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Enumerating palindromes and primitives in rank two free groups

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ABSTRACT

Let F be a free group of rank two. An element of F is *primitive* if it, along with another group element, generates the group. If $F = \langle A, B \rangle$, then a word $W(A, B)$, in A and B, is a *palindrome* if it reads the same forwards and backwards. It is known that in a rank two free group, for any fixed set of two generators a primitive element will be conjugate either to a palindrome or to the product of two palindromes, but known iteration schemes for all primitive words give only a representative for the conjugacy class. Here we derive a new iteration scheme that gives either the unique palindrome in the conjugacy class or expresses the word as a unique product of two unique palindromes that have already appeared in the scheme. We denote these words by $E_{p/q}$ where p/q is rational number expressed in lowest terms. We prove that $E_{p/q}$ is a palindrome if pq is even and the unique product of two unique palindromes if pq is odd. We prove that the pair (X, Y) (or (X^{-1}, Y^{-1})) generates the group if and only if X is conjugate to $E_{p/q}$ and Y is conjugate to $E_{r/s}$ where $|ps - rq| = 1$. This improves a previously known result that held only for pq and rs both even. The derivation of the enumeration scheme also gives a new proof of the known results about primitive words.

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1. Introduction

Let $F = \langle A, B\rangle$ be a rank two free group. It is well known that up to conjugacy and inverses, primitive elements in a rank two free group can be indexed by the rational numbers and that pairs of primitive words that generate the group can be obtained by the number theory of the Farey tessellation of the hyperbolic plane. It is also well known that up to conjugacy, a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group [1,20].

In this paper we give new proofs of the above results. The proofs yield a new enumerative scheme for conjugacy classes of primitive words, still indexed by the rationals (Theorem 2.1). We denote the words representing each conjugacy class by $E_{p/q}$. In addition to proving that the enumeration scheme gives a unique representative for each conjugacy class containing a primitive, we prove that the words in this scheme are either palindromes or the canonically defined product of a pair of palindromes that have already appeared in the scheme and thus give a new proof of the palindrome/product result. Further we show that if X and Y are primitive, then up to replacing one and/or both by their inverse, there are words $E_{p/q}$ and $E_{r/s}$ respectively conjugate to X and Y, and these generate the group if and only if $|pq - rs| = 1$ (Theorem 2.2). This improves the previous known result that held only for pairs of palindromes.

In this paper, we use continued fractions and the Farey tessellation of the hyperbolic plane to find the enumeration scheme and to prove that it actually enumerates all primitives and all primitive pairs.

Pairs of primitives that generate the group and their relation to continued fractions and the Farey tessellation of the hyperbolic plane arise in the discreteness algorithm for $PSL(2,\mathbb{R})$ representations of two generator groups [4,5,11,13]. This new enumerative scheme is useful in extending discreteness criteria to $PSL(2,\mathbb{C})$ representations where the hyperbolic geometry of palindromes plays an important role [8,10].

2. The main result

We are able to state and use our main result with very little notation, only the definition of continued fractions. Namely, we let p and q be relative prime integers positive integers. Write

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_k}}}} = [a_0, a_1, \ldots, a_k]$$

where the a_i are integers with $a_j > 0$, $j = 1 \ldots k$, $a_0 \geqslant 0$.

Enumeration scheme for positive rationals. Set

$$E_{0/1} = A^{-1}, \qquad E_{1/0} = B, \quad \text{and} \quad E_{1/1} = BA^{-1}.$$

Suppose p/q has continued fraction expansion $[a_0, a_1, \ldots, a_{k-1}, a_k]$. Consider the two rationals defined by the continued fractions $[a_0, a_1, \ldots, a_{k-1}]$ and $[a_0, a_1, \ldots, a_{k-1}, a_k - 1]$. One is smaller than p/q and the other is larger; call the smaller one m/n and the larger one r/s so that $m/n < p/q < r/s$. The induction step in the scheme is given by

Case 1. pq – odd:

$$E_{p/q} = E_{r/s} E_{m/n}.$$

Case 2. pq – even:

$$E_{p/q} = E_{m/n} E_{r/s}.$$

We have a similar scheme for negative rationals described in Section 4. With both schemes we can state our main result as

Theorem 2.1 *(Enumeration of primitives by rationals). Up to conjugacy and taking formal inverses, the primitive elements of a two generator free group can be enumerated by the rationals using continued fraction expansions. The resulting primitive words are cyclically reduced and denoted by $E_{p/q}$ so that:*

- *For pq even, $E_{p/q}$ is a palindrome. It is the unique palindrome in its conjugacy class.*
- *For pq odd, $E_{p/q}$ is a product of palindromes that have already appeared in the scheme; that is, $E_{p/q} = E_{m/n}E_{r/s}$ and both $E_{m/n}$ and $E_{r/s}$ are palindromes.*

Remark 2.1. Note that although there are several ways a word in the pq odd conjugacy class can be factored as products of palindromes, in this theorem we specifically choose the unique factorization for $E_{p/q}$ that makes the enumeration scheme work.

