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Exclusion and persistence in deterministic and stochastic chemostat models

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Abstract

We first introduce and analyze a variant of the deterministic single-substrate chemostat model. In this model, microbe removal and growth rates depend on biomass concentration, with removal terms increasing faster than growth terms. Using a comparison principle we show that persistence of all species is possible in this scenario. Then we turn to modelling the influence of random fluctuations by setting up and analyzing a stochastic differential equation. In particular, we show that random effects may lead to extinction in scenarios where the deterministic model predicts persistence. On the other hand, we also establish some stochastic persistence results.

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1. Introduction

Chemostat models hold a special position in mathematical biology because they are closely related to real-life bioreactors. They have proven their value by giving precise predictions about exclusion for competing species, which subsequently could be validated in experiments. The basis for the theoretical investigation was laid by Monod [28,29] in the 1940s, with later substantial contributions, among others, by Novick and Szilard [30] in 1950 and by Herbert et al. [18] in 1956. The mathematical analysis of chemostat systems, in particular the derivation of the exclusion principle, was pursued in a number of publications, among them Aris and Humphrey [1], Hsu [19], Hsu et al.

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[20], and Butler and Wolkowicz [5]. McGehee and Armstrong [27], on the other hand, discussed resource–consumer systems where the exclusion principle does not hold. The monograph by Smith and Waltman [31] contains a unified presentation of the results up to 1995. The mathematical predictions regarding the exclusion principle were validated in experiments by Hansen and Hubbell [17]. This was a particular highlight in the interplay of mathematics and biology.

Modelling and analyzing chemostat systems continues to be a very active field, and extensions and generalizations have been proposed in recent years. In the spatially homogeneous setting, we mention—*pars pro toto*—the delay models introduced and analyzed by Ellermeyer [8], and Wolkowicz et al. [34], as well as the work toward a general exclusion principle by Li [26] and the consideration of time-dependent substrate inflow by Ellermeyer et al. [9]. Reaction-diffusion equations modelling an unstirred (thus spatially heterogeneous) chemostat system were studied recently by Wu and Wolkowicz [35]. Gard [14] discusses stochastic ecosystem models, with chemostat models among them.

In the present paper we will reconsider the basic modelling of bioreactors: It seems that the generally accepted mathematical derivation of exclusion principles rests on debatable model assumptions. A consultation of the fundamental contributions by Monod [29], Novick and Szilard [30] and Herbert et al. [18] shows that all these authors start from two assumptions: First, an exponential distribution governs death and removal of microbes, leading to a first-order transport term in the associated differential equation. Second, the authors assume the existence of a specific growth rate for microbe biomass, which depends on substrate concentration but is independent of microbe concentration. This leads to a growth term of first order in the microbe concentration. While Herbert et al. [18] remark that “theoretical discussions of bacterial growth usually start from the familiar ‘exponential growth’ equation”, they give no further theoretical justification. From a modelling perspective, caution is in order here: This basic assumption seems to have its roots in reasoning similar to the modelling of chemical reactions. Thus it can be seen as analogous to some type of mass action kinetics for a dynamical process, which is reliably applicable only for low concentrations. Of course, there is some experimental support for this model assumption, but there are also the experimental observations by Contois [6], which indicate that it has to be reconsidered for high concentrations. Likewise, the independence of the removal rate from the microbe concentrations seems to be theoretically justified only in the case of low microbe concentrations. The “washout only” scenario e.g., of Herbert et al. [18] ignores death terms altogether. Although the linear washout term can then be rigorously justified, one has to keep in mind that “washout only” is certainly an idealization. On the other hand, the assumption of linearity in the microbe concentration for both uptake and removal is crucial in the standard derivation of the exclusion principle. Thus one may say that this strong mathematical result is based on problematic model assumptions. This may point to an inherent weakness of the model.

Therefore it seems appropriate to consider a broader class of models, with more general uptake and removal terms. We indicate two possible effects which may lead to nonlinear effects in uptake and removal. First, the specific uptake rate of a microbe may decrease with population density, due to obstruction of individual substrate uptake and growth.

This is consistent with Contois' observations [6]; he suggests an inhibitory effect of metabolic products as the cause. Second, the specific death rate may increase with population density, for instance due to additional competition effects. We refer to the paper by Fredrickson and Stephanopoulos [11]. It should be noted that Braun [4] in his recent doctoral thesis also discussed the basic model assumptions concerning removal rates, and proposed a possible mechanism leading to nonlinear terms: Colonization when few microbes are present in the reactor, gradually replaced by washout at higher densities.

In view of the above, the consideration of nonlinear effects in substrate uptake and removal seems to be warranted. Of course any model and its predictions will have to be judged in the light of the experimental evidence.

In addition, the consideration of stochastic effects is helpful for a proper understanding of the behavior of a biological system. We recall the famous paper by Beddington and May [2] about harvesting populations in a randomly fluctuating environment, and also the stochastic equation for evolutionary games derived by Fudenberg and Harris [13], which was recently investigated by Imhof [22]. We also refer to the study in a biological context by Foster and Young [10]. Their results illustrate that stochastic effects may change the behavior in a substantial manner. Therefore, a thorough study of stochastic chemostat models is justified.

The paper is organized as follows. In Section 2 we introduce and discuss a variant of the standard deterministic chemostat model. The principal difference is that the removal terms increase faster with microbe concentration than the uptake terms. As it turns out, in the single species scenario the qualitative behavior is virtually indistinguishable from the standard chemostat model, but there may be fundamentally different behavior when several microbe species are present. Using a comparison principle we prove that persistence of all species may occur in the multiple species setting if the additional removal is mainly caused by intra-specific effects. In Section 3 we turn to the stochastic setting. Our basic approach is analogous to Beddington and May [2]. Following a detailed and rigorous derivation of the models and of the stochastic differential equations, we discuss some properties of this class of equations. The stochastic setting encompasses both the classical chemostat models and the variant introduced in Section 2. The results for the single-species model may be summarized as follows: If the deterministic counterpart admits persistence, and the stochastic effects are not too strong, then one will have a recurrent system and, with some additional assumptions, the stochastic solution can be expected to remain close to the interior deterministic stationary point. In the two-species case we prove a transience result and show that under certain conditions the stochastic model leads to extinction even though the deterministic counterpart predicts persistence.

2. A generalization of the deterministic chemostat model

2.1. The model

We introduce a variant of the single-substrate chemostat with one or several microbe species. The system modelling a single microbe species is:

$$\dot{x}_0 = r - \delta x_0 - a(x_0, x_1),$$

$$\dot{x}_1 = a(x_0, x_1) - s(x_1). \quad (1)$$

Here x_0 stands for substrate concentration, and x_1 denotes the biomass concentration of a microbe species feeding on the substrate. We are only interested in the positive quadrant $x_0 \geq 0$, $x_1 \geq 0$. Throughout this paper, all functions are supposed to have continuous partial derivatives—one-sided if applicable—on the closed positive orthant. The constants and functions involved in this equation satisfy the following conditions:

- The substrate inflow rate r and the relative substrate outflow rate δ are positive constants.
- The substrate uptake rate $a(x_0, x_1)$, which is equal to the microbe growth rate, is non-negative and strictly increasing in both variables. Furthermore, $a(x_0, 0) = a(0, x_1) = 0$ for all x_0 and x_1 ; thus substrate uptake occurs only when substrate and microbes are present.
- The microbe removal rate $s(x_1)$ is non-negative and strictly increasing.

The conditions given so far are not particularly restrictive in the setting of ordinary differential equations. The two essential points are the requirement of monotonicity in the uptake function, and the requirement that substrate uptake leads directly to proportional biomass growth; the proportionality factor being set equal to 1 by scaling. We now add other, more specific conditions.

