which are common multiples of \( a \) and \( b \) has a g.c.d., which is the l.c.m. of \( a \) and \( b \). We will denote \( a \vee b \) as the l.c.m.

The center of \( B^+ \) can be determined in general using the following

3.3. Lemma. If \( w \) is central in \( B^+ \) and \( s, t \in S \) are such that \( s < w \) and \( st \neq ts \) then \( t < w \).

Proof. Under the assumption of the lemma, \( s \) and \( t \) both divide \( wt = tw \). Thus if we define \( \delta \) by \( \Delta_{(s,t)} = t\delta \) then \( \delta < w \). Now \( st \neq ts \) implies \( st < \delta < w \), so there exists \( x \) such that \( w = stx \). Now \( sw = ws \) gives \( sstx = stxs \) and cancelling \( s \) we get \( w = stx = txs \) so \( t < w \).

This lemma implies that the center of \( B^+ \) is generated by the \( \Delta^2_i \) (or possibly \( \Delta_i \), when the automorphism of Proposition 3.1 is trivial on \( W_i \)) where \( I \) runs over the connected components of the Coxeter diagram of \( W \) such that the parabolic subgroup \( W_i \) is finite.
4. NORMAL FORMS

We say that a decomposition \((g_1, \ldots, g_n)\) is the normal form of \(g_1 \cdots g_n \in B^+\) if no \(g_j\) is equal to 1 and for any \(i\) we have \(g_i = \alpha(g_i \cdots g_n)\) (so \(g_{i+1} \cdots g_n = \omega(g_i \cdots g_n)\)).

4.1. Proposition (cf. [C1, 2.5]). The normality of a form can be seen locally: \((g_1, \ldots, g_k)\) is a normal form if and only if \((g_i, g_{i+1})\) is for all \(i\). This implies that any segment \((g_i, \ldots, g_j)\) of a normal form is normal.

Proof. Indeed, using the formula \(\alpha(gh) = \alpha(g\alpha(h))\) we have for all \(i\) that \(\alpha(g_i \cdots g_n) = \alpha(g_i g_{i+1})\).

4.2. Corollary. A form \((g_1, \ldots, g_k)\) is normal if and only if for any \(i\) we have \(\mathcal{H}(g_i) \supset \mathcal{L}(g_{i+1})\).

Thus we have a new, very convenient description of \(B^+\) (we don’t need to quotient by an equivalence relation):

4.3. Corollary. \(B^+\) can be identified to the set of sequences \(\{(g_1, \ldots, g_k), g_i \in W - \{1\}\} \mathcal{H}(g_i) \supset \mathcal{L}(g_{i+1})\) for any \(i\).

Let \(\Gamma\) be a group of diagram automorphisms of \(W\) (i.e., automorphisms of \(W\) which stabilize \(S\)). Then the set of fixed points \(W^\Gamma\) is still a Coxeter group, with the Coxeter generating set the set of longest elements in the parabolic subgroups \(W_\sigma\) where \(\sigma\) runs over the orbits \(S/\Gamma\) such that the group \(W_\sigma\) is finite (the idea of the proof is essentially due to Steinberg; see [H, M] for a proof in the general case). In this situation, \(\Gamma\) acts naturally on \(B\) (by acting on the generators), and from Corollary 4.3 we get:

4.4. Corollary. In the above situation, \((B^+)^\Gamma\) identifies to the Braid monoid of \(W^\Gamma\). If \(W\) is of finite type, \(B^\Gamma\) identifies to the Braid group of \(W^\Gamma\).

Proof. By the unicity of \(\alpha\), a normal form \((g_1, \ldots, g_k)\) represents an element of \((B^+)^\Gamma\) if and only if all \(g_i \in W^\Gamma\). The condition \(\mathcal{H}(g_i) \supset \mathcal{L}(g_{i+1})\) translates to the same condition in \(W^\Gamma\), whence the first part of the statement. For the second part, we just notice that if \(\Delta^{-n}b \in B^\Gamma\), with \(b \in B^+\), then since \(\Delta\) is \(\Gamma\)-fixed we must have \(b \in (B^+)\); and \(\Delta\) is also the lift to the braid monoid of \(W^\Gamma\) of the longest element of \(W^\Gamma\).

