Ill-posedness of the 3D-Navier–Stokes equations in a generalized Besov space near $BMO^{-1}$

Tsuyoshi Yoneda

Institute for Mathematics and its Applications, University of Minnesota, 114 Lind Hall 207 Church Street S.E., Minneapolis, MN 55455, USA

Received 18 September 2009; accepted 2 February 2010

Available online 11 February 2010

Communicated by J. Bourgain

Abstract

The ill-posedness of the 3D-Navier–Stokes equations in a generalized Besov space which is smaller than $B_{\infty,q}^{-1} (q > 2)$ is considered. In 2008, Bourgain–Pavlović proved that the 3D-Navier–Stokes equation is ill-posed in $B_{\infty,\infty}^{-1}$ by showing norm inflation phenomena of the solution for some initial data. On the other hand, in 2008, Germain proved that the flow map is not $C^2$ in the space $B_{\infty,q}^{-1}$ for $q > 2$. However he did not treat ill-posed problem in such spaces. Thus our result is an extension of these previous results.

© 2010 Elsevier Inc. All rights reserved.

Keywords: Navier–Stokes equations; Ill-posedness; Besov spaces

1. Introduction

We consider the nonstationary incompressible Navier–Stokes equations in $\mathbb{R}^3$:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \quad \text{div } u = 0 \quad \text{in } x \in \mathbb{R}^3, \ t \in (0, T), \\
\left. u \right|_{t=0} &= u_0,
\end{align*}
$$

(1)

where $u = u(t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(t) = p(x,t)$ denote the velocity vector field and the pressure of fluid at the point $(x,t) \in \mathbb{R}^3 \times (0, T)$, respectively, while $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$ is a given initial velocity vector field.
In this paper we are concerned with the ill-posedness of the Cauchy problem for (1). More precisely for a given function space $X = X(\mathbb{R}^3)$ we say that the Cauchy problem is well-posed in $X$ if there exists a space $Y \subset C([0, T), X)$ such that for all $u_0 \in X$ there exists a unique solution $u \in Y$ for (1), and the flow map $u_0 \mapsto u = \Phi(u_0)$ is continuous from $X$ to $C([0, T), X)$. Also we say that the Cauchy problem is ill-posed in $X$ if it is not. The classical results on the existence theorem of the mild solution were shown by Kato [9] and Giga and Miyakawa [7]. Making use of the iteration procedure, they constructed a global solution in the class $C([0, \infty); L^n(\mathbb{R}^n)) \cap C((0, \infty); L^p(\mathbb{R}^n))$ for $n < p \leq \infty$, when an initial data $u_0$ is small enough in $L^n(\mathbb{R}^n)$. To construct a solution in more general classes of initial data is very important problem. Giga and Miyakawa [8], Kato [10] and Taylor [20] proved the well-posedness in certain Morrey spaces. Cannone [3] and Kozono and Yamazaki [13] investigated this problem in Besov spaces. In particular, Koch and Tataru [12] obtained the global solvability for (1), when the initial data $u_0$ is small enough in $\dot{B}^{0,\infty}_{1,1}$. $\dot{B}^{0,\infty}_{1,1}$ includes above function spaces and it has been considered as the largest space of initial data (see Lemarié-Rieusset [14]). On the other hand, Montgomery-Smith [16] introduced an equation similar to Navier–Stokes equation and proved ill-posedness in the Besov space $B^{-1}_{\infty,\infty}$, which is larger than $\dot{B}^{-1}_{1,1}$. In 2008, Bourgain and Pavlović [2] showed that (1) is ill-posed in $B^{-1}_{\infty,\infty}$ by showing norm inflation phenomena of the solution for some initial data. More precisely, they proved that for any $\delta > 0$ there exists an initial data $u_0$ with $\|u_0\|_{B^{-1}_{\infty,\infty}} < \delta$ such that the corresponding solution $u$ satisfies $\|u(t)\|_{B^{-1}_{\infty,\infty}} > 1/\delta$ for some $t < \delta$. This shows that the flow map $\Phi$ is not continuous. On the other hand, Germain [4] proved that the flow map is not $C^2$ in the Besov spaces $B^{-1}_{\infty,q}$ for $q > 2$. However he did not treat ill-posed problem in such spaces. The purpose of the present paper is to show ill-posedness of 3D-Navier–Stokes equations in a generalized Besov space $V$ which is strictly smaller than $B^{-1}_{\infty,q}$ ($q > 2$). Thus our result is an extension of both Bourgain–Pavlović’s and Germain’s results.