- *Saturation with respect to substrate availability:* There is a strictly increasing function $\bar{a}(x_1)$ such that $\bar{a}(0) = 0$ and $a(x_0, x_1) \leq \bar{a}(x_1)$ for all x_0 and x_1 .
- *Microbe removal is at least as strong as washout:* $s(x_1) \geq \delta x_1$. Thus, the microbes cannot “cling” to the reactor.
- *Removal beats growth:* With increasing microbe population, the growth rate increases more slowly than the removal rate for any fixed substrate concentration, and the ratio tends to zero. Thus for each $x_0 > 0$, the map $x_1 \mapsto a(x_0, x_1)/s(x_1)$ is strictly decreasing to 0. We also require such a property for the upper bound of the growth rate, thus $\bar{a}(x_1)/s(x_1)$ is strictly decreasing to 0 as $x_1 \rightarrow \infty$. In this assumption lies the principal difference to the standard chemostat model.

The extension of the model for two microbe species, under the hypothesis of exploitative competition for a single substrate, is:

$$\begin{aligned} \dot{x}_0 &= r - \delta x_0 - a_1(x_0, x_1) - a_2(x_0, x_2), \\ \dot{x}_1 &= a_1(x_0, x_1) - s_1(x_1, x_2), \\ \dot{x}_2 &= a_2(x_0, x_2) - s_2(x_1, x_2). \end{aligned} \quad (2)$$

Again, x_0 stands for substrate concentration, while x_1 and x_2 denote the biomass concentrations of the respective microbe species. In addition to the terminology and conditions introduced above, we require:

- The substrate uptake rates $a_i(x_0, x_i)$ are non-negative and strictly increasing in both variables, and satisfy $a_i(x_0, 0) = a_i(0, x_i) = 0$. Since we assume exploitative

competition, the uptake rate does not depend on the concentrations of other microbe species.

- Each microbe removal rate $s_i(x_1, x_2)$ is non-negative, strictly increasing in x_i , and increasing in the other variable. Moreover, $s_1(0, x_2) = s_2(x_1, 0) = 0$ for all x_1 and x_2 . Note that inter-species effects on the removal terms are admissible in this setting.
- *Saturation with respect to substrate availability*: There are strictly increasing functions $\bar{a}_i(x_i)$ such that $\bar{a}_i(0) = 0$ and $a_i(x_0, x_i) \leq \bar{a}_i(x_i)$ for all x_0 and x_i , for $i = 1, 2$.
- *Removal beats growth*: The intra-species removal effects incorporate at least washout, and grow faster than the uptake rates. Define strictly increasing functions $\bar{s}_1(x_1) = s_1(x_1, 0)$ and $\bar{s}_2(x_2) = s_2(0, x_2)$, thus $s_i(x_1, x_2) \geq \bar{s}_i(x_i)$ for $i = 1, 2$. We require that $\bar{s}_i(x_i) \geq \delta x_i$ for all x_1 and x_2 , and moreover that for each $x_0 > 0$, the map $x_i \mapsto a_i(x_0, x_i)/s_i(x_1, x_2)$ is strictly decreasing to 0, and $\bar{a}_i(x_i)/\bar{s}_i(x_i)$ decreases strictly to 0 as $x_i \rightarrow \infty$ for all i .

The generalization to more than two microbe species is straightforward. We will focus our attention to the two-species scenario mainly for reasons of notational convenience.

2.2. Stationary points

In this subsection we discuss stationary points of the systems for one, respectively two, microbe species. Several of the results presented here were independently obtained, for a slightly different model, by Braun in his doctoral thesis [4].

We start with an investigation of the single-consumer system (1), keeping the notation and the assumptions from above. Before discussing the stationary points and dynamics in detail, we observe an important consequence of the monotonicity assumptions: In the open positive quadrant,

$$\operatorname{div} \left(\frac{1}{s(x_1)} \begin{pmatrix} r - \delta x_0 - a(x_0, x_1) \\ a(x_0, x_1) - s(x_1) \end{pmatrix} \right) = \frac{-\delta - \frac{\partial a}{\partial x_0}}{s} + \frac{\partial(a/s)}{\partial x_1} < 0.$$

Thus, by Bendixson's criterion the phase portrait of this system contains neither a cycle nor a graphic. Moreover, the inequality

$$\frac{d}{dt}(x_0 + x_1) = r - \delta x_0 - s(x_1) \leq r - \delta(x_0 + x_1)$$

implies that all solutions starting in the positive orthant exist for all $t > 0$, and eventually enter some compact set. These features are fundamental for the following result on local and global dynamics, which is closely related to Braun [4, Proposition 6.3.2].

Proposition 1. (a) *The single-species equation (1) always admits the trivial stationary point $(r/\delta, 0)$. This point is globally asymptotically stable for the system in the closed positive quadrant if $\frac{\partial a}{\partial x_1}(r/\delta, 0) \leq s'(0)$, and unstable if $\frac{\partial a}{\partial x_1}(r/\delta, 0) > s'(0)$.*

(b) *There exists a stationary point in the interior of the positive quadrant if and only if $\frac{\partial a}{\partial x_1}(r/\delta, 0) > s'(0)$. If such a stationary point exists, it is unique and locally asymptotically stable. Moreover, every solution starting in the open positive quadrant converges to this stationary point as $t \rightarrow \infty$.*

Proof. (i) If there is an interior stationary point (y_0, y_1) then $0 < s(y_1) = r - \delta y_0$, so $y_0 < r/\delta$. Moreover, y_1 solves the equation

$$a(y_0, x_1)/s(x_1) = 1. \tag{*}$$

Using monotonicity we find

$$1 < \lim_{x_1 \rightarrow 0} a(y_0, x_1)/s(x_1) < \lim_{x_1 \rightarrow 0} a(r/\delta, x_1)/s(x_1).$$

By our differentiability assumptions, $s'(0)$ exists, and $s'(0) \geq \delta > 0$ due to $s(x_1) \geq \delta x_1$ for all x_1 . Now l'Hospital's rule provides the necessary condition

$$\frac{\partial a}{\partial x_1}(r/\delta, 0) > s'(0)$$

for the existence of an interior stationary point.

(ii) Conversely, suppose that this necessary condition is satisfied. Then there is a $v_1 \in]0, r/\delta[$ such that

$$\lim_{x_1 \rightarrow 0} a(y_0, x_1)/s(x_1) > 1$$

for all $y_0 > v_1$. Therefore equation (*) has a unique solution $y_1 = \phi_1(y_0)$ for all such y_0 , and one also sees, by monotonicity properties of a and s , that ϕ_1 is a strictly increasing function. Thus the entry y_0 of a strictly positive stationary point is a zero of the strictly decreasing map

$$]v_1, r/\delta[\rightarrow \mathbf{R}, \quad x_0 \mapsto r - \delta x_0 - a(x_0, \phi_1(x_0)),$$

and uniqueness follows.

(iii) The functional matrix of the vector field at $(r/\delta, 0)$ is equal to

$$\begin{pmatrix} -\delta & * \\ 0 & \frac{\partial a}{\partial x_1}(r/\delta, 0) - s'(0) \end{pmatrix}$$

since $a(x_0, 0) = 0$ implies $\frac{\partial a}{\partial x_0}(z_0, 0) = 0$ for any z_0 . Therefore the stationary point $(r/\delta, 0)$ is unstable in case $\frac{\partial a}{\partial x_1}(r/\delta, 0) > s'(0)$, and its stable manifold is then con-

tained in the invariant x_0 -axis. By the Poincaré–Bendixson theorem, the theorem of Butler and McGehee (see [31]) and Bendixson’s criterion, the omega limit set of any point in the open positive quadrant cannot contain $(r/\delta, 0)$.