In addition we have

Theorem 2.2. *Let $\{E_{p/q}\}$ denote the words in the enumeration scheme for rationals. If $(p/q, p'/q')$ satisfies $|pq' - qp'| = 1$, the pair $(E_{p/q}, E_{p'/q'})$ generates the group. Conversely, the pair of elements (X, Y) generates the group if and only if there is a rational p/q such that X is conjugate to $E_{p/q}$ and a rational p'/q' such that Y is conjugate to $E_{p'/q'}$, where $|pq' - qp'| = 1$.*

These theorems will be proved in Section 4. In order to prove the theorems and the related results we need to review some terminology and background.

3. Preliminaries

The main object here is a two generator free group which we denote by $F = \langle A, B\rangle$. A word $W = W(A, B) \in F$ is, of course, an expression of the form

$$A^{m_1} B^{n_1} A^{m_2} \cdots B^{n_r} \tag{1}$$

for some set of $2r$ integers $m_1, \ldots, m_r, n_1, \ldots, n_r$ with $m_2, \ldots, m_r, n_1, \ldots, n_{r-1}$ non-zero.

Definition 1. An element of F is *primitive* if there is another element V such that W and V generate F. V is called a *primitive associate* of W and the unordered pair W and V is called a *pair of primitive associates* or a *primitive pair* for short.

If $F = \langle A, B\rangle$, a primitive element $W = W(A, B)$ is also referred to as a *primitive word* when we are thinking of it as a word in A and B.

There are various versions of the following theorem in the literature, for example in [2,3,7,14,15, 17–19]. We use a version convenient for what we need below.

Theorem 3.1. *A word $W(A, B) \in F$ is primitive if and only if after freely reducing the word and interchanging A and B and p and q, it is conjugate to a word of the form*

$$B^{\epsilon} A^{n_1} B^{\epsilon} \cdots A^{n_p} \tag{2}$$

where $\epsilon = \pm 1$, all of the n_j have the same sign, and $|n_j| \geqslant 1$, $1 \leqslant j < p$, $|n_j - n_{j+1}| \leqslant 1$, and j is taken modulo p. The exponents satisfy $\sum_{i=1}^{p} n_i = q$ with p and q relatively prime and $0 \leqslant |p/q| < 1$.

Moreover, to every rational p/q there is a unique primitive word of this form.

The elements of the ordered set $\{n_1, \ldots, n_p\}$ are called the *primitive exponents* of the primitive word. Note that the primitive exponents are defined only up to cyclic permutation.

In the next subsections we summarize terminology and facts about the Farey tessellation and continued fraction expansions for rational numbers. Details and proofs can be found in [21–23]. See also [24].

3.1. Preliminaries: the Farey tessellation

In what follows when we use r/s to denote a rational number, we assume that r and s are integers with $s > 0$, and that r and s are relatively prime, that is, that $(r,s) = 1$. We let $\mathbb{Q}$ denote the rational numbers, and we identify the rationals with points on the extended real axis on the Riemann sphere. We use the notation $1/0$ to denote the point at infinity.

We need the concept of Farey addition for fractions.

Definition 2. If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ with $|ps - qr| = 1$, the *Farey sum* is

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

Two fractions are called *Farey neighbors* if $|ps - qr| = 1$.

When we write $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$ we tacitly assume the fractions are Farey neighbors.

Remark 3.1. It is a simple computation to see that both pairs of fractions

$$\left(\frac{p}{q}, \frac{p}{q} \oplus \frac{r}{s}\right) \quad \text{and} \quad \left(\frac{p}{q} \oplus \frac{r}{s}, \frac{r}{s}\right)$$

are Farey neighbors if $(p/q, r/s)$ are, and that if $\frac{p}{q} < \frac{r}{s}$, then

$$\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}.$$

It is easy to calculate that the Euclidean distance between finite Farey neighbors is strictly less than one unless they are adjacent integers. This implies that unless one of the fractions is $0/1$, both neighbors have the same sign.

One creates the Farey diagram in the upper half-plane by marking each fraction by a point on the real line and joining each pair of Farey neighbors by a semi-circle orthogonal to the real line. The point here is that because of the above properties none of the semi-circles intersect in the upper half plane. This gives a tessellation of the hyperbolic plane where the semi-circles joining a pair of neighbors, together with the semi-circles joining each member of that pair to the Farey sum of the pair, form an ideal hyperbolic triangle. The tessellation is called the Farey tessellation and the vertices are precisely the points that correspond to rational numbers. A vertex corresponding to p/q is labeled $v_{p/q}$. It is termed *even* or *odd* according to the parity of pq. See Fig. 1.

