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ON HEREDITARY PROPERTIES

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Abstract: A property of a space is called hereditary if each subspace of the space possesses this property. In this paper, we consider some properties which are not hereditary in general and we ask: *when* are they hereditary? A typical result is: a regular space is a Lindelöf p-space hereditarily if and only if the space has a countable base. Some other new results of this kind are obtained.

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hereditary property	space of point-countable type
p-space	Fréchet–Urysohn space

Let P be a topological property and X a topological space. Suppose that each subspace of the space X has P. Assumptions of this kind often prove to be very strong. A number of cases is known in which from a supposition of this type, unexpected corollaries follow nontrivially. It is remarkable that the arguments in these considerations do not have any common features at all. Moreover, they do not fit into any general scheme.

Here are some results of this type.

(1) Each hereditarily-k-space ("very-k-space") is a Fréchet–Urysohn space [2].

(2) Each hereditarily normal diadic bicompactum is metrizable [10, Theorem 25, p. 244].

(3) If a bicompactum X satisfies the countable chain condition (ccc) hereditarily (i.e., $c(Y) \leqslant \aleph_0$ for each $Y \subset X$) then its tightness [6] is countable ($t(X) \leqslant \aleph_0$) (B. Shapirovskiĭ; another proof of this assertion was found independently by Arhangel'skiĭ [6]).

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It seems interesting that the above three assertions are similar in their construction: an assumption that a space X has a property P hereditarily (where P is not always hereditary) turns out to be equivalent to an assumption that X has a much stronger well known property P* which is always inherited by subspaces, i.e., is hereditary.

In the first assertion, the Role of P* is played by the Frechet–Urysohn property, in the second assertion the role of P* is played by metrizability, in the third assertion P* is the condition $t(X) \leqslant \aleph_0$.

This empirical rule will never take the shape of a general theorem, probably, but surely we shall find other interesting assertions of this nature in the future.

In what follows, we take into account regular T_1-spaces only. We put:

(i) $X \in A_p$ if each subspace of the space X is of point-countable type [3, Definition 3.8, p.34].

(ii) $X \in F_p$ if each subspace of the space X is a Lindelöf p-space [4, Definition 1.4, p.57].

(iii) $X \in F_{pp}$ if each subspace of the space X is a paracompact p-space [4, Definition 1.4, p.57].

Let τ be a cardinal number such that $\tau \geqslant \aleph_0$. Let A_τ denote the Alexandroff bicompactification (with one point) of a discrete space of cardinality τ.

The *Souslin number* $c(X)$ of a topological space X is defined as the least upper bound of cardinalities of disjoint systems of open subsets of the space X. This is a well-known topological invariant. It will play an important role in our subsequent considerations. Let us mention that $c(A_\tau) = \tau$ for all $\tau \geqslant \aleph_0$.

We assume that the Continuum Hypothesis holds: $2^{\aleph_0} = \aleph_1$.

Theorem 1. *If $X \in A_p$, then there is a subspace $Y \subset X$ such that the closure $[Y]$ of Y is X, Y is an open subset of the space X and Y is first countable.*

Proof. Let Z be the set of all points at which the space X is first countable, i.e., $Z = \{x \in X: \chi(x, X) \leqslant \aleph_0\}$. We put $Y = \operatorname{Int} Z$. The set Y has all the desired properties. Indeed, we must only verify that $[Y] = X$. Let us suppose the contrary. Then $U = X \setminus [Y]$ is a nonempty open subset of X such that $X \setminus Z$ is dense in U. We define $X' = (X \setminus Z) \cap U$; then $X' \neq \emptyset$. There exists a nonempty bicompactum $\varphi \subset X'$ such that $\chi(\varphi, X') \leqslant \aleph_0$. Observing that X' is dense in U, it follows that $\chi(\varphi, U) \leqslant \aleph_0$. Clearly $\chi(\varphi, X) = \chi(\varphi, U)$. Thus $\chi(\varphi, X) \leqslant \aleph_0$.