(iv) Now assume $\frac{\partial a}{\partial x_1}(r/\delta, 0) \leq s'(0)$. Then $(r/\delta, 0)$ is the only stationary point in the closed positive quadrant, and the omega limit set of any point in this quadrant equals $\{(r/\delta, 0)\}$: The limit set is not empty since the semi-trajectory lies in a compact set, and it cannot contain more than one point due to Bendixson’s criterion.

(v) If $\frac{\partial a}{\partial x_1}(r/\delta, 0) > s'(0)$ then the omega limit set of any point in the open positive quadrant does not contain $(r/\delta, 0)$ by (iii). The Poincaré–Bendixson theorem and Bendixson’s criterion imply that there must be another stationary point, necessarily in the open positive orthant, which is unique by (ii) and, again by Bendixson, the only point in this limit set. \square

Remark. (a) If there is an interior stationary point (y_0, y_1) then it seems appropriate to call y_0 the “break-even concentration” of the microbe species, in analogy to the classical case.

(b) If there is an interior stationary point (y_0, y_1) then the linearization of the vector field at this point equals

$$\begin{pmatrix} -\delta - \frac{\partial a}{\partial x_0}(y_0, y_1) & -\frac{\partial a}{\partial x_1}(y_0, y_1) \\ \frac{\partial a}{\partial x_0}(y_0, y_1) & \frac{\partial a}{\partial x_1}(y_0, y_1) - s'(y_1) \end{pmatrix}.$$

The lower right entry of this matrix is negative: The map

$$x_1 \mapsto a(a_0, x_1)/s(x_1)$$

has negative derivative, and evaluation of this derivative at (y_0, y_1) , using $a(y_0, y_1) = s(y_1)$, shows the asserted inequality. The matrix therefore has negative trace and positive determinant, which shows asymptotic stability by linearization. This local argument was not needed in the proof above but will be useful later on.

To summarize: The global dynamics of the model variant with one microbe species is very similar to the behavior of the standard chemostat system. The case of two competing microbe populations may turn out to be different, however.

The following can be said about stationary points:

Proposition 2. (a) *The two-species equation (2) has at most three stationary points on the boundary of the positive orthant. The trivial stationary point $(r/\delta, 0, 0)$ always exists. This point is locally asymptotically stable if*

$$\frac{\partial a_1}{\partial x_1}(r/\delta, 0) - \bar{s}'_1(0) < 0 \quad \text{and} \quad \frac{\partial a_2}{\partial x_2}(r/\delta, 0) - \bar{s}'_2(0) < 0,$$

and unstable if one of these quantities is > 0 .

(b) If $\frac{\partial a_1}{\partial x_1}(r/\delta, 0) - \bar{s}'_1(0) > 0$ then there exists a stationary point $(y_0, y_1, 0)$ with $y_1 > 0$.

If $\frac{\partial a_2}{\partial x_2}(r/\delta, 0) - \bar{s}'_2(0) > 0$ then there exists a stationary point $(z_0, 0, z_2)$ with $z_2 > 0$.

(c) Assume that s_1 depends only on x_1 and s_2 depends only on x_2 , thus additional effects on removal are purely intra-specific. In this case, if $z_0 > y_0$ then the second of the stationary points listed in (b) is unstable. This fact allows a biological interpretation similar to the standard model: The species with the higher “break-even concentration” will not survive.

Proof. (i) The assertions about the trivial stationary point follow from straightforward calculations. The two quantities occurring in the stability condition of part (a) are eigenvalues of the linearization at this point.

(ii) The existence assertions of part (b), as well as the fact that there cannot exist more than one stationary point of each respective type, follow from Proposition 1 and its proof. The functional matrix at $(y_0, y_1, 0)$ is equal to

$$\begin{pmatrix} -\delta - \frac{\partial a_1}{\partial x_0}(y_0, y_1) & -\frac{\partial a_1}{\partial x_1}(y_0, y_1) & * \\ \frac{\partial a_1}{\partial x_0}(y_0, y_1) & \frac{\partial a_1}{\partial x_1}(y_0, y_1) - \bar{s}'_1(y_1) & * \\ 0 & 0 & \frac{\partial a_2}{\partial x_2}(y_0, 0) - \frac{\partial s_2}{\partial x_2}(y_1, 0) \end{pmatrix}$$

since

$$\frac{\partial a_2}{\partial x_0}(y_0, 0) = \frac{\partial s_2}{\partial x_1}(y_1, 0) = 0.$$

In view of Proposition 1, stability depends on the sign of the lower right entry. A similar observation applies to a stationary point of type $(z_0, 0, z_2)$.

(iii) Now assume that s_i depends only on x_i for $i = 1, 2$, and that $y_0 < z_0$. Then $\lim_{x_1 \rightarrow 0} a_1(y_0, x_1)/s_1(x_1) > 1$ implies $\lim_{x_1 \rightarrow 0} a_1(z_0, x_1)/s_1(x_1) > 1$, and $\frac{\partial a_1}{\partial x_1}(z_0, 0) > s'_1(0)$ by l’Hôpital. Therefore, the point $(z_0, 0, z_2)$ is unstable. \square

Remark. If s_i depends only on x_i for $i = 1, 2$ then there is at most one stationary point in the open positive orthant. The conditions

$$\frac{\partial a_1}{\partial x_1}(r/\delta, 0) - s'_1(0) > 0 \quad \text{and} \quad \frac{\partial a_2}{\partial x_2}(r/\delta, 0) - s'_2(0) > 0$$

are necessary for its existence. If such a point exists then it is locally asymptotically stable. Moreover, if the existence conditions for both points $(y_0, y_1, 0)$ and $(z_0, 0, z_2)$ are satisfied then one can always force the existence of an interior stationary point by increasing substrate inflow r while keeping all the other parameters and functions

unchanged. This follows from a variant of Braun [4, Proposition 6.3.3], and also from the results in the following subsection.

2.3. Comparison principles and persistence

The investigation of stationary points becomes more involved, and eventually intractable, when models become more complex. In general, a complete and detailed analysis will no longer be feasible. But other tools and methods are available for our scenario: We will state and prove a comparison principle for the two species system in 2.1, which in turn will lead to a persistence theorem.

Proposition 3. *The system*

$$\begin{aligned}
 \dot{\bar{x}}_0 &= r - \delta\bar{x}_0, \\
 \dot{\bar{x}}_1 &= \bar{a}_1(\bar{x}_1) - \bar{s}_1(\bar{x}_1), \\
 \dot{\bar{x}}_2 &= \bar{a}_2(\bar{x}_2) - \bar{s}_2(\bar{x}_2), \\
 \dot{\underline{x}}_0 &= r - \delta\underline{x}_0 - a_1(\underline{x}_0, \bar{x}_1) - a_2(\underline{x}_0, \bar{x}_2), \\
 \dot{\underline{x}}_1 &= a_1(\underline{x}_0, \underline{x}_1) - s_1(\underline{x}_1, \bar{x}_2), \\
 \dot{\underline{x}}_2 &= a_2(\underline{x}_0, \underline{x}_2) - s_2(\bar{x}_1, \underline{x}_2)
 \end{aligned} \tag{3}$$

yields componentwise upper and lower estimates for the two-consumer system (2), in the following sense: If $u(t)$ is a solution of (2) and $(\bar{u}(t), \underline{u}(t))$ solves (3), and there is some t_0 such that $\bar{u}(t_0) \geq u(t_0) \geq \underline{u}(t_0)$ then the inequalities

$$\bar{u}(t) \geq u(t) \geq \underline{u}(t)$$

hold for all $t \geq t_0$.