The Farey tessellation is invariant under the semi-group generated by $z \mapsto z+1$ and $z \mapsto 1/z$.

Fix any point ζ on the positive imaginary axis. Given a fraction, $\frac{p}{q}$, there is an oriented hyperbolic geodesic γ connecting ζ to $\frac{p}{q}$. We assume γ is oriented so that moving from ζ to a positive rational is the positive direction. This geodesic γ will intersect some number of triangles.

Definition 3. The *Farey level* or the *level* of p/q, denoted by $Lev(p/q)$, is the number of triangles traversed by γ.

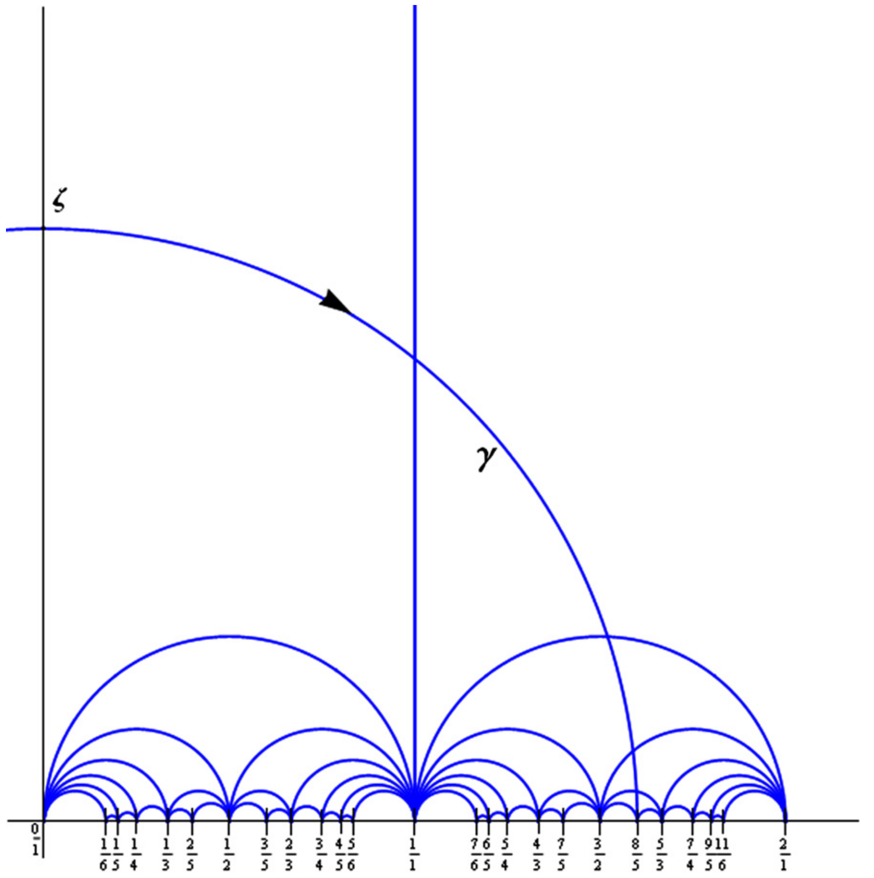

Fig. 1. The Farey tesselation with the curve γ.

Note that the definition is independent of the choice of ζ and that our definition implies $Lev(p/q) = Lev(-p/q)$.

The geodesic γ will enter any given triangle along an edge. This edge will connect two vertices and γ will exit the triangle along an edge connecting one of these two vertices and the third vertex of the triangle, the *new* vertex. Since γ is oriented, the edge through which γ exits a triangle is either a left edge or a right edge depending upon whether the edge γ cuts is to the right or left of the new vertex.

Definition 4. We determine a *Farey sequence for* $\frac{p}{q}$ inductively by choosing the new vertex of the next triangle in the sequence of triangles traversed by γ. The sequence ends at p/q.

Given p/q, we can find the smallest rational m/n and the largest rational r/s that are neighbors of p/q. These neighbors have the property that they are the only neighbors with lower Farey level. That is, $m/n < p/q < r/s$ and $Lev(p/q) > Lev(m/n)$, $Lev(p/q) > Lev(r/s)$, and if u/v is any other neighbor $Lev(p/q) < Lev(u/v)$.

Definition 5. We call the smallest and the largest neighbors of the rational p/q the *distinguished neighbors* or the *parents* of p/q.

Note that we can tell whether a distinguished neighbor r/s is smaller or larger than p/q by the sign of $rq - ps$.