We note that at this stage we can show easily (following [D1, 4.24]):

4.5. Corollary. Assume that \(B\) is of finite type. Then \(x, y \in B^+\) are conjugate under \(B\) if and only if there exists \(a_1, \ldots, a_k \in B_{\text{red}}^+\) such that \(x_1 = x, x_{k+1} = y\) and \(x_i a_i = a_i x_{i+1}\). In particular the conjugacy problem in \(B\) is decidable.
Proof. Multiplying a conjugating element by a suitable power of $\pi$ we may assume that $x$ and $y$ are conjugate by some $a \in B^+$, so that $xa = ay$.

So $\alpha(a) < xa$, which implies by the defining property of $\alpha$ that $\alpha(a) < x\alpha(a)$, so $x_1 = \alpha(a)^{-1} x a \alpha(a)$ is in $B^+$. We take $a_1 = \alpha(a)$, $x_2 = a_1^{-1} \in x_1 a_1$ and we go on in this way taking $a_2, \ldots, a_k$ such that $(a_1, \ldots, a_k)$ is the normal form of $a$.

4.6. Lemma (cf. [C1, 3.1]). Let $w \in B_{\text{red}}^+$ and $(g_1, \ldots, g_k)$ be the normal form of $g \in B^+$. Then we can write $g_1 = g_1^{n_1} g_1^{s_1}, \ldots, g_k = g_k^{n_k} g_k^{s_k}$ such that the normal form of $wg$ is $(wg_1^{n_1}, g_1^{s_1}, g_1^{n_1} g_1^{s_1} 1, \ldots, g_k^{n_k} g_k^{s_k} g_k^{n_k} 1 g_k^{s_k})$ if $g_k \neq 1$ and $(wg_1^{n_1}, g_1^{s_1}, g_1^{n_1} g_1^{s_1} 1, \ldots, g_k^{n_k} g_k^{s_k} g_k^{n_k} 1 g_k^{s_k})$ if $g_k = 1$.

Proof. We apply repeatedly $\alpha(gh) = \alpha(g \alpha(h))$: we start with $\alpha(wg) = \alpha(wg_1) = wg_1^{s_1}$ (this defines $g_1$ and $g_1^{s_1} = g_1^{-1} g_1$), then $\alpha(g_1^{s_1} g_2 \cdots g_k) = \alpha(g_1^{s_1} g_2) = \alpha(g_1^{s_1} g_2) = \alpha(g_1^{s_1} g_2)$ etc.

4.7. Proposition. Let $(g_1 \cdots g_n)$ be the normal form of $g$. Then $n = \nu(g)$.

Proof. We proceed by induction on $\nu(g)$. Let $n(g)$ be the number of terms in the normal form of $g$. Let $h_1 \cdots h_k$ be a decomposition of $g$ of length $\nu(g)$ (so that $w, h_1, \ldots, h_k$ are reduced and $k = \nu(g) - 1$). By induction we have $n(h_1 \cdots h_k) = k$. Let $g_1 \cdots g_k$ be the normal form of $h_1 \cdots h_k$. Then by Lemma 4.6 we have $n(g) \leq k + 1$. Since we cannot have $n(g) < \nu(g)$ we must have $n(g) = k + 1 = \nu(g)$.

Note that Lemma 4.6 can be rephrased: either $\nu(wg) = \nu(g)$ and $\omega^{\nu(wg)}(wg) > \omega^{\nu(g)}(g)$ (when $g_k = 1$) or $\nu(wg) = \nu(g) + 1$ and $\omega^{\nu(wg)}(wg) > \omega^{\nu(g)}(g)$.

4.8. Proposition. For $g, h \in B^+$ we have $\max(\nu(g), \nu(h)) \leq \nu(gh) \leq \nu(g) + \nu(h)$.

Proof. This is clear by induction on $\nu(g)$, applying repeatedly Lemma 4.6.