Now we give a sketch of the proof. First we introduce a generalized Besov space $V$ which is smaller than $B^{-1}_{\infty,q}$ ($q > 2$). The idea is proposed by [18]. Second, we introduce initial data which is composed by a sum of $r$ cosine functions. The idea of setting of the initial data is proposed by [2] and [4]. We take a lacunary frequency set, and the norm of initial data in $V$ is controlled by $r$. Third, we extract an inflation term from second approximation. Fourth, we estimate the remainder term $y$. The remainder term satisfies certain integral equation composed by first and second approximations including an inflation term. We also control the remainder term by $r$. Since we set refined initial data from Bourgain–Pavlović’s setting, we can get better estimate of second approximation than their estimate. According to their setting of initial data, using $\dot{B}^{-1}_{1,1}$ norm to estimate remainder term $y$ is important. Since we got better estimate of second approximation, we can use the bilinear estimate of a class of bounded uniformly continuous functions (equipped with the $L^\infty$ norm).

2. Preliminaries

Before presenting our results, we define the Besov spaces, a generalized Besov space which is smaller than $\dot{B}^{-1}_{\infty,q}$ ($q > 2$), the bilinear operator, $j$-th approximation and the remainder term.

Now, we recall the Littlewood–Paley decomposition $\psi, \varphi_j \in \mathcal{S}$, $j = 0, 1, \ldots$, such that

$$\text{supp } \hat{\psi} \subset \{|\xi| \leq 5/6\}, \quad \text{supp } \hat{\psi} \subset \{3/5 \leq |\xi| \leq 5/3\}, \quad \varphi_j(x) = 2^{nj}\varphi(2^jx),$$
\[ 1 = \hat{\psi}(\xi) + \sum_{j=0}^{\infty} \hat{\varphi}_j(\xi) \quad (\xi \in \mathbb{R}^n), \]

\[ 1 = \sum_{j=-\infty}^{\infty} \hat{\varphi}_j(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\}). \]  

(2)

where \( \hat{f} \) denotes the Fourier transform of \( f \).

**Definition 1.** (See Besov space cf. [1,19].) The inhomogeneous and homogeneous Besov spaces \( B_{p,q}^s \) and \( \dot{B}_{p,q}^s \) are defined as follows:

\[ B_{p,q}^s \equiv \{ f \in \mathcal{S}' ; \| f \|_{B_{p,q}^s} < \infty \}, \quad \dot{B}_{p,q}^s \equiv \{ f \in \mathcal{S}' ; \| f \|_{\dot{B}_{p,q}^s} < \infty \}, \]

where

\[ \| f \|_{B_{p,q}^s} = \| \psi * f \|_p + \left( \sum_{j=0}^{\infty} 2^{js} \| \varphi_j * f \|_p^q \right)^{1/q}, \]

\[ \| f \|_{\dot{B}_{p,q}^s} = \left( \sum_{j=-\infty}^{\infty} 2^{js} \| \varphi_j * f \|_p^q \right)^{1/q} \]

for \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \).

Note that

\[ \left\{ f \in \mathcal{S}'; \| f \|_{\dot{B}_{p,q}^s} < \infty, \ f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}' \right\} \cong \dot{B}_{p,q}^s / \mathcal{P}, \]  

(3)

holds if

\[ s < n/p, \quad \text{or} \quad s = n/p \text{ and } q = 1. \]  

(4)

For details, see [13]. Here, \( \mathcal{P} \) denotes the set of all polynomials. Hence, when \( s, p, \) and \( q \) satisfy (4), we may modify the definition of homogeneous Besov space as

\[ \dot{B}_{p,q}^s \equiv \left\{ f \in \mathcal{S}' ; \| f \|_{\dot{B}_{p,q}^s} < \infty, \ f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}' \right\}. \]  

(5)

Hereinafter we use (5) as the definition of \( \dot{B}_{p,q}^s \) when \( s, p, \) and \( q \) satisfy (4). Then, if \( s, p, \) and \( q \) satisfy (4), \( \dot{B}_{p,q}^s \) is a Banach space and

\[ \| f \|_{\dot{B}_{p,q}^s} = 0 \quad \text{if and only if} \quad f = 0 \text{ in } \mathcal{S}'. \]

We now define a generalized Besov space \( V \) which is bigger than \( B_{-1}^{-1} \) but smaller than \( B_{-1}^{-1} \) with \( q > 2 \). The idea of the definition is based on [18].
Definition 2 (A generalized Besov space $V$).