Each subspace of the space φ is of point-countable type. Hence φ is a very-k-space [2; 3, Theorem 3.7′, p. 35]. We know that any very-k-space is Fréchet–Urysohn [2]. It was proved in [5] that if $2^{\aleph_1} > 2^{\aleph_0}$, then each bicompact Hausdorff space which is Fréchet–Urysohn satisfies the First Axiom of Countability at some point. Hence $\chi(x, \varphi) \leqslant \aleph_0$ for some $x \in \varphi$. But $\chi(\varphi, X) \leqslant \aleph_0$ and $\chi(x, \varphi) \leqslant \aleph_0$ imply that $\chi(x, X) \leqslant \aleph_0$ [8; 11]. It follows that $x \in Z$, which contradicts the fact that $x \in X' \subset X \setminus Z$. The proof is complete.

Remark 1. Each subspace of the space A_τ is either discrete or bicompact. In any case, each subspace of A_τ is a space of point-countable type. For $\tau > \aleph_0$, the set $\{x \in A_\tau : \chi(x, A_\tau) \leqslant \aleph_0\}$ does not coincide with A_τ.

If $X \in A_{\mathrm{p}}$ and X is separable then each point x of X is contained in a countable subspace of X which is dense in X (of course this subspace is dependent on the point x). But any countable space of point-countable type has a countable base. In addition, X is regular, and so X is first countable. A finer assertion holds.

Theorem 2. *If $X \in A_{\mathrm{p}}$ and $c(X) \leqslant \aleph_0$, then X is first countable.*

Proof. By Theorem 1, there exists an open subspace U of X which is dense in X and satisfies the First Axiom of Countability. Choose any $z \in X$. We will show that $\chi(z, X) \leqslant \aleph_0$ (i.e., X is first countable at z). We may suppose that $z \in X \setminus U$. By Zorn's Lemma, the regularity of X, and the conditions $c(U) \leqslant c(X) \leqslant \aleph_0$ (we take into account that U is open in X), there exists a countable disjoint family γ of open subsets of X such that two more properties are fulfilled:

(j_1). If $V \in \gamma$, then $[V] \subset U \subset X \setminus \{z\}$.

(j_2). $[\bigcup\{V : V \in \gamma\}] = X$.

Let us consider the subspace $X_z = \{z\} \cup \bigcup\{V : V \in \gamma\}$ of the space X. By (j_1), $\{z\} = \bigcap\{X_z \setminus [V] : V \in \gamma\}$ – i.e., the point z is G_δ in X_z. Since X_z is of point-countable type, this implies that X_z is first countable at z. Observing that $[X_z] = X$ and that X is regular, it follows that $\chi(z, X) \leqslant \aleph_0$. The proof is complete.

Remark 2. By a slight change of the above argument, one can show that if $X \in A_{\mathrm{p}}$ and $\chi(x, X) = \tau$ for some $x \in X$, then there is a $Y \subset X$ such that $x \in Y$, x is non-isolated in Y and Y is homeomorphic to A_τ.

Corollary 2.1. *If $X \in A_p$ and $c(X) \leqslant \aleph_0$, then $|X| \leqslant 2^{\aleph_0}$.*

By Theorem 2, X is first countable. Since $c(X) \leqslant \aleph_0$, it follows [16] that $|X| \leqslant 2^{\aleph_0}$.

We have $F_p \subset F_{pp} \subset A_p$ (see [3, Definition 3.8, p. 34] and [4, Theorem 2.6, p. 61]. The condition "$X \in F_p$" is of course much stronger than the condition "$X \in A_p$" – each first-countable space belongs to A_p. In any case, the following results may be unexpexted.

Theorem 3. *$X \in F_p$ if and only if X has countable base.*

We begin the proof of Theorem 3 with three lemmas.

Lemma 1. *If $F \subset X \in F_p$ and F is bicompact, then F is perfectly normal.*

Indeed, a bicompact Hausdorff space is hereditarily Lindelöf if and only if it is perfectly normal [1, Chapter 2, Section 3, no. 12].

