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The Longest Cycles in a Graph G with Minimum Degree at Least $|G|/k$

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N. Alon [*J. Graph Theory* **10** (1986), 123–127] proved that if the minimum degree of a graph G is not less than $\lceil |V(G)|/k \rceil$, then the circumference of G is at least $\lfloor |V(G)|/(k-1) \rfloor$. In this article, we show a stronger theorem: if $d_G(x)+d_G(y) \geqslant \lceil 2|V(G)|/k \rceil$ for any nonadjacent distinct vertices x and y of $V(G)$, then G has a cycle of length at least $\lceil |V(G)|/(k-1) \rceil$. (The lower bound $\lceil |V(G)|/(k-1) \rceil$ is sharp.)

1. INTRODUCTION

In this article we consider only finite simple graphs. We denote the set of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively. We often write $|G|$ for $|V(G)|$. An edge joining x and y is represented as xy. We write $d_G(v)$ for the degree of a vertex v in G and $\Gamma_G(v)$ for the set of the neighbors of v (the subscript G is omitted when not necessary). Let $c(G)$ mean the circumference of G, i.e., the size of a largest cycle of G. Let $\delta(G)$ denote the minimum degree of G and

$$\bar{\delta}(G) := \begin{cases} 2(n-1) & \text{(if } G \text{ is a complete graph of order } n) \\ \min\{d_G(x)+d_G(y) \mid x, y \in V(G), xy \notin E(G)\} & \text{(otherwise).} \end{cases}$$

Other terminologies or notation not defined here can be found in [3].

Let G be a graph of order n and let k be an integer with $n > k \geqslant 2$. N. Alon proved the following theorem.

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THEOREM 1.1 (Alon [1]). *Suppose that* $\delta(G) \geq \lceil n/k \rceil$. *Then* $c(G) \geq \lfloor n/(k-1) \rfloor$.

He conjectured that the proposition obtained by substituting ceiling for floor brackets in the conclusion of Theorem 1.1 is also true. As the commented in his paper, this conjecture follows if $|V(G) \gg k$ (we have verified that his arguent in the proof of Theorem 1.1 works for this conjecture as well if $|V(G)| \geq 2k(k-1)$). But the problem seemed difficult in the case where $|V(G)|$ is small.

We have solved this conjecture affirmatively and obtained the following Ore-type version of it:

THEOREM 1.2. *If* $\bar{\delta}(G) \geq \lceil 2n/k \rceil$, *then* $c(G) \geq \lceil n/(k-1) \rceil$.

In the case where the theorem is "meaningful," i.e., if $\lceil n/k \rceil < \lceil n/(k-1) \rceil$, our new bound $\lceil n/(k-1) \rceil$ is indeed best possible. For in that case, if we write $n = (k-1)r + s$ with $1 \leq s \leq k-1$ (r and s are integers), then the graph $H = (s-1)K_{r+1} \cup (K_1 + (k-s)K_r)$ satisfies $\delta(H) \geq \lceil n/k \rceil$ and attains the equality $c(H) = \lceil n/(k-1) \rceil$.

2. LEMMAS

LEMMA 2.1 (Bermond [2]). *Let G be a 2-connected non-hamiltonian graph. Then* $c(G) \geq \bar{\delta}(G)$.

LEMMA 2.2. $c(G) \geq \bar{\delta}(G)/2 + 1$.

Proof. Let K be a connected component of G of maximum order, and let P be a maximal path of K. The endvertices of P are denoted by x and y. Note that all the neighbors of x (and y) lie on the path. If $xy \in E(G)$, then K is hamiltonian. Since $\bar{\delta}(G) \leq 2(|K|-1)$ by definition, the lemma is true when $xy \in E(G)$. Suppose $xy \notin E(G)$. Without loss of generality, we may assume that $d(x) \geq d(y)$, which means $d(x) \geq \bar{\delta}(G)/2$. Because a cycle of length at least $\bar{\delta}(G)/2 + 1$ can be easily constructed by using one of the edges incident to x, the lemma is also true in this case as well. ∎

LEMMA 2.3. *Let H be a 2-connected graph and let A be a nonempty subset of* $V(H)$. *Suppose* $d_H(x) + d_H(y) \geq l$ *for any two nonadjacent (in H) vertices* x, y *of* $V(H) - A$. *Then one of the following statements holds*:

(i) $c(H) \geq l - |A| + 1$,

(ii) *there exists a cycle of H visiting all the vertices of* $V(H) - A$ *and at least one vertex of A*.

Proof. If $|V(H)-A| \leqslant 1$, it is obvious that (ii) holds, since H is 2-connected. Thus we may assume $|V(H)-A| \geqslant 2$. Let P be a path such that its endvertices belong to $V(H)-A$ and such that P contains at least one vertex of A. We choose P so that the number of the vertices of $V(H)-A$ contained in P may be maximum. We represent the vertices of P as $a_0, a_1, ..., a_s$ and suppose P visits them in this order.