Proof. We use an argument from [24]. Considering the product system

$$\begin{aligned}
 \dot{\bar{x}}_0 &= r - \delta\bar{x}_0, \\
 \dot{\bar{x}}_1 &= \bar{a}_1(\bar{x}_1) - \bar{s}_1(\bar{x}_1), \\
 \dot{\bar{x}}_2 &= \bar{a}_2(\bar{x}_2) - \bar{s}_2(\bar{x}_2), \\
 \dot{x}_0 &= r - \delta x_0 - a_1(x_0, x_1) - a_2(x_0, x_2), \\
 \dot{x}_1 &= a_1(x_0, x_1) - s_1(x_1, x_2), \\
 \dot{x}_2 &= a_2(x_0, x_2) - s_2(x_1, x_2), \\
 \dot{\underline{x}}_0 &= r - \delta\underline{x}_0 - a_1(\underline{x}_0, \bar{x}_1) - a_2(\underline{x}_0, \bar{x}_2), \\
 \dot{\underline{x}}_1 &= a_1(\underline{x}_0, \underline{x}_1) - s_1(\underline{x}_1, \bar{x}_2), \\
 \dot{\underline{x}}_2 &= a_2(\underline{x}_0, \underline{x}_2) - s_2(\bar{x}_1, \underline{x}_2),
 \end{aligned}$$

the assertion is equivalent to positive invariance of the cone defined by

$$\bar{x}_i \geq x_i \quad \text{and} \quad x_i \geq \underline{x}_i \quad (0 \leq i \leq 2).$$

Positive invariance of the cone follows from standard arguments. For instance, at a boundary point of the cone satisfying $\bar{x}_0 = x_0$ one has

$$\dot{\bar{x}}_0 - \dot{x}_0 = a_1(x_0, x_1) + a_2(x_0, x_2) \geq 0,$$

and at a boundary point satisfying $x_1 = \underline{x}_1$ we find

$$\dot{x}_1 - \dot{\underline{x}}_1 = (a_1(x_0, x_1) - a_1(\underline{x}_0, x_1)) + (s_1(x_1, \bar{x}_2) - s_1(x_1, x_2)) \geq 0$$

by monotonicity of the functions involved. \square

It should be noted that the correspondence between monotonicity and positive invariance of certain subsets in product systems has been observed before; see Gouze and Hadeler [16]. Concerning the behavior of the comparison system, we have:

Proposition 4. (a) *If $\bar{a}_i(x_i) - \bar{s}_i(x_i)$ has positive values for some x_i ($i = 1, 2$) then the subsystem of (3) for $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$ has a unique strictly positive stationary point $(\bar{y}_0, \bar{y}_1, \bar{y}_2)$, and every solution of the subsystem starting in the open positive orthant converges toward this point as $t \rightarrow \infty$.*

(b) *If r is sufficiently large then the four-dimensional subsystem for $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \underline{x}_0)$ has a unique strictly positive stationary point $(\bar{y}_0, \bar{y}_1, \bar{y}_2, \underline{y}_0)$, and every solution of the subsystem starting in the open positive orthant converges toward this point as $t \rightarrow \infty$. Moreover, with increasing r the value of \underline{y}_0 increases and is unbounded, while \bar{y}_1 and \bar{y}_2 are unchanged.*

(c) *If $a_1(\underline{y}_0, x_1) > s_1(x_1, \bar{y}_2)$ for some x_1 , and $a_2(\underline{y}_0, x_2) > s_2(\bar{y}_1, x_2)$ for some x_2 then the system*

$$\dot{\underline{x}}_1 = a_1(\underline{y}_0, \underline{x}_1) - s_1(\underline{x}_1, \bar{y}_2),$$

$$\dot{\underline{x}}_2 = a_2(\underline{y}_0, \underline{x}_2) - s_2(\bar{y}_1, \underline{x}_2)$$

has a unique strictly positive stationary point $(\underline{y}_1, \underline{y}_2)$. Moreover, every solution of (3), which starts in the open positive orthant, converges toward $(\bar{y}_0, \bar{y}_1, \bar{y}_2, \underline{y}_0, \underline{y}_1, \underline{y}_2)$ as $t \rightarrow \infty$.

Proof. The assertion of part (a) follows directly from the hypotheses: The subsystem is a product of equations with a single dependent variable, and monotonicity ensures that there is at most one strictly positive stationary point, while the additional hypotheses, together with $\bar{a}_i(x_i)/\bar{s}_i(x_i) \rightarrow 0$ for $x_i \rightarrow \infty$, ensure existence.

The fundamental ingredient for the remainder of the proof is a theorem by Markus on asymptotically autonomous systems; see Thieme [33]. A consequence of this theorem for the asymptotic behavior of the four-dimensional subsystem in part (b) is that for any solution $(\bar{u}_0, \bar{u}_1, \bar{u}_2, \underline{u}_0)$ starting in the open positive orthant, the limiting behavior of $\underline{u}_0(t)$ can be determined from the autonomous equation

$$\dot{x}_0 = r - \delta x_0 - a_1(x_0, \bar{y}_1) - a_2(x_0, \bar{y}_2).$$

The right-hand side of this equation is a strictly decreasing function of x_0 , with value r at $x_0 = 0$, and thus admits exactly one zero \underline{y}_0 which increases with r and is unbounded.

As for part (c), a variant of the same argument applies. \square

It should be emphasized that the hypotheses in this proposition have a sensible biological interpretation. The hypotheses in part (a) simply ensure that each microbe species will survive on its own. The hypothesis in part (c) means that for some sufficiently high substrate concentration growth beats removal for each species, regardless of effects of the other species. In other words, inter-species competition is not too strong.

The main result of this section now follows directly from the two propositions above.

Theorem 1. *Let the two-consumer system (2) be given, and assume that the hypotheses of Proposition 4 are satisfied. Then both microbe species persist in the system: There is a constant $\rho > 0$ such that every solution $u(t)$ starting in the open positive orthant satisfies $\liminf_{t \rightarrow \infty} u_i(t) \geq \rho$ for $i = 1, 2$.*

Remark. If the hypotheses of Proposition 4 are satisfied then there exists a stationary point in the open positive orthant: The orthant contains a compact and convex positively invariant subset, and the existence of a stationary point follows from Brouwer's fixed point theorem. Braun [4] directly showed the existence and local asymptotic stability of such a stationary point for a slightly different model.

Example. Consider the particular system

$$\begin{aligned} \dot{x}_0 &= r - \delta x_0 - b_1(x_0)x_1 - b_2(x_0)x_2, \\ \dot{x}_1 &= x_1 (b_1(x_0) - \alpha_{11}x_1 - \alpha_{12}x_2), \\ \dot{x}_2 &= x_2 (b_2(x_0) - \alpha_{21}x_1 - \alpha_{22}x_2) \end{aligned} \tag{4}$$

with strictly increasing and bounded functions b_i , and constants $\alpha_{11} > 0$, $\alpha_{22} > 0$, $\alpha_{12} \geq 0$, $\alpha_{21} \geq 0$.

Letting $\beta_i = \lim_{x_0 \rightarrow \infty} b_i(x_0)$ for $i = 1, 2$, a straightforward verification shows that the hypotheses of Proposition 4 are satisfied if $\beta_1 > \delta$, $\beta_2 > \delta$, and

$$\beta_2 - \delta - \frac{\alpha_{21}}{\alpha_{11}} (\beta_1 - \delta) > 0 \quad \text{and} \quad \beta_1 - \delta - \frac{\alpha_{12}}{\alpha_{22}} (\beta_2 - \delta) > 0.$$

In addition to the obvious conditions for single-species survival this requirement means that inter-species competition has a lesser effect than intra-species competition: Both α_{21}/α_{11} and α_{12}/α_{22} have to be “small”.