We emphasize that we have two different and independent orderings of the rational numbers: the ordering as rational numbers and the partial ordering by level. Our proofs will often use induction on the level of the rational numbers involved as well as the order relations as rational numbers among parents and grandparents.

Remark 3.2. It follows from Remark 3.1 that if m/n and r/s are the parents of p/q, then any other neighbor of p/q is of the form $\frac{m+pt}{n+qt}$ or $\frac{r+pt}{s+qt}$ for some positive integer t. The neighbors of ∞ are precisely the set of integers.

Finally we note that we can describe the Farey sequence of p/q by listing the number of successive left or right edges of the triangles that γ crosses where a left edge means that there is one vertex of the triangle on the left of γ and two on the right, and a right edge means there is only one vertex of γ on the right. This will be a sequence of integers, the *left-right sequence* $\pm(n_0, n_1, \ldots, n_t)$ where the integers n_i, $i > 0$ are all positive and n_0 is positive or zero. The sign in front of the sequence is positive or negative depending on the orientation of γ.

3.2. Preliminaries: continued fractions

Farey sequences are related to continued fraction expansions of positive fractions; they can also be related to expansions of negative fractions. We review the connection in part to fix our notation precisely. We do not use the classical notation of [12] for negative fractions, but instead use the notation of [9,18,21,22,25] which is standardly used by mathematicians working in Kleinian groups and three manifolds. This notation reflects the symmetry about the imaginary axis in the Farey tessellation which plays a role in our applications. This symmetry is built in to the semi-group action on the tessellation.

For $p/q \geqslant 0$ write

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_k}}}} = [a_0, a_1, \ldots, a_k]$$

where $a_j > 0$, $j = 1 \ldots k$, $a_0 \geqslant 0$. For $0 \leqslant n \leqslant k$ set

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

Remark 3.3. The continued fraction of a rational is ambiguous; the continued fractions $[a_0, a_1, \ldots, a_n]$ and $[a_0, a_1, \ldots, a_n - 1, 1]$ both represent the same rational. Therefore, if we have $[a_0, a_1, \ldots, a_{n-1}, 1]$ we may replace it with $[a_0, a_1, \ldots, a_{n-1} + 1]$.

Remark 3.4. Note that if $\frac{p}{q} \geqslant 1$ has continued fraction expansion $[a_0, a_1, \ldots, a_n]$, then $\frac{q}{p}$ has expansion $[0, a_0, \ldots, a_n]$ while if $\frac{p}{q} < 1$ has continued fraction expansion $[0, a_1, \ldots, a_n]$, then $\frac{q}{p}$ has expansion $[a_1, a_2, \ldots, a_n]$.

Remark 3.5. The distinguished neighbors or parents of p/q have continued fractions

$$[a_0, a_1 \ldots, a_{k-1}] \quad \text{and} \quad [a_0, a_1, \ldots, a_{k-1}, a_k - 1].$$

The Farey sequence contains the approximants $\frac{p_n}{q_n}$ as a subsequence. They can be computed recursively from the continued fraction for p/q as follows:

$$p_0 = a_0, \qquad q_0 = 1 \quad \text{and} \quad p_1 = a_0a_1 + 1, \qquad q_1 = a_1,$$
$$p_j = a_jp_{j-1} + p_{j-2}, \qquad q_j = a_jq_{j-1} + q_{j-2}, \quad j = 2, \dots, k.$$

One can calculate from these recursion formulas that the approximants are alternately to the right and left of p/q. The points of the Farey sequence between $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$ have continued fraction expansions

$$[a_0, a_1, \dots a_j + 1], \quad [a_0, a_1, \dots, a_j + 2], \quad \dots, \quad [a_0, a_1, \dots a_j + a_{j+1} - 1].$$

Thus a_j is the number of points on the same side of p/q before jumping to the other side.

We extend the continued fraction notion to negative fractions by defining the continued fraction of $\frac{p}{q} < 0$ to be the negative of that for $|\frac{p}{q}|$. That is, by setting

$$\frac{p}{q} = -[a_0, a_1 \dots, a_k] = [-a_0, -a_1 \dots, -a_k] \quad \text{where} \left|\frac{p}{q}\right| = [a_0, a_1, \dots a_k].$$

We also set $Lev(p/q) = Lev(|p/q|)$.

In [12] the continued fraction $[a_0, a_1, \dots, a_k]$ of $p/q < 0$ is defined so that $a_0 < 0$ is the largest integer in p/q and $[a_1, \dots a_k] = p/q - a_0$ is the continued fraction of a positive rational. With this notation the symmetry about the origin which plays a role in our applications is lost.

We note that for any pair of neighbors, unless one of them is $0/1$ or $1/0$, they both have the same sign and thus have equal a_0 entries. Since we almost always work with neighbors the difference between our notation and the classical one does not play a role.