(1) $\dot{B}^{s,\alpha}_{\infty,\infty}$ denotes the set of all $f \in \mathcal{S}'/\mathcal{P}$ for which the norm

$$\|f\|_{\dot{B}^{s,\alpha}_{\infty,\infty}} = \sup_{j \in \mathbb{Z}} 2^{js} (|j| + 1)^\alpha \|\varphi_j * f\|_\infty < \infty. \quad (6)$$

(2) $V := \dot{B}^{-1,1/2}_{\infty,2} + \dot{B}^{-1,1/2}_{\infty,2}$ is the set of all $f \in \mathcal{S}'/\mathcal{P}$ that is written as a sum $f = f_1 + f_2$, where $f_1 \in \dot{B}^{-1,1/2}_{\infty,2}$ and $f_2 \in \dot{B}^{-1,1/2}_{\infty,2}$. The norm is defined by

$$\|f\|_V := \|f\|_{\dot{B}^{-1,1/2}_{\infty,2} + \dot{B}^{-1,1/2}_{\infty,2}} = \inf_{f = f_1 + f_2} \|f_1\|_{\dot{B}^{-1,1/2}_{\infty,2}} + \|f_2\|_{\dot{B}^{-1,1/2}_{\infty,2}} \quad (7)$$

for $f \in \dot{B}^{-1,1/2}_{\infty,2} + \dot{B}^{-1,1/2}_{\infty,2}$, where $f_1, f_2$ run over all admissible representation $f = f_1 + f_2$ with $f_1 \in \dot{B}^{-1,1/2}_{\infty,2}$ and $f_2 \in \dot{B}^{-1,1/2}_{\infty,2}$.

Let us investigate the above generalized Besov space. The following proposition is also based on [18].

Proposition 3.

(1) $\dot{B}^{-1,2}_{\infty,2} \subset V \subset \dot{B}^{-1}_{\infty,q}$ for all $q > 2$.

(2) There is no inclusion relationship between $\dot{B}^{-1}_{\infty,2}$ and $\dot{B}^{-1,1/2}_{\infty,\infty}$.

Proof of Proposition 3. (1) is easy to check from the definition of the norm. So let us consider (2). Let $\delta_z$ be the Dirac delta function massed at $z \in \mathbb{R}^3$. Define

$$f = \sum_{j=1}^{\infty} a_j \delta_{2^j} \quad \text{for } \{a_j\}_{j=1}^{\infty} \subset \mathbb{R}. \quad (8)$$

Then we have

$$\|f\|_{\dot{B}^{-1,2}_{\infty,2}} \simeq \left(\sum_{j \in \mathbb{Z}} 2^{-2j} |a_j|^2\right)^{1/2}, \quad \|f\|_{\dot{B}^{-1,1/2}_{\infty,\infty}} \simeq \sup_{j \in \mathbb{Z}} 2^{-j} \sqrt{j+1} |a_j|. \quad (9)$$

So if we take $a_j = \frac{2^j}{\sqrt{j+1}}$ for $j \geq 0$, $a_j = 0$ for $j < 0$, then $\|f\|_{\dot{B}^{-1,1/2}_{\infty,\infty}} \simeq 1$, $\|f\|_{\dot{B}^{-1,2}_{\infty,2}} = \infty$. Therefore, $\dot{B}^{-1,2}_{\infty,2}$ is not included in $\dot{B}^{-1,1/2}_{\infty,\infty}$.

Let $\delta_{jk}$ be Kronecker’s delta. For fixed $k \in \mathbb{N}$, if we take $a_j = \frac{\delta_{jk} 2^j}{\sqrt{j+1}}$ for $j \geq 0$, $a_j = 0$ for $j < 0$, then we have $\|f\|_{\dot{B}^{-1,1/2}_{\infty,\infty}} \simeq 1$, $\|f\|_{\dot{B}^{-1,2}_{\infty,2}} = \frac{1}{k}$. Since $k$ is arbitrary, $\dot{B}^{-1,1/2}_{\infty,\infty}$ is not included in $\dot{B}^{-1,2}_{\infty,2}$. \(\square\)

Now we define the bilinear operator and $j$-th approximation of the solution to the Navier–Stokes equations.
Definition 4 (Bilinear operator). For \( w_1, w_2 \in (L^1(0, T : L^\infty))^3 \) with \( w_1 \otimes w_2 \in (L^1(0, T : L^\infty))^{3 \times 3} \), let

\[
B(w_1, w_2) := \int_0^t \nabla \cdot e^{(t-\tau)\Delta} P(w_1(\tau) \otimes w_2(\tau)) \, d\tau,
\]

where \( P \) is the Helmholtz projection.

The bilinear estimate is important to consider ill-posed problem. According to [2], they used the bilinear estimate in \( BMO^{-1} \) provided by Koch and Tataru [12]. In this paper we use the following estimate. See [5,15] for example (for elementary proof, see [6]).