4.9. Lemma. If $g \in B^+$, $w \in B_{\text{red}}^+$, and $\nu(gw) > \nu(g)$ then $w > \omega^{\nu(g)}(gw)$.

Proof. This is by induction on $k = \nu(g)$. If $(g_1, \ldots, g_k)$ is the normal form of $g$, then we must have $\nu(g_2 \cdots g_k w) > \nu(g_2 \cdots g_k)$, otherwise $\nu(gw)$ could not be greater than $\nu(g)$. So the normal form of $g_2 \cdots g_k w$ is by induction $(x_1, \ldots, x_k)$ where $w > x_k$. By Lemma 4.6 the last term of the normal form of $g_1 x_1 \cdots x_k$ is a right divisor of $x_k$ (so of $w$) — otherwise we would have $\nu(gw) = \nu(g_1 x_1 \cdots x_k) = k$, a contradiction.

4.10. Lemma. Let $(g_1, \ldots, g_k)$ be the normal form of $g \in B^+$ and let $x \in B^+$, $x < g$. Then $\nu(x) \leq k$ and $x < g_1 \cdots g_{\nu(x)}$.
Proof. The fact that $\nu(x) \leq k$ comes from, e.g., Proposition 4.8. We prove the second part by induction on $h = \nu(x)$. We have $x = x'w$ with $\nu(x') = h - 1$ and $w \in B_{\text{red}}^+$. By induction, there exists $a \in B^+$ such that $x'a = g_1 \cdots g_{h-1}$, so $g = x'ag_{h} \cdots g_{k}$. Therefore $x = x'w < g$ implies $w < ag_{h} \cdots g_{k}$. Since $w \in B_{\text{red}}$ we get $w < \alpha(ah_{1} \cdots g_{k}) = \alpha(\alpha(gh_{1} \cdots g_{k})) = \alpha(\alpha(g_{h})), \ \text{so} \ \ w < \alpha(g_{h})$ thus $x = x'w < x'ag_{h} = g_{1} \cdots g_{h}$. 

5. CHARNEY'S RESULT

I thank J.-Y. Hée for providing me with the proof of Proposition 5.1 given below (as well as Lemma 4.10 above, used along the way) which replaces my proof which needed to assume $B$ of finite type.

5.1. Proposition. Let $g \in B^+$, $w \in B_{\text{red}}^+$; then $\nu(gw) = \nu(g)$ if and only if $\omega(\omega^{-1}(g)w) \in B_{\text{red}}^+$. 

Proof. Let $(g_{1}, \ldots, g_{k})$ be the normal form of $g$. We first notice that the “if” part, which says that $g_{k}w \in B_{\text{red}}^+$ implies $\nu(gw) = \nu(g)$, results from Proposition 4.8 which gives $k = \nu(g) \leq \nu(gw) \leq \nu(g_{1} \cdots g_{k}) + \nu(g_{k}) = k$.

Conversely, assume $\nu(gw) = k$ and let $(g_{1}, \ldots, g_{k})$ be the normal form of $gw$. By Lemma 4.10, there exists $a \in B^+$ such that $g_{1} \cdots g_{k}a = h_{1} \cdots h_{k}$. From $g_{1} \cdots g_{k-1}g_{k}w = h_{1} \cdots h_{k-1}h_{k} = g_{1} \cdots g_{k-1}a$, we get $g_{k}w \in B_{\text{red}}^+$. Let $\alpha'$ be the left l.c.m. of $h_{i}$ and $w$, so there exists $g'$ such that $g_{k}w = g'\alpha'$ and $h' = h_{1} \cdots h_{k}$ such that $\alpha' = h'h_{k}$. Let $x = g_{1} \cdots g_{k-1}g'$. Then by construction $g' < g_{k}$, thus $x < g$; on the other hand $xh' = x\alpha' = gw = h_{1} \cdots h_{k}$, thus $xh' = h_{1} \cdots h_{k-1}$, so $\nu(x) \leq k - 1$ and by Lemma 4.10 we have $x < g_{1} \cdots g_{k-1}$, which implies $g' = 1$ and $g_{k}w = \alpha' \in B_{\text{red}}^+$. 