It is not difficult to check that the continued fraction expansion and the left-right sequence of a rational agree.

4. Enumerating primitives: palindromes and products

In the enumeration scheme we define here, the group element B^{-1} never appears. It is therefore clear that either a word or its inverse is in the scheme, but both cannot be.

We first work with positive rationals. We do this merely for ease of exposition and to simplify the notation. We then indicate the minor changes needed for negative rationals.

Enumeration scheme for positive rationals. Set

$$E_{0/1} = A^{-1}, \qquad E_{1/0} = B, \quad \text{and} \quad E_{1/1} = BA^{-1}.$$

If p/q has continued fraction expansion $[a_0, a_1, \dots, a_{k-1}, a_k]$, consider the parent fractions $[a_{0,1}, \dots, a_{k-1}]$ and $[a_0, \dots, a_{k-1}, a_k - 1]$ (see Remark 3.5). Choose labels m/n and r/s for the parents so that $m/n < p/q < r/s$. Set

Case 1. pq – odd:

$$E_{p/q} = E_{r/s}E_{m/n}.$$

Case 2. pq – even:

$$E_{p/q} = E_{m/n}E_{r/s}.$$

Note that in Case 1 (pq odd) the word indexed by the larger fraction is on the left and in Case 2 (pq even) it is on the right.

Enumeration scheme for negative rationals. Now assume $p/q < 0$. We use the reflection in the imaginary axis to obtain the enumeration scheme. The reflection sends A^{-1} to A. This reverses the order of the distinguished neighbors on the real line.

Set

$$E_{0/1} = A \quad \text{and} \quad E_{1/0} = B.$$

These are trivially palindromes. At the next level we have $E_{-1/1} = BA$.

To give the induction scheme: we assume $m/n, r/s$ are the distinguished neighbors of p/q and they satisfy $m/n > p/q > r/s$, $p/q = (m+r)/(n+s)$ and set

Case 1. pq – odd:

$$E_{p/q} = E_{r/s} E_{m/n}.$$

Case 2. pq – even:

$$E_{p/q} = E_{m/n} E_{r/s}.$$

Note that now in Case 1 (pq odd) the word indexed by the larger fraction is on the right and in Case 2 (pq even) it is on the left.

Proofs of Theorems 2.1 and 2.2

Before we give the proofs we note that there are other enumeration schemes for relating rational numbers to primitive words given, for example, in [3,6,15]. Some of these also use Farey neighbors to find primitive pairs. In general, these schemes produce a different word in the conjugacy class of, or the inverse conjugacy class of, our $E_{p/q}$.

Theorem 2.2 tells us that words in the enumeration scheme labeled with neighboring Farey fractions give rise to primitive pairs. Note that although cyclic permutations are obtained by conjugation, we cannot necessarily *simultaneously* conjugate a primitive pair coming from the other enumeration schemes to get to the corresponding primitive pair coming from Theorem 2.2. Here by simultaneously we mean conjugate the two elements of the primitive pair by the same single element of the group.

Proof of Theorem 2.1. The proof uses the connection between continued fractions and the Farey tessellation. We call a rational p/q or the word $E_{p/q}$ *odd* if pq is odd and *even* otherwise. We observe that in every Farey triangle with vertices $m/n, p/q, r/s$ one of the vertices is *odd* and the other two are *even*. To see this simply use the fact that $mq - np$, $ps - rq$, $ms - nr$ are all congruent to 1 modulo 2. (This also gives the equivalence of parity cases for pq and the $p + q$ used by other authors.) In a triangle where pq is even, it may be that the smaller distinguished neighbor is even and the larger odd or vice versa and we take this into account in discussing the enumeration scheme. We note that in general if X and Y are palindromes, then so is $(XY)^t X$ for any positive integer t.

We give the proof assuming $p/q > 0$. The proof proceeds by induction on the Farey level. The idea behind the proof is that each rational has a pair of parents (distinguished neighbors) and each parent in turn has two parents so there are at most four *grandparents* to consider. The parents and grandparents may not all be distinct. The cases considered below correspond to the possible ordering of the grandparents as rational numbers and also the possible orders of their levels.

To deal with negative rationals we use the reflection in the imaginary axis. The reflection sends A^{-1} to A. We again have distinguished neighbors m/n and r/s, and using the reflection our assumption is $m/n > p/q > r/s$. In the statement of the theorem, m/n is now the larger neighbor. Using our definition of the Farey level of p/q as the Farey level of $|p/q|$, the proof is exactly the same as for positive rationals.

If p/q is odd, by induction we get the product of distinguished neighbor palindromes.

If p/q is even we need to show that we get palindromes.

The set up shows that we have palindromes for level 0, ($\{0/1, 1/0\}$) and the correct product for level 1, $\{1/1\}$.