Proposition 5. There exists a constant \( c > 0 \) such that

\[
\| \nabla e^{t\Delta} P f \|_\infty \leq ct^{-1/2} \| f \|_\infty \quad \text{for } t > 0, \ f \in L^\infty.
\]

Corollary 6. Let \( w_1, w_2 \in (L^1(0, T : L^\infty))^3 \) be such that \( w_1 \otimes w_2 \in (L^1(0, T : L^\infty))^{3 \times 3} \). Then we have the following estimate:

\[
\| B(w_1, w_2) \|_\infty \leq \int_0^t \frac{C}{(t-\tau)^{1/2}} \| w_1(\tau) \|_\infty \| w_2(\tau) \|_\infty \, d\tau.
\]

Now we define \( j \)-th approximation of the solution \( u \), and the remainder term \( y \).

Definition 7. Let

\[
\begin{align*}
  u_1 &= u_1(t) := e^{t\Delta} u_0, \\
  u_j &= u_j(t) := \sum_{k_1+k_2=j, 1 \leq k_1, k_2 \leq j-1} B(u_{k_1}, u_{k_2}) \quad \text{for } j \geq 2, \\
  y &= y(t) := \sum_{k \geq 3} u_k.
\end{align*}
\]

For example, \( u_2 = B(u_1, u_1) \) and \( u_3 = B(u_1, u_2) + B(u_2, u_1) \).

Remark 8. By a formal calculation, we see that \( u = u_1 + u_2 + y \). Moreover, the remainder term satisfies the following integral equation:

\[
y = B(y, y) + B(y, u_2 + u_1) + B(u_2 + u_1, y) + B(u_2, u_2) + B(u_2, u_1) + B(u_1, u_2) \quad (10)
\]
on \((0, \infty)\) with the initial condition \( y(0) = 0 \).

Throughout this paper, we only treat periodic functions, since the following embedding inequality is necessary for (13). Let \( BUC \) be the space of all bounded uniformly continuous functions equipped with the \( L^\infty \)-norm.
Remark 9. A fundamental observation shows that for $L > 0$, if an initial data $u_0 \in BUC$ is a periodic function in $[0, L)^3$ and its mean value is zero, then the solution $u(t) \in BUC$ $(t > 0)$ to Eq. (1) is also periodic in $[0, L)^3$ and its mean value is also zero. Moreover we have the following embedding inequality:

$$\|u(t)\|_V \leq C_L \|u(t)\|_\infty$$  \hspace{1cm} (11)

for $t > 0$.

According to the above remark, homogeneous and inhomogeneous type function spaces are equivalent in this paper. We always denote by $C > 0$ universal constants unless no confusion occurs.

3. Main result

Recall that $BUC$ is the space of all bounded uniformly continuous functions equipped with the $L^\infty$-norm. The main result is as follows.

Theorem 10. For any $\delta > 0$, there exist real-valued initial data $u_0 \in BUC$ with $\text{div } u_0 = 0$ in $S'$ and $\|u_0\|_V < \delta$ such that the corresponding solution $u$ exists in $C([0, T) : BUC)$ $(T < \delta)$ with $\|u(T)\|_V > 1/\delta$.

Remark 11. The same result is true if we replace $V$ by $\dot{B}^{-1}_{\infty,q}$ or $\dot{F}^{-1}_{\infty,q}$ (Triebel–Lizorkin spaces) for $q > 2$. The proof is quite similar to the case of $V$. Thus we omit its detail.

Remark 12. The solution $(u, p)$ is unique in $L^\infty((0, T) \times \mathbb{R}^3) \times L^1_{\text{loc}}((0, T) : \mathcal{L}_{1,\phi})$, where $\mathcal{L}_{1,\phi}$ is generalized Campanato spaces which include $\text{BMO}$ (see [11,17]).

Proof of Theorem 10. First, we set initial data which bring norm inflation phenomena. In order to set initial data, we need several definitions. For sufficiently small $\epsilon > 0$, let $\Gamma_1, \Gamma_2 : \mathbb{N} \mapsto \mathbb{R}$ be such that

$$\Gamma_1(m) := \sum_{s=1}^{m} s^{-1}, \quad \Gamma_2(m) := \Gamma_1^{1/\epsilon}(m)$$

for $m \in \mathbb{N}$. We take sufficiently small $C_1 = C_1(\epsilon) > 0$ and sufficiently large $C_2 = C_2(\epsilon) > 0$, and let us take $\Gamma_3 : \mathbb{N} \mapsto \mathbb{N}$ satisfying

$$C_1 \Gamma_1^{-3\epsilon}(m) \leq \Gamma_3(m) \leq C_2 \Gamma_1^{-3\epsilon}(m)$$

for $m \in \mathbb{N}$. Let us set coefficients $v^0 := (0, 0, 1)$ and $v^1 := (0, 1, 0)$, and let $\{k_s^0\}_{s=1}^r \subset \mathbb{N}$ and $\{k_s^1\}_{s=1}^r \subset \mathbb{N}$ be frequency sets satisfying the following property,