Generally, the hypotheses of Proposition 4 and Theorem 1 may be summarized from a biological perspective as follows: According to our basic assumption, removal beats growth for both species in case of high concentrations. But on the other hand one hypothesis of Proposition 4 implies that growth beats removal of each species when its concentration is low, regardless of the concentration of the other species. In this sense, inter-species competition is weak. An additional hypothesis requires that sufficient substrate be present. Given such a scenario, the two species will coexist. For the classical chemostat model, introducing washout at high concentrations for both species will primarily affect the better exploiter and thus create room for the weaker one to persist. Thus the possibility of coexistence is intuitively clear. The exclusion principle appears in a different light when washout is density-dependent: Even if the hypotheses of Proposition 4 and Theorem 1 are satisfied, there may be extinction of all but one species due to insufficient substrate availability. Moreover, one will not expect coexistence, for instance, if the additional competition terms have a strong inter-species component. In such cases, the break-even concentrations may still be relevant; see Proposition 2.

We note that the comparison and persistence results can be directly generalized to the case of more than two microbe species.

3. A stochastic chemostat model

3.1. The model

We now turn to a continuous time chemostat model which takes random effects into account. This will lead to stochastic generalizations of Eqs. (1) and (2), but our assumptions on the functions involved are somewhat less restrictive than those in Section 2. In particular, we include the classical chemostat model in our considerations.

There are different possible approaches to include random effects in the model, both from a biological and from a mathematical perspective. Our basic approach is analogous to that of Beddington and May [2]. A different path was taken by Stephanopoulos et al. [32] who superimposed a one-dimensional white noise process on the dilution rate, in order to investigate stochastic effects for the classical model in the case of equal, or nearly equal break-even concentrations. It seems appropriate to give a detailed derivation of the model, including the technical issues involved. This is also motivated by the observation that the intuitive choice of diffusion coefficients by Foster and Young [10]

turned out to be problematic. The work of Stephanopoulos et al. [32] also involves an ad hoc choice.

Let us first consider a model for a single microbe species. We will show in detail that a reasonable stochastic analogue of the deterministic equation (1) is given by

$$\begin{aligned}dX_0 &= \{r - \delta X_0 - a(X_0, X_1)\} dt + \sigma_0 X_0 dW_0(t), \\dX_1 &= \{a(X_0, X_1) - s(X_1)\} dt + \sigma_1 X_1 dW_1(t)\end{aligned}\quad (5)$$

with independent Brownian motions W_0 and W_1 .

To derive this, we first consider a discrete time Markov chain. For a fixed time increment $\Delta t > 0$ we define a process $X^{(\Delta t)}(t) = (X_0^{(\Delta t)}(t), X_1^{(\Delta t)}(t))^T$ for $t = 0, \Delta t, 2\Delta t, \dots$. Here $X_0^{(\Delta t)}(t)$ denotes substrate concentration and $X_1^{(\Delta t)}(t)$ denotes biomass concentration of the microbe species. Let $X^{(\Delta t)}(0) = \xi$ with a deterministic initial value $\xi \in (0, \infty)^2$. Let $\{R_i^{(\Delta t)}(k)\}_{k=0}^\infty$, $i = 0, 1$, be two sequences of random variables. Suppose that these variables are jointly independent and that within each sequence the variables are identically distributed such that

$$ER_i^{(\Delta t)}(k) = 0, \quad E\left[R_i^{(\Delta t)}(k)\right]^2 = \sigma_i^2 \Delta t, \quad i = 0, 1; \quad k = 0, 1, \dots, \quad (6)$$

where $\sigma_0, \sigma_1 \geq 0$ are constants that reflect the size of the stochastic effects. The variable $R_1^{(\Delta t)}(k)$ is supposed to capture the effect of random influences on the biomass concentration of the microbe species during the period $[k\Delta t, (k+1)\Delta t)$. We assume that $X_1^{(\Delta t)}$ grows within that time period according to the deterministic equation (1) and, in addition, by the random amount $R_1^{(\Delta t)}(k)X_1^{(\Delta t)}(k\Delta t)$. Random effects on the substrate concentration are similarly modelled by $R_0^{(\Delta t)}(k)$. Specifically, for $k = 0, 1, \dots$ we set

$$\begin{aligned}X_0^{(\Delta t)}((k+1)\Delta t) &= X_0^{(\Delta t)}(k\Delta t) \\&\quad + \Delta t \left\{ r - \delta X_0^{(\Delta t)}(k\Delta t) - a\left(X_0^{(\Delta t)}(k\Delta t), X_1^{(\Delta t)}(k\Delta t)\right) \right\} \\&\quad + R_0^{(\Delta t)}(k)X_0^{(\Delta t)}(k\Delta t)\end{aligned}$$

and

$$\begin{aligned}X_1^{(\Delta t)}((k+1)\Delta t) &= X_1^{(\Delta t)}(k\Delta t) \\&\quad + \Delta t \left\{ a\left(X_0^{(\Delta t)}(k\Delta t), X_1^{(\Delta t)}(k\Delta t)\right) - s\left(X_1^{(\Delta t)}(k\Delta t)\right) \right\} \\&\quad + R_1^{(\Delta t)}(k)X_1^{(\Delta t)}(k\Delta t).\end{aligned}$$

For positive arguments, we require a and s to be strictly increasing in their respective variables, but we will not generally assume that removal beats growth, as we did in

Section 2. In particular, the classical chemostat is included in this model. Moreover, we set $a(x_0, x_1) = 0$ if $x_0 \leq 0$ or $x_1 \leq 0$, and $s(x_1) = 0$ if $x_1 \leq 0$.

We will show that $X^{(\Delta t)}(t)$ converges to a diffusion process as $\Delta t \rightarrow 0$. We first determine the drift coefficients of the diffusion. Let $\Pi^{(\Delta t)}(x, dy)$ denote the transition probabilities of the homogeneous Markov chain $\left\{ X^{(\Delta t)}(k\Delta t) \right\}_{k=0}^{\infty}$, that is

$$\Pi^{(\Delta t)}(x, A) = P \left\{ X^{(\Delta t)}((k + 1)\Delta t) \in A \mid X^{(\Delta t)}(k\Delta t) = x \right\}$$

for all $x = (x_0, x_1) \in \mathbb{R}^2$ and all Borel sets $A \subset \mathbb{R}^2$. By (6),

$$\begin{aligned} \frac{1}{\Delta t} \int (y_0 - x_0) \Pi^{(\Delta t)}(x, dy) &= r - \delta x_0 - a(x) + \frac{x_0}{\Delta t} ER_0^{(\Delta t)}(0) \\ &= r - \delta x_0 - a(x), \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{1}{\Delta t} \int (y_1 - x_1) \Pi^{(\Delta t)}(x, dy) &= a(x) - s(x_1) + \frac{x_1}{\Delta t} ER_1^{(\Delta t)}(0) \\ &= a(x) - s(x_1). \end{aligned} \tag{8}$$

To determine the diffusion coefficients consider the moments

$$g_{ij}^{(\Delta t)}(x) = \frac{1}{\Delta t} \int (y_i - x_i)(y_j - x_j) \Pi^{(\Delta t)}(x, dy), \quad i, j = 0, 1.$$

By (6),

$$\begin{aligned} \left| g_{11}^{(\Delta t)}(x) - \sigma_1^2 x_1^2 \right| &= \left| \frac{1}{\Delta t} E \left[\Delta t \{a(x) - s(x_1)\} + R_1^{(\Delta t)}(0)x_1 \right]^2 - \sigma_1^2 x_1^2 \right| \\ &= \left| \Delta t \{a(x) - s(x_1)\}^2 + 2 \{a(x) - s(x_1)\} x_1 ER_1^{(\Delta t)}(0) \right. \\ &\quad \left. + \frac{1}{\Delta t} x_1^2 E \left[R_1^{(\Delta t)}(0) \right]^2 - \sigma_1^2 x_1^2 \right| \\ &= \Delta t \{a(x) - s(x_1)\}^2. \end{aligned}$$

Therefore, since a and s are bounded on compact sets,

$$\lim_{\Delta t \rightarrow 0+} \sup_{\|x\| \leq K} \left| g_{11}^{(\Delta t)}(x) - \sigma_1^2 x_1^2 \right| = 0 \tag{9}$$

for all $0 < K < \infty$. Similarly,

$$\lim_{\Delta t \rightarrow 0^+} \sup_{\|x\| \leq K} \left| g_{00}^{(\Delta t)}(x) - \sigma_0^2 x_0^2 \right| = 0, \quad \lim_{\Delta t \rightarrow 0^+} \sup_{\|x\| \leq K} \left| g_{01}^{(\Delta t)}(x) \right| = 0 \tag{10}$$

for all $0 < K < \infty$.