Assume the scheme works for all rationals with level less than N and assume $Lev(p/q) = N$. Since $m/n, r/s$ are distinguished neighbors of p/q both their levels are less than N.

Suppose $p/q = [a_0, a_1, \ldots, a_k]$. Then $Lev(p/q) = \sum_0^k a_j = N$, and by Remark 3.5 the continued fractions of the parents of p/q are

$$[a_0, a_1, \ldots, a_{k-1}] \quad \text{and} \quad [a_0, a_1, \ldots, a_{k-1}, a_k - 1].$$

Assume we are in the case where pq is even.

Suppose first that

$$Lev(r/s) > Lev(m/n)$$

so that we have

$$m/n = [a_0, a_1, \ldots, a_{k-1}] \quad \text{and} \quad r/s = [a_0, a_1, \ldots, a_{k-1}, a_k - 1]. \tag{3}$$

Then the smaller distinguished neighbor of r/s is m/n and the larger distinguished neighbor is

$$w/z = [a_0, a_1, \ldots, a_k - 2]. \tag{4}$$

The smaller distinguished neighbor of m/n is

$$u/v = [a_0, \ldots, a_{k-2}, a_{k-1} - 1] \tag{5}$$

and the larger distinguished neighbor is

$$x/y = [a_0, \ldots, a_{k-2}]. \tag{6}$$

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = E_{w/z} E_{m/n}$$

and

$$E_{p/q} = E_{m/n} E_{r/s} = E_{m/n}(E_{w/z} E_{m/n})$$

which is a palindrome.

If mn is odd we have, by the induction hypothesis,

$$E_{m/n} = E_{x/y} E_{u/v}$$

and by Eqs. (3), (4), (5) and (6)

$$E_{r/s} = E_{m/n}^{(a_k-1)} E_{x/y} = (E_{x/y} E_{u/v})^{(a_k-1)} E_{x/y}$$

so that

$$E_{p/q} = E_{m/n} E_{r/s} = (E_{x/y} E_{u/v})^{a_k} E_{x/y}$$

is a palindrome.

If

$$Lev(r/s) < Lev(m/n),$$

we have

$$m/n = [a_0, a_1, \ldots, a_{k-1}, a_k - 1] \quad \text{and} \quad r/s = [a_0, a_1, \ldots, a_{k-1}]. \tag{7}$$

Then the larger distinguished neighbor of m/n is r/s and the smaller distinguished neighbor is

$$w/z = [a_0, a_1, \ldots, a_k - 2]. \tag{8}$$

The larger distinguished neighbor of r/s is

$$x/y = [a_0, \ldots, a_{k-2}, a_{k-1} - 1] \tag{9}$$

and the smaller distinguished neighbor is

$$u/v = [a_0, \ldots, a_{k-2}]. \tag{10}$$

If mn is odd we have, by the induction hypothesis

$$E_{m/n} = E_{r/s}E_{w/z}$$

and

$$E_{p/q} = E_{m/n}E_{r/s} = E_{r/s}(E_{w/z}E_{r/s})$$

which is a palindrome.

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = E_{x/y}E_{u/v}$$

and by Eqs. (7), (8), (9) and (10)

$$E_{m/n} = E_{u/v}E_{r/s}^{(a_k-1)} = E_{u/v}(E_{x/y}E_{u/v})^{(a_k-1)}$$

so that

$$E_{p/q} = E_{m/n}E_{r/s} = E_{u/v}(E_{x/y}E_{u/v})^{a_k}$$

is a palindrome.

It is clear that the $E_{p/q}$ words are primitive because they are products of associate primitive pairs and such a product, together with either element of the original primitive pair form a new primitive pair. We need, however, to show that we obtain an $E_{p/q}$ word corresponding to every conjugacy class of primitive words in F (or to the class of its inverse) and that this representative of the conjugacy class is the unique palindrome in the class when the word is even.

We work with positive rationals between 0 and 1 and note that our comments can easily be adjusted to apply to all rationals.

We observe that in generating the words $E_{p/q}$ for the positive rationals between 0 and 1, there is no cancelation because the A's all appear with negative exponents and the B's all appear with exponent one. Thus there is no difference between concatenation and multiplication followed by free reduction. It follows that the $E_{p/q}$ have minimal length in their conjugacy class.

In fact, every $E_{p/q}$ with $0 \leqslant p/q < 1$ begins and ends with a power of A^{-1} and the powers of A^{-1} are all separated by a factor of B with exponent $+1$. It is easy to check this for all words of level up through three. The rest follows by induction on the level of p/q: if $E_{p/q} = E_{r/s}E_{m/n}$ with the levels of m/n and r/s greater than three, the statement holds by the induction hypothesis. Assume that one of the factors is at a level smaller than three. If $E_{p/q}$ is a palindrome, whatever its level, it must start and end with an A^{-1}. The only time the factor $E_{m/n}$ or $E_{r/s}$ will not begin and end with an A^{-1} is if it is $E_{1/1}$. In that case we must have pq even, $E_{m/n} = E_{1/1} = BA^{-1}$ and $E_{r/s}$ beginning and ending with A^{-1} terms.