$$k_s^0 := 2^{3s} T^{-1/2} a_0, \quad k_s^1 := k_s^0 + a_1 \quad (s = 1, \ldots, r),$$

where $a_0 = (1, 0, 0)$, $a_1 = (0, 0, 1)$ and $T = (1/\Gamma_3^2(r))$ for $r \in \mathbb{N}$. The constant $T$ with $r \gg 1$ will be the existence time. To obtain the embedding inequality (11), we require $\Gamma_3$ to be integer-valued.
Remark 13. It is easy to see that $|k_0^s| \geq 4 \sum_{s'=1}^{s-1} |k_0^{s'}|$, $|k_0^1|^2 = 64T^{-1}$, $1 \leq |k_0^s| \leq |k_1^s| \leq 2|k_0^s|$ for $s = 1, \ldots, r$, and

$$
\begin{align*}
& a^0 \cdot v^0 = 0, \\
& a^1 \cdot v^0 = 1, \\
& a^0 \cdot v^1 = 0, \\
& a^1 \cdot v^1 = 0.
\end{align*}
$$

If the space dimension is two, we cannot obtain the above properties. Thus, ill-posed result is obtained only for the space dimension $n \geq 3$.

We set the initial data as follows:

$$ u_0(x) := \frac{1}{\Gamma_2(r)} \sum_{s=1}^{r} |k_0^s|^{s-1/2} (v^0 \cos(k_0^s \cdot x) + v^1 \cos(k_1^s \cdot x)). $$

Remark 14. We state properties of the initial data.

- In [2], the coefficients of $v^0$ and $v^1$ depend on $s$.
- A direct calculation yields $\text{div } u_0 = 0$.
- Since $\{k_0^s\}_{s=1}^{r}$ and $\{k_1^s\}_{s=1}^{r}$ are lacunary, and homogeneous and inhomogeneous type function spaces are equivalent, we see

$$
\|u_0\|_{L^{1,\infty}} \leq \sup_{j \in \mathbb{Z}} 2^{-j} \sqrt{j+1} \|\varphi_j \ast u_0\|_{\infty} \\
= \sup_{j \in \mathbb{N}} 2^{-j} \sqrt{j+1} \|\varphi_j \ast u_0\|_{\infty} \\
\leq \frac{1}{\Gamma_2(r)} \sup_{j \in \mathbb{N}} \frac{\sqrt{j+1}}{2^{j}} \sum_{s=1}^{r} \frac{|k_0^s|^1}{s^{1/2}} (\|\varphi_j \ast \cos(k_0^s \cdot x)\|_{\infty} + \|\varphi_j \ast \cos(k_1^s \cdot x)\|_{\infty}) \\
\leq \frac{1}{\Gamma_2(r)} \sup_{j \in \mathbb{N}} \frac{\sqrt{j+1}}{j^{1/2}} \rightarrow 0 \quad \text{as } r \rightarrow \infty.
$$

Our strategy is to decompose second approximation $u_2$ as Bourgain and Pavlović [2] did. Since

$$ u_2 = B(u_1, u_1) = \frac{1}{\Gamma_2^2(r)} \sum_{s,s'=1}^{r} \sum_{\sigma \in \{0, 1\}^3} \int_{0}^{t} e^{(t-\tau)\Delta} \mathbf{P} U_{s,s'}^{\sigma}(\tau, x) d\tau, $$

where

$$
U_{s,s'}^{\sigma}(\tau, x) = U_{s,s'}^{\sigma_1, \sigma_2, \sigma_3}(\tau, x) \\
:= (1/2) |k_s^0| |k_{s'}^{0'}| (ss')^{-1/2} e^{-((k_s^0)^2 + (k_{s'}^{0'})^2)\tau} \\
\times v^{\sigma_1} \sin((k_s^0 + (-1)^{\sigma_3} k_{s'}^{0'}) \cdot x) v^{\sigma_2} \cdot (k_s^0 + (-1)^{\sigma_3} k_{s'}^{0'}),
$$
we can decompose $u_2$ as $u_2 = u_2^0 + u_2^1 + \tilde{u}_2$, where

$$
\begin{align*}
    u_2^0 &:= \left(1/\Gamma_2^2(r)\right) \sum_{s=1}^{r} \sum_{\sigma_1, \sigma_2 \in \{0,1\}} \int_0^t e^{(t-\tau)\Delta} P U_{s,s}^{(\sigma_1, \sigma_2, 0)}(\tau, x) d\tau, \\
    u_2^1 &:= \left(1/\Gamma_2^2(r)\right) \sum_{s=1}^{r} \sum_{\sigma_1, \sigma_2 \in \{0,1\}} \int_0^t e^{(t-\tau)\Delta} P U_{s,s}^{(\sigma_1, \sigma_2, 1)}(\tau, x) d\tau, \\
    \tilde{u}_2 &:= \left(1/\Gamma_2^2(r)\right) \sum_{s=1}^{r} \sum_{\sigma \in \{0,1\}} \sum_{s' \prec s} \int_0^t e^{(t-\tau)\Delta} P(U_{s',s}^\sigma(\tau, x) + U_{s,s'}^\sigma(\tau, x)) d\tau.
\end{align*}
$$

We note $u_2^1$ is an inflation term.