Proposition 5.1 can be rephrased as follows:

5.2. Proposition. If there exists $i$ such that $\nu(\omega^{i}(g)w) = \nu(\omega^{i}(g))$ then for any $i$ we have $\nu(\omega^{i}(g)w) = \nu(\omega^{i}(g))$.

Proof. If $(g_{1}, \ldots, g_{k})$ is the normal form of $g$, we can rephrase Proposition 5.2 as: if there exists $i$ such that $k - i = \nu(g_{i} \cdots g_{k})$ then for any $i$ we have $k - i = \nu(g_{i} \cdots g_{k})$. By Proposition 5.1 applied to $g_{i} \cdots g_{k}$ and $w$, the hypothesis implies that $g_{k}w \in B_{\text{red}}^+$ which in turn implies for any $i$ (applying Proposition 5.1 to $g_{i} \cdots g_{k}$) that $\nu(g_{i} \cdots g_{k}w) = k$. 

The next proposition says a bit more than [C1, 3.3]. Note that it implies that the equality $\nu(\omega^{i}(g)w) = \nu(\omega^{i}(g))$ in Proposition 5.2 can be replaced by $\nu(\omega^{i}(gw)) = \nu(\omega^{i}(g))$. 

5.3. Proposition. For any $i \in \mathbb{N}$, $g \in B^+$ and $w \in B^+_\text{red}$ then $\omega^{i+1}(gw) = \omega \omega (g) w$.

Proof. It is sufficient to check the proposition for $i = 1$. Indeed from the case $i = 1$, $\omega^2(gw) = \omega \omega (g) w$ and from the case $i = 1$, $\omega^i(gw) = \omega (\omega^{i-1}(gw))$ we get $\omega^{i+1}(gw) = \omega^2(\omega^{i-1}(gw)) = \omega (\omega(g) w)$ which is the case for $i$.

We proceed by induction on $k = \nu(g)$. The result is clear for $k \leq 1$. Assume $k \geq 2$ and let $(g_1, \ldots, g_k)$ be the normal form of $g$. Assume first $\nu(gw) = k$. Then by Corollary 5.1, $g_kw \in B^+_\text{red}$ which implies

$$
\omega^2((g_1 \cdots g_{k-1})(g_kw)) = \omega(\omega(g_1 \cdots g_{k-1})g_kw)
= \omega(g_2 \cdots g_kw) = \omega (\omega(g) w),
$$

where the first equality is by the induction hypothesis. Otherwise let $(\eta_0, \ldots, \eta_k)$ be the normal form of $gw$. Then by Lemma 4.9 we have $w \succ \eta_k$, say $w = \beta \eta_k$. Then $g_b \beta$ must be in $B^+_\text{red}$ else by Proposition 5.1, $k = \nu(\eta_0 \cdots \eta_{k-1} = g_1 \cdots g_k \beta) > \nu(g_1 \cdots g_k) = k$, a contradiction. We thus get by induction $\eta_2 \cdots \eta_{k-1} = \omega^2(\eta_0 \cdots \eta_{k-1}) = w(w/g_1 \cdots g_{k-1} \beta)$ so there exists $a$ such that the normal form of $g_2 \cdots g_k \beta$ is $(a, \eta_2, \ldots, \eta_{k-1})$. If $k > 2$ this implies that $(a, \eta_2, \ldots, \eta_{k-1}, \eta_k)$ is the normal form of $g_2 \cdots g_kw$, since it verifies the local criterion for normality. Thus $\omega(g_2 \cdots g_kw) = \eta_2 \cdots \eta_k$ as we had to prove. If $k = 2$ it remains to prove that $(a, \eta_2)$ is normal. In this case, let $(a_2, a_2)$ be the normal form of $a \eta_2$. By definition $\eta_2 > a_2$. We have $g_1 a_2 a_2 = g_1 a_2 = g_1 g_2 w = \eta_0 \eta_2 \eta_2$ so $\nu(g_2 a_2 a_2) > \nu(a_2 a_2)$ so by Lemma 4.6, $a_2 > \omega^2(g_1 a_2 a_2) = \eta_2$. So $a_2 = \eta_2$.

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