First of all, we will prove that if H has a cycle containing all the vertices of P, (ii) follows. Let C be such a cycle. It is clear that C contains at least one vertex of A. Thus we only have to show that all the vertices of $V(H)-A$ are contained in C. Suppose not. Then there exists a vertex $v \in V(H)-A$ and v is not contained in C. Since H is connected, H has a path Q linking v and $V(C)$. Let u be a terminus of Q in $V(C)$ and let u^+, u^- be vertices neighboring u on C. If one of u^+ or u^- belongs to $V(H)-A$ (we may assume that u^+ belongs to $V(H)-A$), the path formed by concatenating $C-uu^+$ and Q has more vertices of $V(H)-A$ than P; this is a contradiction. Suppose both belong to A. We denote by w the vertex of $V(H)-A$ that we meet when we start from u, visit u^+, and proceed along C in this direction. Then the path obtained by concatenating Q and $u, u^-, ..., w$ has more vertices of $V(H)-A$ than P; we again have a contradiction. Therefore, if H has a cycle visiting all the vertices of P, the lemma is true.

Next we will show that we may assume $|\Gamma(a_0) \cap V(P)| + |\Gamma(a_s) \cap V(P)| \geqslant l-(|A|-1)$. We may assume that $a_0 a_s \notin E(H)$; otherwise H has a cycle which visits all the vertices of P. Thus $d(a_0)+d(a_s) \geqslant l$. Similarly, we may also assume that a_0 and a_s are not adjacent to the same vertex of $V(H)-V(P)$. Since the maximality of $|V(P)-A|$ implies $(\Gamma(a_0) \cup \Gamma(a_s))-V(P) \subseteq A$ and since P contains at least one vertex of A, we have

$$|\Gamma(a_0)-V(P)|+|\Gamma(a_s)-V(P)| = |(\Gamma(a_0) \cup \Gamma(a_s))-V(P)| \leqslant |A-V(P)| \leqslant |A|-1.$$

Therefore, $|\Gamma(a_0) \cap V(P)| + |\Gamma(a_s) \cap V(P)| \geqslant l-(|A|-1)$.

The rest of the argument corresponds almost word for word to that in [4, problem 10.27, p. 68], but we will describe the outline for the convenience of the reader. We define integers k and h by

$$k = \max\{\alpha \mid a_0 a_\alpha \in E(H)\}$$
$$h = \min\{\beta \mid a_\beta a_s \in E(H)\}.$$

If $k \leqslant h$, then at most one vertex on P is adjacent to both a_0 and a_s. Since we can show, by vrtue of the 2-connectedness of H, that H has a cycle containing a_0, a_s, and all the vertices on C adjacent to a_0 or a_s, (i) follows in this case. Suppose that $k > h$. We can again construct a cycle C' containing

a_0, a_s, and all the vertices on P adjacent to a_0 or a_s. Now that we have already settled the case where H has a cycle visiting all the vertices of P, we may assume that if $a_0 a_j \in E(H)$ for some j, $a_{j-1} a_s \notin E(H)$. Thus in C' there exist at least as many vertices, including a_0 and a_s, nonadjacent to a_s as adjacent to a_0. Therefore, $|C'| \geqslant l-(|A|-1)$, and the proof of the lemma is complete. ▌

3. Proof of Theorem 1.2

Suppose that the theorem is false and G is a counterexample. We choose G edge-maximal, that is, for any nonadjacent vertices $x, y \in V(G)$, $c(G+xy) \geqslant \lceil n/(k-1) \rceil$.

First we claim that G is connected; otherwise, we can add some bridges to G so that the resulting graph may be connected and may satisfy the condition of the theorem. Since the cycles in the new graph are also the ones in G, the new graph does not have a cycle of length not less than $\lceil n/(k-1) \rceil$. This contradicts the maximality of G.

Next, we will show that $n > k(k-1)$. Lemma 2.2 implies that $c(G) \geqslant \lceil \lceil 2n/k \rceil /2 \rceil + 1 = \lceil n/k \rceil + 1$. Since G is a counterexample, $\lceil n/k \rceil + 1 < \lceil n/(k-1) \rceil$ holds. Therefore,

$$\frac{n}{k} + 1 < \frac{n}{k-1}.$$

Hence,

$$n > k(k-1).$$

A vertex of G is called *internal* if v is not a cutvertex of G. Thus all the neighbors of an internal vertex are contained in the block to which the vertex belongs. We call a block of G *essential* if it has an internal vertex; otherwise we call it *nonessential.* Any vertex of a nonessential block is a cutvertex of G.

We claim that G does not have more than $k-1$ essential blocks. Let s denote the number of the essential blocks of G and let $B_1, B_2, \ldots, B_s$ be the essential blocks. Then,

$$\begin{aligned} n &\geqslant \left| \bigcup_{i=1}^{s} V(B_i) \right| \\ &\geqslant \sum_{i=1}^{s} |B_i| - (s-1). \end{aligned}$$

Note that internal vertices belonging to different essential blocks are not adjacent to each other. Thus the average order of an essential block is not less than $n/k+1$. Therefore,

$$\sum_{i=1}^{s} |B_i| - (s-1) \geqslant \left(\frac{n}{k}+1\right)s-(s-1)$$
$$=\frac{n}{k}s+1.$$

If $s \geqslant k$, we have a contradiction. Thus we have $s \leqslant k-1$.