**Remark 15.** By (12), we see that

$$
v_{\sigma_2} \cdot \left(k_{s_1}^{\sigma_1} + (-1)^{\sigma_1} k_{s_2}^{\sigma_2}\right) = \begin{cases} 
1 & \text{for } (\sigma_1, \sigma_2) = (1, 0), \\
0 & \text{otherwise.}
\end{cases}
$$

**Lemma 16.** We have the following key estimates of $u_2^1$, $u_2^0$, $\tilde{u}_2$, $u_1$ and $y$.

- **The estimate of the inflation term $u_2^1$.**
  We have the following inequalities:

  $$\|u_2^1(t)\|_{\infty} \leq C \frac{\Gamma_1(r)}{\Gamma_2^2(r)} = C \Gamma_1^\epsilon(r)$$

  for $t > 0$,

  $$\|u_2^1(t)\|_{V} \geq C \frac{\Gamma_1(r)}{\Gamma_2^2(r)} = C \Gamma_1^\epsilon(r)$$

  for $T/64 \leq t \leq 1$ with sufficiently large $r$.

- **The estimate of the first approximation and another part of the second approximation (exclude the inflation term).**
  We have the following inequalities:

  $$\|u_2^0(t)\|_{\infty} \leq \frac{C}{\Gamma_2^2(r)},$$

  $$\|\tilde{u}_2(t)\|_{\infty} \leq \frac{C}{\Gamma_2^2(r)},$$

  $$\|e^{t\Delta}u_0\|_{\infty} \leq \frac{C}{\Gamma_2(r)t^{1/2}}.$$
for $0 < t < T = (1/Γ^2_3(r))$ and
\[
\|e^tΔu_0\|_{∞} ≤ \frac{Γ_3(r)}{Γ_2(r)} = Γ^{3/2 - 1}_{1} (r) \quad \text{for } T/64 < t < T.
\]

- The estimate of the remainder term.

Let $ρ_1(r) := Γ_1(r)/Γ^2_2(r) = Γ^{3/2 - 1}_{1} (r)$ and $ρ_2(r) := Γ^2_1(r)/(Γ^4_2(r)Γ_3(r)) ≤ CΓ^{-ε}_1 (r)$. (Note that $ρ_1(r), ρ_2(r) → 0$ as $r → ∞$.) There exists the remainder term $y ∈ C([0, T] : BUC)$, $y(0) = 0$ satisfying the following estimate:
\[
\|y(t)\|_{∞} ≤ 2C(ρ_1(r) + ρ_2(r))
\]
for $0 < t < T = 1/Γ^2_3(r)$ with sufficiently large $r$.

We postpone to prove the above lemma. We now prove the main theorem. By the embedding inequality (11) and Lemma 16, we have the following estimate:
\[
\|u(t)\|_V \geq \|u^1_2(t)\|_V - \|u_0^0\|_∞ - \|u^2_0\|_∞ - \|e^tΔu_0\|_∞ - \|y(t)\|_∞
\]
\[\geq CΓ^ε_1 (r) - \frac{C}{Γ^2_2(r)} - \frac{Γ_3(r)}{Γ_2(r)} - 2C(ρ_1(r) + ρ_2(r)) ≥ CΓ^ε_1 (r).
\]
for $T/64 < t < T \ (T = 1/Γ^2_3(r))$ with sufficiently large $r$. This is the desired estimate. □