Since the $E_{p/q}$ are primitive words, each is conjugate to one of the form of Eq. (2) in Theorem 3.1. Again because there is no cancelation, the ordered set of exponents for the A^{-1} terms that appear in $E_{p/q}$ are the primitive exponents, $\{v_1, \ldots, v_{p'}\}$ for some integer $p' > 0$ and there is a q' with $\sum_{i=1}^{p'} v_i = q'$. As there is no cancelation when we concatenate $E_{r/s}E_{m/n}$, the exponents sums of the A^{-1} terms add as do those of the B terms. Thus we must have $p' = p$ and $q' = q$.

We have thus established a bijection between the conjugacy classes of primitives (up to inverse) and the primitive words $E_{p/q}$. (See also [9].)

We claim that the palindromes we obtain are unique in their conjugacy class. We assume $E_{p/q}$ is a palindrome so that pq is even. Again, we write things as if p/q is in $[0, 1]$ and leave it to the reader to make the adjustments for other rationals. As above, we have only factors of A^{-1} and B. Let V be an arbitrary word in F written in the form of Eq. (2). Suppose $W = VE_{p/q}V^{-1}$ is any word in the conjugacy class of $E_{p/q}$. We consider all of the possible cases: if V begins or ends with a power of B, W will have powers of B other than $+1$. If V is a single power of A, then W will not be a palindrome. Assume V has an initial (or terminal) segment that coincides with the inverse of an initial (or terminal) segment of $E_{p/q}$. Then either W is of a form that does not have minimal length, V is not a palindrome or $VE_{p/q}V^{-1} = E_{p/q}$. □

Proof of Theorem 2.2. The proof of the first statement is by induction on the maximum of the levels of p/q and p'/q'. Again we proceed assuming $p/q > 0$; reflecting in the imaginary axis we obtain the proof for negative rationals.

At level 1, the theorem is clearly true: (A^{-1}, B), (A^{-1}, BA^{-1}) and (BA^{-1}, B) are all primitive pairs.

Assume now that the theorem holds for any pair both of whose levels are less than N and let $(p/q, p'/q')$ be a pair of neighbors with $Lev(p/q) = N$. Let $m/n, r/s$ be the distinguished neighbors of p/q and assume $m/n < p/q < r/s$. Then $m/n, r/s$ are neighbors and both have level less than N so that by the induction hypothesis $(E_{m/n}, E_{r/s})$ is a pair of primitive associates. It follows that all of the pairs

$$(E_{m/n}, E_{m/n}E_{r/s}), \qquad (E_{m/n}, E_{r/s}E_{m/n}),$$
$$(E_{m/n}E_{r/s}, E_{r/s}) \quad \text{and} \quad (E_{r/s}E_{m/n}, E_{r/s})$$

are pairs of primitive associates since we can retrieve the original pair $(E_{m/n}, E_{r/s})$ from any of them. Since $E_{p/q} = E_{r/s}E_{m/n}$ or $E_{p/q} = E_{m/n}E_{r/s}$ we have proved the theorem if p'/q' is one of the distinguished neighbors.

If p'/q' is not one of the distinguished neighbors, then either $p'/q' = (tp + m)/(tq + n)$ for some $t > 0$ or $p'/q' = (tp + r)/(tq + s)$ for some $t > 0$. Assume for definiteness $p'/q' = (tp + m)/(tq + n)$; the argument is the same in the other case.

Note that the pairs $p/q, (jp + m)/(jq + n)$ are neighbors for all $j = 1, \ldots, t$. We have already shown $E_{m/n}, E_{p/q}$ is a pair of primitive associates. The argument above applied to this pair shows that $E_{m/n}, E_{(p+m)/(q+m)}$ is also a pair of primitive associates. Applying the argument t times proves the theorem for the pair $E_{p'/q'}, E_{p/q}$.

We have established that all pairs $E_{p'/q'}, E_{p/q}$ with $|pq' - q'p| = 1$ are primitive pairs.