Assuming that $E \left[R_i^{(\Delta t)}(k) \right]^4 = o(\Delta t)$ for $i = 0, 1$, one may verify that for all $0 < K < \infty$,

$$\lim_{\Delta t \rightarrow 0^+} \sup_{\|x\| \leq K} \frac{1}{\Delta t} \int \|y - x\|^3 \Pi^{(\Delta t)}(x, dy) = 0. \tag{11}$$

Finally, extend the definition of $X^{(\Delta t)}(t)$ to all $t \geq 0$ by setting $X^{(\Delta t)}(t) = X^{(\Delta t)}(k\Delta t)$ for $t \in [k\Delta t, (k + 1)\Delta t)$. According to [7, Theorem 7.1, p. 297, Lemma 8.2, p. 306], we can conclude from Eqs. (7)–(11):

Proposition 5. *As $\Delta t \rightarrow 0$, $X^{(\Delta t)}(t)$ converges weakly to the solution $X(t)$ of the stochastic differential equation (5) with initial condition $X(0) = \xi$, provided a unique solution exists.*

Weak convergence refers to the space of right continuous functions with left limits, endowed with the Skorokhod topology, see [7, Section 8.6] for details.

Throughout P_ξ denotes the probability measure corresponding to $X(t)$ with initial value ξ , and E_ξ denotes the expectation with respect to P_ξ .

Proposition 6. *Assume that there exist constants $\varepsilon, c, C > 0$ and $0 < r' < r$ such that*

$$\begin{aligned} a(x_0, x_1) &\leq Cx_0x_1 + r', \quad \text{for all } x \text{ with } 0 \leq x_0 \leq \varepsilon, \\ s(x_1) &\leq Cx_1, \quad \text{for all } 0 \leq x_1 \leq \varepsilon, \\ (a(x_0, x_1) - s(x_1))x_1 &\leq c(1 + \|x\|^2), \quad \text{for all } x. \end{aligned}$$

Then for every initial value $\xi \in (0, \infty)^2$, there exists a strong solution $X(t)$ of (5) defined for all $t \geq 0$, pathwise uniqueness holds and

$$P_\xi \left\{ X(t) \in (0, \infty)^2 \text{ for all } t \geq 0 \right\} = 1.$$

The proof follows from Proposition A.1 in the Appendix. In the following we will always assume that the hypotheses of Proposition 6 are satisfied.

Remark. It should be emphasized that the conditions given in Proposition 6 are not at all restrictive, and are satisfied for the classical deterministic chemostat models as well

as the variant introduced in Section 2. The first two assumptions are a consequence of the basic differentiability and boundedness properties in Section 2. For the third assumption to be violated, growth would have to beat washout with increasing microbe concentration, and do so in a very strong manner.

A similar discussion, starting from the deterministic equation (2), gives rise to the stochastic differential equation

$$\begin{aligned} dX_0 &= \{r - \delta X_0 - a_1(X_0, X_1) - a_2(X_0, X_2)\} dt + \sigma_0 X_0 dW_0(t), \\ dX_1 &= \{a_1(X_0, X_1) - s_1(X_1, X_2)\} dt + \sigma_1 X_1 dW_1(t), \\ dX_2 &= \{a_2(X_0, X_2) - s_2(X_1, X_2)\} dt + \sigma_2 X_2 dW_2(t), \end{aligned} \tag{12}$$

where W_0, W_1, W_2 are independent Brownian motions, and $\sigma_0, \sigma_1, \sigma_2 \geq 0$. To ensure that $X(t)$ stays in the open positive orthant we assume for the remainder of this paper that there exist constants $\varepsilon, c, C > 0$ and $0 < r' < r$ such that

$$\begin{aligned} a_1(x_0, x_1) + a_2(x_0, x_2) &\leq Cx_0(x_1 + x_2) + r', \quad \text{for all } x \text{ with } 0 \leq x_0 \leq \varepsilon, \\ s_i(x_1, x_2) &\leq Cx_i, \quad \text{for all } x \text{ with } 0 \leq x_i \leq \varepsilon, \\ (a_i(x_0, x_i) - s_i(x_1, x_2)) x_i &\leq c \left(1 + \|x\|^2\right) \quad \text{for all } x, \quad i = 1, 2. \end{aligned}$$

Now consider the transformed process $Y(t)$, where $Y_i(t) = \log X_i(t), i = 0, 1, 2$. By Ito's lemma,

$$\begin{aligned} dY_0 &= \left\{ \left[r - a_1(e^{Y_0}, e^{Y_1}) - a_2(e^{Y_0}, e^{Y_2}) \right] e^{-Y_0} - \delta - \frac{\sigma_0^2}{2} \right\} dt + \sigma_0 dW_0(t), \\ dY_1 &= \left\{ \left[a_1(e^{Y_0}, e^{Y_1}) - s_1(e^{Y_1}, e^{Y_2}) \right] e^{-Y_1} - \frac{\sigma_1^2}{2} \right\} dt + \sigma_1 dW_1(t), \\ dY_2 &= \left\{ \left[a_2(e^{Y_0}, e^{Y_2}) - s_2(e^{Y_1}, e^{Y_2}) \right] e^{-Y_2} - \frac{\sigma_2^2}{2} \right\} dt + \sigma_2 dW_2(t). \end{aligned} \tag{13}$$

The diffusion matrix of Y is everywhere nonsingular, provided $\sigma_0 > 0, \sigma_1 > 0, \sigma_2 > 0$. From [3] we get:

Lemma 1. *Assume that $\sigma_0 > 0, \sigma_1 > 0, \sigma_2 > 0$. Then the solution $X(t)$ of the stochastic two-consumer system (12) is either recurrent or transient. Thus either for every nonempty open set $U \subset (0, \infty)^3$ and every $\xi \in (0, \infty)^3$,*

$$P_\xi \{X(t) \in U \text{ for some sequence of } t\text{'s increasing to infinity}\} = 1$$

or for every ξ ,

$$P_\xi \left\{ \lim_{t \rightarrow \infty} \sum_{i=0}^2 \left(X_i(t) + \frac{1}{X_i(t)} \right) = \infty \right\} = 1.$$

In particular, if $X(t)$ is recurrent then

$$P_\xi \left\{ \limsup_{t \rightarrow \infty} X_1(t) > 0 \right\} = P_\xi \left\{ \limsup_{t \rightarrow \infty} X_2(t) > 0 \right\} = 1.$$

That is, both microbe species persist.

The same dichotomy into transience and recurrence applies to the stochastic system (5) for a single microbe species.