4. Proof of the key lemma

In this section, we prove Lemma 16. First we estimate the inflation term $u^1_2$. It is easy to see that
\[
U^{001} s,s \equiv 0 \quad \text{and} \quad U^{111} s,s \equiv 0 \quad \text{for } s = 1, \ldots, r.
\]
Since $v^1 \cdot a_1 = 0$ and $v^0 \cdot a_1 = 1$, we also see that $U^{011} s,s \equiv 0$ for $s = 1, \ldots, r$ and $Pv^1(v^0 \cdot a_1) \sin(a_1 \cdot x) = v^1 \sin x_3$. Thus we have the following equality:
\[
u^1_2(x,t) = \frac{1}{Γ^2_2(r)} \sum_{s=1}^{r} \int_{0}^{t} e^{(t-τ)Δ}P U^{(1,0,1)}_{s,s}(τ, x) \, dτ
\]
\[= \frac{1}{Γ^2_2(r)} \sum_{s=1}^{r} s^{-1} \int_{0}^{t} e^{-(t-τ)|a_1|^2} e^{-(|k^1_s|^2 + |k^0_s|^2)τ} \, dτ P v^1(v^0 \cdot a_1) \sin(a_1 \cdot x).
\]
We now estimate lower and upper bound of $u^1_2$. By Remark 13, we have
\[
|\int_{0}^{t} e^{-(t-τ)|a_1|^2} e^{-(|k^1_s|^2 + |k^0_s|^2)τ} \, dτ| = e^{−τ|k^0_s|^2} 2 \frac{1 - e^{-(|k^1_s|^2 + |k^0_s|^2)τ}}{|k^0_s|^2 + |k^1_s|^2 - 1} \geq e^{−τ(1 - e^{-2|k^1_s|^2})} \geq C
\]
for \( t \in [T/64, 1] = [1/|k_1^0|^2, 1] \). Since we easily estimate \( \sin x_3 \) in \( V \), we have
\[
\| u_1^1(t) \|_V \geq C \frac{\Gamma_1(r)}{\Gamma_2^2(r)}
\]
for \( t \in [T/64, 1] = [1/|k_1^0|^2, 1] \) with sufficiently large \( r \). We also have \( \| u_1^1(t) \|_\infty \leq B(\Gamma_1(r)/\Gamma_2^2(r)) \) for some constant \( B > C \) and \( t > 0 \). Thus we complete the estimate of the inflation term.

Next we consider \( \| \tilde{u}_2 \|_\infty, \| u_2^0 \|_\infty \) and \( \| e^{t\Delta}u_0 \|_\infty \). However, we only calculate \( \| \tilde{u}_2 \|_\infty \), since the calculations of \( \| u_2^0 \|_\infty \) and \( \| e^{t\Delta}u_0 \|_\infty \) are easy. Let
\[
J_{s,s'} := \int_0^t e^{-(|k_s^\sigma|^2+|k_s^2|^2)\tau-|k_s^\sigma|+(-1)^s k_s^2 t^2(t-\tau)} \, d\tau
\]
\[
= e^{-|k_s^\sigma|+(-1)^s k_s^2 t^2} + \frac{e^{2(-1)^s k_s^2 t - 1}}{2(-1)^s k_s^2 \cdot k_s^2 t}.
\]
A direct calculation shows that
\[
|\tilde{u}_2^s| \leq \frac{2}{\Gamma_2^2(r)} \sum_{s=1}^r \sum_{s' \leq s} (ss')^{-1/2} |k_s^0| |k_s^0| |J_{s,s'}|.
\]

By Remark 15, we only need to consider just two cases, the case \( \sigma = (\sigma_1, \sigma_2, \sigma_3) = (1, 0, 1) \) and the case \( \sigma = (\sigma_1, \sigma_2, \sigma_3) = (1, 0, 0) \). If \( \sigma_3 = 1 \), the function \( \frac{e^{2(1)^s k_s^1 t - 1}}{2(1)^s k_s^1 \cdot k_s^2 t} \) is uniformly bounded with respect to \( t > 0 \). But if \( \sigma_3 = 0 \), it is not. Thus we need to distinguish the cases \( \sigma_3 = 0 \) and \( \sigma_3 = 1 \).

**The case \( \sigma = (1, 0, 1) \).** Since \( s' \leq (s-1) < s \), we see that
\[
-|k_s^\sigma t + (-1)^s k_s^2 t^2| = -|2^{3s} T^{-1/2} a_0 - 2^{3s} T^{-1/2} a_0 - a_1|^2
\]
\[
\leq -|2^{3s} - 2^{3s'}| T^{-1/2} a_0 |^2 \leq -|2^{3s} - 2^{3(s-1)}| T^{-1/2} a_0 |^2
\]
\[
= -\left( \frac{49}{64} \right) |k_s^0|^2.
\]
Thus we have \( |J_{s,s'}| \leq Cte^{-C|k_s^0|^2 t} \) for \( t > 0 \).