We classify the vertices of G into three parts as follows:

$C_0 := \{v \mid \in V(G),\ v$ is contained in no essential blocks of $G\}$,

$C_1 := \{v \mid v \in V(G),\ v$ is contained in just one essential block of $G\}$,

$C_2 := \{v \mid v \in V(G),\ v$ is contained in at least two essential blocks of $G\}$.

Since G has at most $k-1$ essential blocks, $|C_2| \leqslant k-2$ holds.

We will show that $C_0 = \phi$. Suppose there exists a vertex $v \in C_0$. There are at least two nonessential blocks B' and B'' sharing v. Since G is edge-maximal, $c(G+uw) \geqslant \lceil n/(k-1) \rceil > k$ for any vertices $u \in V(B') - \{v\}$ and $w \in V(B'') - \{v\}$. This means $|V(B') \cup V(B'') - \{v\}| \geqslant k$, which implies that G has at least k endblocks. However, this is impossible because any endblock is essential. Thus we have $C_0 = \phi$.

For each essential block B, we define an integer $f(B)$ by

$$f(B) = \begin{cases} |V(B)| & (\text{if } B \cap C_2 = \phi) \\ |V(B) - C_2| + 1 & (\text{otherwise}). \end{cases}$$

Let $B_1, B_2, ..., B_s$ be the essential blocks of G. Then, $\sum_{i=1}^{s} f(B_i) \geqslant n$ holds because in the left-hand side of this inequality each vertex of C_1 is counted once and each of C_2 at least once. Thus there exists an essential block B that satisfies

$$f(B) \geqslant \left\lceil \frac{n}{k-1} \right\rceil.$$

Since

$$|B| \geqslant f(B) \geqslant \frac{n}{k-1} > k \geqslant 2,$$

B is 2-connected (i.e., not an acyclic block). Let $t = |V(B) - C_2|$. Then since $|V(B) \cap C_2| \leqslant |C_2|k-2$, we have

$$t = |B| - |V(B) \cap C_2| > k - (k-2) = 2.$$

Let $x_1, x_2, ..., x_t$ be the vertices of $V(B) - C_2$, and let

$$b_i = |\{v \mid v \in V(G) - V(B), vx_i \in E(G)\}|.$$

We arrange the vertices of $V(B) - C_2$ so that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_t$ holds. Then $d_B(x) + d_B(y) \geqslant 2n/k - (b_1 + b_2)$ for any distinct nonadjacent vertices x and y of $V(B) - C_2$. Note that if $u \in V(B) - C_2$ has a neighbor v outside B, then v is a cutvertex of G, for $u \in C_1$. Since each of such cutvertices v as well as each vertex in $V(B) \cap C_2$, "gives rise to" at least one endblock, and since any endblock is essential, we obtain

$$\begin{aligned} k - 2 &\geqslant \text{(the number of essential blocks other than } B) \\ &\geqslant |V(B) \cap C_2| + \sum_{i=1}^{t} b_i \\ &\geqslant |V(B) \cap C_2| + (b_1 + b_2). \end{aligned}$$

If $V(B) \cap C_2 = \phi$, then for any distinct nonadjacent vertices x and y,

$$\begin{aligned} d_B(x) + d_B(y) &\geqslant \frac{2n}{k} - (b_1 + b_2) \\ &\geqslant \frac{2n}{k} - (k-2) \end{aligned}$$

holds. Since B is 2-connected and

$$\begin{aligned} \frac{2n}{k} - (k-2) - \frac{n}{k-1} &\geqslant \frac{1}{k(k-1)} \{2(k-1)n - k(k-1)(k-2) - kn\} \\ &= \frac{k-2}{k(k-1)} \{n - k(k-1)\} \geqslant 0, \end{aligned}$$

Lemma 2.1 implies that G has a cycle of length not less than $\lceil n/(k-1) \rceil$ (this conclusion obviously holds in the case where B is hamiltonian).

We may now assume that $V(B) \cap C_2 \neq \phi$. We apply Lemma 2.3 with $H = B$ and $A = V(B) \cap C_2$. If (ii) of the conclusion of the lemma holds, B has a cycle of length at least $f(B)$. Since $f(B) \geqslant \lceil n/(k-1) \rceil$, we have a contradiction. If (i) holds, we have a cycle of length at least

$$\begin{aligned} \frac{2n}{k} - (b_1 + b_2) - \{(k-2) - (b_1 + b_2)\} + 1 &= \frac{2n}{k} - (k-2) + 1 \\ &> \frac{n}{k-1}. \end{aligned}$$

This completes the proof of Theorem 1.2.

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