This lemma is our starting point for more detailed investigations. In the recurrent case we have the obvious biological interpretation of persistence and coexistence in the two-consumer setting. In the transient case, one possible scenario is that the consumer, at least one species, is washed out. However, transience would also occur, for instance, if one concentration grows beyond all bounds or the concentrations alternate between ever higher and lower values in a suitable succession. Thus two of the scenarios, which this technical result leaves open, correspond directly to the deterministic setting, while the remaining ones seem to admit no sensible biological interpretation. But this lemma is only a starting point, and in any case, a more detailed investigation is necessary.

3.2. Persistence for a single microbe species

It seems reasonable to expect that the solution $X(t)$ of the single-species system (5) is recurrent if the substrate inflow and uptake rates are sufficiently large and the death and outflow rates are sufficiently small, at least when the biomass concentration becomes low. This notion is confirmed, and made precise, in the following theorem.

Let L denote the second-order partial differential operator associated with $X(t)$ (see [23, pp. 281–284]), that is,

$$\begin{aligned} Lg(x_0, x_1) = & [r - \delta x_0 - a(x_0, x_1)] \frac{\partial g(x_0, x_1)}{\partial x_0} + \frac{\sigma_0^2}{2} x_0^2 \frac{\partial^2 g(x_0, x_1)}{\partial x_0^2} \\ & + [a(x_0, x_1) - s(x_1)] \frac{\partial g(x_0, x_1)}{\partial x_1} + \frac{\sigma_1^2}{2} x_1^2 \frac{\partial^2 g(x_0, x_1)}{\partial x_1^2}. \end{aligned}$$

Theorem 2. *Let the stochastic single-consumer system (5) be given, and assume that removal beats growth for the corresponding deterministic model. Suppose that $\sigma_0 > 0$*

and $\sigma_1 > 0$. Suppose that there exists $c_1 > 0$ such that

$$a(x_0, x_1) \leq c_1 x_0 x_1, \quad \text{for all } x_0, x_1 > 0, \tag{14}$$

$$\lim_{x_1 \rightarrow \infty} \frac{s(x_1)}{x_1} = \infty, \tag{15}$$

and that there exist $\lambda \in (0, 1)$ and $\varepsilon_1, \eta > 0$ such that

$$\inf_{\substack{0 < x_0 \\ 0 < x_1 \leq \varepsilon_1}} \lambda \frac{r}{x_0} + (1 - \lambda) \frac{a(x_0, x_1)}{x_1} > \eta + \lambda \left(\delta + \frac{\sigma_0^2}{2} \right) + (1 - \lambda) \left(s'(0) + \frac{\sigma_1^2}{2} \right). \tag{16}$$

Then $X(t)$ is recurrent.

Proof. We use the following recurrence criterion. The process $X(t)$ is recurrent if there exists a compact set $K \subset (0, \infty)^2$ such that $P_x \{ \tau_K < \infty \} = 1$ for all $x \in (0, \infty)^2 \setminus K$, where $\tau_K = \inf \{ t > 0 : X(t) \in K \}$, see, e.g., [3]. We verify this condition by showing that there exists a non-negative function $g \in C^2(0, \infty)^2$ such that $\sup_{x \notin K} Lg(x) < 0$ for some compact set K . This is sufficient by [7, Theorem 5.3, p. 268].

Let $\gamma > 0$ and $\varepsilon_2 > 0$ be so small that

$$\frac{s(x_1)}{x_1} < s'(0) + \frac{\eta}{3}, \quad \text{for all } 0 < x_1 \leq \varepsilon_2, \tag{17}$$

$$r\gamma + c_1 \varepsilon_2 < \frac{\eta}{3}. \tag{18}$$

Define

$$g(x_0, x_1) = \gamma x_0 - \lambda \log x_0 + \gamma x_1 - (1 - \lambda) \log x_1 + \rho,$$

where $\rho > 0$ is so large that $g(x_0, x_1) \geq 0$ for all $x_0, x_1 > 0$. Then

$$\begin{aligned} Lg(x_0, x_1) &= \left[r - \delta x_0 - a(x_0, x_1) \right] \left(\gamma - \frac{\lambda}{x_0} \right) + \lambda \frac{\sigma_0^2}{2} \\ &\quad + [a(x_0, x_1) - s(x_1)] \left(\gamma - \frac{1 - \lambda}{x_1} \right) + (1 - \lambda) \frac{\sigma_1^2}{2}. \end{aligned} \tag{19}$$

Thus for all $x_0, x_1 > 0$,

$$\begin{aligned} Lg(x_0, x_1) &\leq r\gamma + \lambda \left[-\frac{r}{x_0} + \delta + \frac{a(x_0, x_1)}{x_0} + \frac{\sigma_0^2}{2} \right] \\ &\quad + (1 - \lambda) \left[-\frac{a(x_0, x_1)}{x_1} + \frac{s(x_1)}{x_1} + \frac{\sigma_1^2}{2} \right]. \end{aligned}$$

Consequently, if $x_1 \leq \min\{\varepsilon_1, \varepsilon_2\}$, then by (14), (17), (16), and (18),

$$\begin{aligned} Lg(x_0, x_1) &\leq r\gamma + \lambda \left[-\frac{r}{x_0} + \delta + \frac{\sigma_0^2}{2} \right] + c_1 x_1 \\ &\quad + (1 - \lambda) \left[-\frac{a(x_0, x_1)}{x_1} + s'(0) + \frac{\sigma_1^2}{2} \right] + \frac{\eta}{3} \\ &\leq r\gamma - \eta + c_1 \varepsilon_2 + \frac{\eta}{3} \leq -\frac{\eta}{3}. \end{aligned}$$

We next show that $Lg(x_0, x_1)$ is negative if x_1 is large. By (15) and since $\bar{a}(x_1)/s(x_1) \rightarrow 0$ as $x_1 \rightarrow \infty$, there exists $M_1 > 0$ such that for all $x_1 > M_1$,

$$\lambda c_1 x_1 \leq \frac{\gamma}{4} s(x_1), \quad \bar{a}(x_1) \leq \frac{1}{4} s(x_1), \quad \frac{1 - \lambda}{x_1} \leq \frac{\gamma}{2}.$$

From these inequalities and (14) it follows that, for x_1 sufficiently large,

$$\lambda \frac{a(x_0, x_1)}{x_0} \leq \frac{\gamma}{4} s(x_1), \quad [a(x_0, x_1) - s(x_1)] \left(\gamma - \frac{1 - \lambda}{x_1} \right) \leq -\frac{3\gamma}{8} s(x_1).$$

Inserting this into (19), we obtain

$$Lg(x_0, x_1) \leq r\gamma + \lambda\delta + \frac{\gamma}{4} s(x_1) + \lambda \frac{\sigma_0^2}{2} - \frac{3\gamma}{8} s(x_1) + (1 - \lambda) \frac{\sigma_1^2}{2}.$$

It is now obvious that there exists $M_2 > M_1$ such that $Lg(x_0, x_1) \leq -1$ for all $x_0 > 0$ and $x_1 > M_2$.

It remains to consider the case where x_0 is small or large. For all $x_0 > 0$ and $x_1 \in [\min\{\varepsilon_1, \varepsilon_2\}, M_2]$,

$$\frac{a(x_0, x_1)}{x_0} \leq c_1 x_1 \leq c_1 M_2, \quad \frac{s(x_1)}{x_1} \leq c_2$$

for some $c_2 < \infty$. Hence

$$Lg(x_0, x_1) \leq r\gamma + \lambda\delta + \lambda c_1 M_2 + \lambda \frac{\sigma_0^2}{2} - \delta x_0 \gamma - \lambda \frac{r}{x_0} + c_2 + (1 - \lambda) \frac{\sigma_1^2}{2}.$$

This shows that there exist constants $0 < \varepsilon_0 < M_0$ such that $Lg(x_0, x_1) \leq -1$ if $x_0 \in (0, \varepsilon_0] \cup [M_0, \infty)$ and $x_1 \in [\min\{\varepsilon_1, \varepsilon_2\}, M_2]$.