Lemma 2. *If $X \in F_p$, $F \subset X$ and F is bicompact, then either $|X| \leqslant \aleph_0$ or $|X| = 2^{\aleph_0}$.*

This follows from Lemma 1 (see [1, Chapter 2, Section 3, Theorem 6]).

Lemma 3. *Let $X \in F_p$ and γ be the family of all bicompact subspaces of X. Then $|\gamma| \leqslant 2^{\aleph_0}$.*

Proof. There exists a perfect map f of the space X onto some separable metrizable space Y (see [4, Theorem 6.1, p. 77]. Let λ be the family of all bicompact subspaces of Y. Clearly $|\lambda| \leqslant 2^{\aleph_0}$. We put $\mu = \{f^{-1}P : P \in \lambda\}$. Then $|\mu| = |\lambda| \leqslant 2^{\aleph_0}$.

Let $\gamma(Q) = \{F \subset X : F \subset Q$ and F is bicompact$\}$ for each $Q \in \mu$. Since f is perfect, each $Q \in \mu$ is bicompact. By Lemma 1, all elements of μ are perfectly normal bicompacta. But for each perfectly normal bicompactum, the cardinality of the family of all bicompact subspaces of this bicompactum does not exceed $2^{\aleph_0}$; we conclude that $|\gamma(Q)| \leqslant 2^{\aleph_0}$ for each $Q \in \mu$.

Let $\gamma = \bigcup \{\gamma(Q) : Q \in \mu\}$. Then $|\gamma| \leqslant 2^{\aleph_0}$. Let F be any bicompact subspace of X. Then $\varphi = fF$ is bicompact and $\varphi \subset Y$. Thus $\varphi \in \lambda$ and $f^{-1}\varphi \in \mu$. Since $F \subset f^{-1}\varphi$, it follows that $F \in \gamma(f^{-1}\varphi) \subset \gamma$.

We have shown that γ is the family of all bicompact subspaces of X. The proof of Lemma 3 is complete.

Lemma 4. *If $X \in F_p$, then there exist $X_1 \subset X$ and $X_2 \subset X$ such that $X = X_1 \cup X_2$ and the following property* (∗) *is fulfilled:*

(∗) *If F is a bicompact subspace of X_1, or F is a bicompact subspace of X_2, then $|F| \leqslant \aleph_0$.*

Indeed, Because of Lemmas 2 and 3 we can apply [15, Chapter 3, Section 40, Theorem 2], where the family of all uncountable bicompact subspaces of X plays the role of M.

Now we are prepared for the proof of Theorem 3.

Proof of Theorem 3. Clearly, every metrizable space with countable base belongs to F_p [4, Theorem 2.3, p. 60]. Let X_1 and X_2 satisfy the conditions of Lemma 4. We fix the perfect maps (which are onto) $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, where Y_1 and Y_2 are some metrizable spaces (such maps exist because X_1 and X_2 are Lindelöf p-spaces (see [4, Theorem 6.1, p. 77]). We may suppose that $Y_1 \cap Y_2 = \emptyset$. Neither X_1 nor X_2 contains uncountable bicompact subspaces. Thus $|f_1^{-1}y| \leqslant \aleph_0$, $|f_2^{-1}y| \leqslant \aleph_0$ for each $y \in Y_1 \cup Y_2$. For each $y \in Y_1$ (resp. $y \in Y_2$), we fix a map of the set N of all natural numbers onto the set $f_1^{-1}y$ (resp. onto $f_2^{-1}y$); of course we do not suppose the map to be one-to-one. The image of $n \in \mathrm{N}$ under this map we denote $x_n(y)$. Let $Z_n^1 = \{x_n(y) : y \in Y_1\}$, $Z_n^2 = \{x_n(y) : y \in Y_2\}$, for all $n \in \mathrm{N}$. Then $X_1 = \bigcup\{Z_n^1 : n \in \mathrm{N}\}$ and $X_2 = \bigcup\{Z_n^2 : n \in \mathrm{N}\}$. The restriction $f_1 | Z_n^1$ of the map f_1 onto Z_n^1 is a one-to-one continuous map of the space Z_n^1 onto some metrizable space. The same holds for f_2 and Z_n^2, for arbitrary $n \in \mathrm{N}$. Since Z_n^1 (resp. Z_n^2) is a Lindelöf p-space, it follows by [4, Theorem 8.2, p. 81], that Z_n^1 (resp. Z_n^2) is metrizable and has a countable base. Since $X = \bigcup\{Z_n^i : n \in \mathrm{N}, i = 1, 2\}$, we conclude that X has a countable network. Hence by [4, Theorem 4.2, p. 64], X has a countable base (note that X is a p-space). The proof of Theorem 3 is complete.