**The case \( \sigma = (1, 0, 0) \).** A direct calculation yields
\[
J_{s,s'} = e^{-|k_s^\sigma|+|k_s^2|^2t} \frac{e^{2k_s^\sigma k_s^2 t}}{2k_s^\sigma k_s^2} \left( 1 - \frac{e^{-2k_s^\sigma k_s^2 t}}{2k_s^\sigma k_s^2} \right) \leq Cte^{-|k_s^\sigma t|^2 + |k_s^2|^2 t} \leq Cte^{-|k_s^0|^2 t}.
\]
Thus we have
\[
\left| \tilde{u}_2 \right| \leq \frac{C}{\Gamma_2^2(r)} \sum_{s=1}^{r} \sum_{s' < s} (ss')^{-1/2} \left| k_s^0 \right| \left| k_s^0 \right| t e^{-C|k_s^0|^2 t}
\]
\[
\leq \frac{C}{\Gamma_2^2(r)} \sum_{s=1}^{r} \left| k_s^0 \right|^2 t e^{-C|k_s^0|^2 t} \leq \frac{C}{\Gamma_2^2(r)},
\]
where we used \( \sum_{s=1}^{\infty} s t |k_s^0|^2 e^{-C|k_s^0|^2 t} \leq C \sum_{j \in \mathbb{Z}} 2^j \exp(-2^j) < \infty \) for \( t > 0 \) and \( 4 \sum_{s' < s} |k_s^0| \leq |k_s^0| \). Thus we complete the estimate of the term \( \tilde{u}_2 \).

Now we consider the remainder term \( y \) which satisfies (10). Recall that \( \rho_1(r) = \Gamma_1^2(r)/\Gamma_2^2(r) \) and \( \rho_2(r) = \Gamma_1^2(r)/(\Gamma_2^2(r) \Gamma_3^3(r)) \lesssim C \Gamma_1^{-t}(r) \). Let \( \rho_3(r) := \Gamma_1(r)/(\Gamma_2^2(r) \Gamma_3^3(r)) \lesssim C \Gamma_1^{-2t}(r) \). Note that \( \rho_1(r) \to 0, \rho_2(r) \to 0 \) and \( \rho_3(r) \to 0 \) as \( r \to \infty \). Also we recall that \( T = 1/\Gamma_2^2(r) \). By the above estimates of \( \| u_1 \|_{\infty}, \| u_2 \|_{\infty} \) and Corollary 6, and since the worst term to estimate in \( B(u_1, u_2) \) or \( B(u_2, u_1) \) is \( B(u_2^1, u_1) \), we have the following inequality:

\[
\| B(u_1(t), u_2(t)) \|_{\infty} + \| B(u_2(t), u_1(t)) \|_{\infty} \leq C \| B(u_2^1(t), u_1(t)) \|_{\infty} \leq C \rho_1(r).
\]

Since the worst term to estimate in \( B(u_2, u_2) \) is \( B(u_2^1, u_2^1) \) and \( B(y, u_2) \) is \( B(y, u_2^1) \), we have

\[
\| B(u_2(t), u_2(t)) \|_{\infty} \leq C \| B(u_2^1(t), u_2^1(t)) \|_{\infty} \leq C \rho_2(r)
\]

and

\[
\| B(u_1(t) + u_2(t), y(t)) \|_{\infty} + \| B(y(t), u_1(t) + u_2(t)) \|_{\infty} \leq C \| B(y(t), u_2^1(t)) \|_{\infty} \leq C \rho_3(r) \sup_{0 < \tau \leq T} \| y(\tau) \|_{\infty}
\]

for \( 0 < t < T = 1/\Gamma_2^2(r) \) with sufficiently large \( r \). Therefore we have

\[
\sup_{0 < \tau \leq T} \| y(t) \|_{\infty} \leq \left( C \rho_3(r) + \sup_{0 < \tau \leq T} \| y(t) \|_{\infty} \right) \sup_{0 < \tau \leq T} \| y(\tau) \|_{\infty} + C(\rho_1(r) + \rho_2(r)). \quad (14)
\]

By an absorbing argument, we have the following a priori bound,

\[
\sup_{0 < \tau \leq T} \| y(t) \|_{\infty} \leq 2C(\rho_1(r) + \rho_2(r))
\]

for sufficiently large \( r \) in order to satisfy \( C \rho_3(r) + 2C(\rho_1(r) + \rho_2(r)) \leq 1/2 \). By an iteration procedure and the above a priori bound, we obtain existence of \( y(t) \) in \( C([0, T]; BUC) \).

**Acknowledgments**

The author is deeply grateful to Professor Dongho Chae for encouragement and valuable suggestions. This paper developed during a stay of the author at the Department of Mathematics at Sungkyunkwan University. The author would like to thank the Department of Mathematics for support and hospitality. The author is also grateful for Professors Yoshikazu Giga, Kyungkeun...
Kang, Jihoon Lee and Hideyuki Miura for their interest and expository comments. The author also would like to thank Professors Okihiro Sawada and Nataša Pavlović for discussion about our refined initial data. Finally, the author would like to thank Professor Yoshihiro Sawano and the referee for advices about a generalized Besov space.

References