Conversely, suppose the pair (X, Y) generates the group with X conjugate to $E_{p/q}$ and Y conjugate to $E_{p'/q'}$. By Theorem 7.1 of [6], parts 8 and 9, there is a rational number m/n associated to X and a rational number r/s associated to Y such that these rationals are Farey neighbors (that is, $|ms - rn| = 1$). The primitive exponents of X and Y are determined by the continued fraction expansions of m/n and r/s: m is the exponent sum of the A terms in X, r is the sum of the exponents of the A terms in Y and $m+n$ and $r+s$ are respectively the minimal word lengths of the conjugacy classes. Since X is conjugate to $E_{p/q}$ and Y is conjugate to $E_{p'/q'}$, as we saw above at the end of the proof of Theorem 2.1, this implies $m = p, n = q, r = p'$ and $s = q'$ so that $|pq' - p'q| = 1$. □

An immediate corollary is

Corollary 4.1. *The scheme of Theorem* 2.1 *also gives a scheme for enumerating only primitive palindromes and a scheme for enumerating only primitives that are canonical palindromic products.*

5. Examples

Here we compute some examples of words in the enumeration scheme:

Fraction	Parents	Parity	$E_{p/q}$	Parental product	Simplified
1/2	$0/1 \oplus 1/1$	even	$E_{1/2}$	$A^{-1} \cdot BA^{-1}$	$A^{-1}BA^{-1}$
2/1	$1/1 \oplus 1/0$	even	$E_{2/1}$	$BA^{-1} \cdot B$	$BA^{-1}B$
1/3	$1/2 \oplus 0/1$	odd	$E_{1/3}$	$A^{-1}BA^{-1} \cdot A^{-1}$	$A^{-1}BA^{-2}$
2/5	$1/3 \oplus 1/2$	even	$E_{2/5}$	$A^{-1}BA^{-2} \cdot A^{-1}BA^{-1}$	$A^{-1}BA^{-3}BA^{-1}$
1/4	$1/3 \oplus 0/1$	even	$E_{1/4}$	$A^{-1} \cdot A^{-1}BA^{-2}$	$A^{-2}BA^{-2}$
2/7	$1/4 \oplus 1/3$	even	$E_{2/7}$	$A^{-2}BA^{-2} \cdot A^{-1}BA^{-2}$	$A^{-2}BA^{-3}BA^{-2}$

We illustrate how to find the word $E_{31/9}$. The continued fraction is $31/9 = [3, 2, 4]$; note that the distinguished Farey neighbors are $[3, 2] = 7/2$ and $[3, 2, 3] = 24/7$. We form the following words indicating $E_{31/9}$ and its neighbors in boldface

$$E_{1/1} = BA^{-1}, \qquad E_{2/1} = BA^{-1}B, \qquad E_{3/1} = E_{1/0}E_{2/1} = B \cdot BA^{-1}B,$$

$$E_{4/1} = E_{3/1}E_{0/1} = BBA^{-1}B \cdot B,$$

$$\mathbf{E_{7/2}} = E_{3/1}E_{4/1} = \mathbf{BBA^{-1}B \cdot BBA^{-1}BB},$$

$$E_{10/3} = E_{3/1}E_{7/2} = B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2,$$

$$E_{17/5} = E_{7/2}E_{10/3} = B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2,$$

$$\mathbf{E_{24/7}} = E_{17/5} \cdot E_{7/2} = \mathbf{B^2A^{-1}B^3A^{-1}B^2B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^2},$$

$$\mathbf{E_{31/9}} = E_{7/2}E_{24/7} = \mathbf{B^2A^{-1}B^3A^{-1}B^2 \cdot B^2A^{-1}B^3A^{-1}B^2B^2A^{-1}B^3A^{-1}B^3A^{-1}B^2B^2A^{-1}B^3A^{-1}B^2}.$$

6. Farey diagram visualization

We can visualize the relation between primitive pairs and neighboring rationals using the Farey diagram. Suppose the primitive pair (A, B) corresponds to $(r/s, p/q)$. Note that we have done this so that the Farey level of p/q is greater than that of r/s and r/s is the parent of p/q with lowest Farey level. (The other parent is $(r-p)/(q-s)$ and corresponds to $A^{-1}B$.) Draw the curve γ from a point on the imaginary axis to p/q. If p/q is positive, orient it toward p/q; if p/q is negative, orient it towards the imaginary axis.

The left-right sequence and the continued fraction for p/q are the same. Traversing the curve in the other direction reverses left and right. The symmetry about the imaginary axis is reflected in our definition of negative continued fractions.

Given two primitive pairs (A, B) and $(A'B')$ such that B corresponds to p/q and B' corresponds to p'/q' draw the curves γ and γ'. We can find the sequence to go from (A, B) to (A', B') by traversing γ from p/q to the imaginary axis and then traversing γ' to p'/q'. We can also draw an oriented curve from p/q to p'/q' and read off the left-right sequence along this curve to get the continued fraction that gives (A', B') as words in (A, B) directly. We have

Corollary 6.1. *Given any two sets of primitive pairs, (A, B) and (A', B') there is a continued fraction containing either only positive or only negative integers that connects one pair to the other.*

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