Taking into account the spaces A_τ for $\tau > \aleph_0$, we conclude that not every bicompactum belonging to F_{pp} is metrizable ($A_\tau \in F_{pp}$ for all $\tau > \aleph_0$). All metrizable spaces belong to F_{pp} (see [4, Theorem 2.3, p. 60]). Thus F_{pp} strictly contains the class of all metrizable spaces. Nevertheless, there are good reasons to assert that the properties of spaces belonging to F_{pp} resemble in general the properties of spaces belonging to F_p. In any case, the elements of F_p are characterized among all elements of F_{pp} by a very simple, rough condition.

Theorem 4. *$X \in F_p$ if and only if $X \in F_{pp}$ and $c(X) \leqslant \aleph_0$.*

Proof. ($\Rightarrow$). If $X \in F_p$, then X is hereditarily Lindelöf and $c(X) \leqslant \aleph_0$. Besides, every subspace of X is a paracompact p-space. Hence $X \in F_{pp}$. The necessity is proved.

($\Leftarrow$) Now we suppose that $X \in F_{pp}$ and $c(X) \leqslant \aleph_0$. It is sufficient to prove that each open subspace Y of X is Lindelöf.

Let γ be an open covering of Y. We fix a locally finite open covering λ of the space Y refining γ. By Zorn's Lemma and the local finiteness of λ, there exists a family ξ of open pairwise disjoint subsets of Y such that the set $G = \bigcup \{V : V \in \xi\}$ is dense in Y and each $V \in \xi$ has common points with only finitely many elements of λ. Since G is dense in Y, every element of λ intersects some element of ξ. Also all elements of ξ are open in X. Thus λ is countable. Moreover, Y is Lindelöf. This completes the proof.

Corollary 4.1. *Let Y be a subspace of a perfectly normal bicompactum such that all subspaces of the space Y are* p-*spaces. Then Y is a metrizable space with a countable base.*

It is known [9] that each p-space X is of countable type [3, Definition 3.7, p. 34]. That is, for every bicompact subspace F of X there exists a bicompactum $\varphi \subset X$ such that $\chi(\varphi, X) \leqslant \aleph_0$.

Problem 1. Let X be a perfectly normal bicompactum such that every subspace of X is of countable type. Is X metrizable?

The product of countably many paracompact p-spaces is a paracompact p-space [4, Theorem 2.4, p. 60]. It is obvious (after Theorem 3) that if $X \in F_p$ and $Y \in F_p$, then $X \times Y \in F_p$. But it is easy to verify that $A_\tau \times A_\tau \notin F_{pp}$ when $\tau > \aleph_0$. *

Problem 2. Let $X \in F_{pp}$. Is it true that the weight of X is equal to $c(X)$?

(If the answer is "yes", then we get a good generalization of Theorem 4).

* The last sentence was added in proof.

Problem 3. Let $X \in F_{pp}$ and let $f : X \to Y$ be a perfect onto map. Is it true that $Y \in F_{pp}$?
(See [9; 11; 16; 17]).

Problem 4. Let $X \in F_{pp}$. Is there $Y \subset X$ which is metrizable and dense in X?

Problem 5. What are the simplest criteria of metrizability for spaces belonging to F_{pp}?

Problem 6. By which properties are developable spaces characterized among hereditarily p-spaces?

Problem 7. What else can be said about the structure of spaces belonging to F_{pp}?

We conclude with the following theorem.

Theorem 5. *Let $X = \Pi\ \{X_i : i = 1, 2\ ...\}$ be the topological product of countably many spaces and $X \in A_p$. Then X_i is first countable for all but at most finitely many values of i.*

Proof. By Theorem 1, there exists a nonempty open subset U of X such that $\chi(x, X) \leqslant \aleph_0$ for all $x \in U$. We may choose U to be a standard basic set of the product topology : $U = \Pi\ \{U_i : i = 1, 2...\}$, where U_i is an open subset of X_i and $U_i = X_i$ for all but at most finitely many values of i. Clearly, U contains a topological copy of X_i if $U_i = X_i$. Since U is first countable, it follows that if $U_i = X_i$, then X_i is first countable. This completes the proof.

Corollary 5.1. *X is first countable if and only if each subspace of the topological product $X^{\aleph_0}$ of countably many topological copies of X is of point countable type (that is, if $X^{\aleph_0} \in A_p$).*

References

[1] P.S. Aleksandroff and P.S. Urysohn, On bicompact topological spaces, Tr. Mat. Inst. Akad. Nauk SSSR 31 (1950) (in Russian).*

[2] A. Arhangel'skiĭ, A characterization of very-k-spaces, Czech. Math. J. 18 (93) (1968).

[3] A.V. Arhangel'skiĭ, Bicompact sets and topology of spaces, Tr. Mosk. Mat. Soc. 13 (1965) 3–55 (in Russian).*

[4] A.V. Arhangel'skiĭ, On a class of spaces which contains all metric spaces and all locally bicompact spaces, Mat. Sb. 67 (1965) 55–88 (in Russian).*
[5] A.V. Arhangel'skiĭ, The Souslin number and cardinality; characters of points in sequential bicompact spaces, Dokl. Akad. Nauk SSSR 192 (1970) 255–258 (in Russian).*
[6] A.V. Arhangel'skiĭ, On bicompacta satisfying the Souslin condition hereditarily. The tightness and the free sequences, Dokl. Akad. Nauk SSSR 199 (1971) 1227–1230 (in Russian).*
[7] A.V. Arhangel'skiĭ and V.I. Ponomarev, On dynamic bicompacta, Dokl. Akad. Nauk SSSR 182 (1968) 993–996 (in Russian; English Transl.: Soviet Math. Dokl. 9 (1968) 1220–1224).
[8] M.M. Čoban, On the behaviour of metrizability under factor s-mappings, Dokl. Akad Nauk SSSR 166 (1966) 562–565 (in Russian).*
[9] M.M. Čoban, Perfect mappings and spaces of countable type, Vestn. Mosk. Gos. Univ. (6) (1967) 87–93.
[10] B.A. Efimov, Dyadic bicompacta, Tr. Mosk. Mat. Soc. 14 (1965) 211–247 (in Russian).*
[11] V.V. Filippov, On the perfect images of paracompact p-spaces, Dokl. Akad. Nauk SSSR 176 (1967) 533–535 (in Russian).*
[12] T. Ishii, On closed mappings and M-spaces, I, II, Proc. Japan Acad. 43 (1967) 752–756, 757–761.
[13] I. Juhász and A. Hajnal, Discrete subspaces of topological spaces, Proc. Koninkl. Ned. Akad. Wetensch. Ser. A 70 (1967) 343–356.
[14] J.L. Kelley, General topology (Van Nostrand, Princeton, N.J., 1955).
[15] K. Kuratowski, Topology, Vol. 1 (Academic Press, New York, 1966).
[16] K. Morita, Some properties of M-spaces, Proc. Japan Acad. 43 (1967) 869–872.
[17] A.H. Stone, Metrizability of decomposition spaces, Proc. Am. Math. Soc. 7 (1956) 690–700.

* The translation of this article into English was also issued.