Altogether, we have shown that $Lg(x) \leq \max\{-1, -\eta/3\}$ for all $x \notin K$, where $K = [\varepsilon_0, M_0] \times [\min\{\varepsilon_1, \varepsilon_2\}, M_2]$. \square

In a biological interpretation, condition (14) bounds the specific growth rate, and condition (15) stipulates that the removal rate increases stronger than linearly for high concentrations. Thus the theorem does not apply to the classical chemostat. The quite technical condition (16) means, roughly, that substrate inflow and growth terms must be strong enough to beat washout and stochastic effects for low concentration. To illustrate this, consider the special case when $a(x_0, x_1) = c_1 \cdot x_0 x_1$. Then one finds

$$\inf_{\substack{0 < x_0 \\ 0 < x_1 \leq \varepsilon_1}} \lambda \frac{r}{x_0} + (1 - \lambda) \frac{a(x_0, x_1)}{x_1} = 2\sqrt{\lambda(1 - \lambda)} \cdot \sqrt{r \cdot c_1},$$

hence condition (16) can be enforced, for instance, by increasing the inflowing substrate concentration. Note that (16) requires the diffusion coefficients to be small. While such a condition is far from being necessary for recurrence, it will be shown in Theorem 4 that $X(t)$ becomes transient if σ_1 is sufficiently large.

We next investigate the behavior of $X(t)$ near a nontrivial stationary point x^* of the deterministic equation (1). In contrast to the deterministic solutions, the stochastic solutions do not converge to x^* . However, we will prove a stability result to the effect that the time average of $E_x \|X(t) - x^*\|^2$ is small, provided the diffusion coefficients are sufficiently small. We restrict attention to the case where the substrate uptake rate is proportional to biomass concentration, so that $a(x_0, x_1) = \tilde{a}(x_0)x_1$ for some strictly increasing function \tilde{a} . Furthermore, we consider the “washout only” scenario here, so that the microbe removal rate is equal to the washout rate. Then (5) takes the form

$$\begin{aligned} dX_0 &= \{r - \delta X_0 - \tilde{a}(X_0)X_1\} dt + \sigma_0 X_0 dW_0(t), \\ dX_1 &= \{\tilde{a}(X_0) - \delta\} X_1 dt + \sigma_1 X_1 dW_1(t). \end{aligned} \tag{20}$$

Theorem 3. *In the particular stochastic single-consumer system (20) let \tilde{a} be differentiable such that*

$$\begin{aligned} \alpha_0 &:= \inf \{\tilde{a}'(x_0) : x_0 \in [0, \infty)\} > 0, \\ \alpha_1 &:= \sup \{\tilde{a}'(x_0) : x_0 \in [0, \infty)\} < \infty. \end{aligned}$$

Suppose that the corresponding deterministic differential equation has a nontrivial stationary point $x^ = (x_0^*, x_1^*) \in (0, \infty)^2$, thus*

$$r - \delta x_0^* - \tilde{a}(x_0^*)x_1^* = 0, \quad \tilde{a}(x_0^*) - \delta = 0.$$

Set $\delta_0 = \delta - \max\{\sigma_0^2, \sigma_1^2\}$ and suppose that $\delta_0 > 0$. Then for every initial value $x \in (0, \infty)^2$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E_x \int_0^t \|X(s) - x^*\|^2 ds \leq \frac{2}{\alpha_0} \left(\max\{\sigma_0^2, \sigma_1^2\} \frac{r^2 \alpha_1}{\delta^2 \delta_0} + \sigma_1^2 x_1^* \right).$$

Proof. By the mean value theorem we have

$$|\tilde{a}(x_0) - \delta| \geq \alpha_0 |x_0 - x_0^*|, \quad (21)$$

$$|\tilde{a}(x_0) - \delta| \leq \alpha_1 |x_0 - x_0^*| \quad (22)$$

for all $x_0 \geq 0$.

Let L denote the differential operator associated with $X(t)$. Following [15] we define the Liapunov function

$$g(x) = \left(x_0 + x_1 - \frac{r}{\delta}\right)^2 + \frac{4\delta_0}{\alpha_1} \left(x_1 - x_1^* + x_1^* \log \frac{x_1}{x_1^*}\right).$$

It is easily seen that g is non-negative, and

$$\begin{aligned} Lg(x) &= -2\delta \left(x_0 + x_1 - \frac{r}{\delta}\right)^2 + \sigma_0^2 x_0^2 + \sigma_1^2 x_1^2 \\ &\quad + \frac{4\delta_0}{\alpha_1} [\tilde{a}(x_0) - \delta](x_1 - x_1^*) + \frac{2\delta_0}{\alpha_1} \sigma_1^2 x_1^*. \end{aligned}$$

We will show that for all x ,

$$Lg(x) \leq -\frac{\delta_0 \alpha_0}{\alpha_1} \|x - x^*\|^2 + c,$$

where

$$c = 2 \max\{\sigma_0^2, \sigma_1^2\} \frac{r^2}{\delta^2} + \frac{2\delta_0}{\alpha_1} \sigma_1^2 x_1^*.$$

It then follows from [25, Theorem 6, p. 50], that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E_x \int_0^t \|X(s) - x^*\|^2 ds \leq \frac{c \alpha_1}{\delta_0 \alpha_0},$$

proving the assertion. For all x ,

$$\begin{aligned} \sigma_0^2 x_0^2 + \sigma_1^2 x_1^2 &\leq \max\{\sigma_0^2, \sigma_1^2\} (x_0 + x_1)^2 \\ &\leq 2 \max\{\sigma_0^2, \sigma_1^2\} \left[\left(x_0 + x_1 - \frac{r}{\delta}\right)^2 + \frac{r^2}{\delta^2} \right]. \end{aligned}$$

Using that $r/\delta = x_0^* + x_1^*$, we obtain

$$Lg(x) \leq -2\delta_0 (x_0 - x_0^* + x_1 - x_1^*)^2 + \frac{4\delta_0}{\alpha_1} [\tilde{a}(x_0) - \delta](x_1 - x_1^*) + c.$$

Suppose first that $(x_0 - x_0^*)(x_1 - x_1^*) \geq 0$. Then, by (22),

$$\begin{aligned} Lg(x) &\leq -2\delta_0 (x_0 - x_0^* + x_1 - x_1^*)^2 + 4\delta_0 |x_0 - x_0^*| |x_1 - x_1^*| + c \\ &= -2\delta_0 \|x - x^*\|^2 + c. \end{aligned}$$

Suppose next that $(x_0 - x_0^*)(x_1 - x_1^*) < 0$ and $0 \leq x_0 \leq x_0^* + 4x_1^*$. Then, $[\tilde{a}(x_0) - \delta](x_1 - x_1^*) < 0$, and so, by (21),

$$\begin{aligned} Lg(x) &\leq -2\delta_0 (x_0 - x_0^* + x_1 - x_1^*)^2 - \frac{4\delta_0}{\alpha_1} |\tilde{a}(x_0) - \delta| |x_1 - x_1^*| + c \\ &\leq -2\delta_0 \|x - x^*\|^2 + 4\delta_0 \left(1 - \frac{\alpha_0}{\alpha_1}\right) |x_0 - x_0^*| |x_1 - x_1^*| + c \\ &\leq -2\delta_0 \|x - x^*\|^2 + 2\delta_0 \left(1 - \frac{\alpha_0}{\alpha_1}\right) \|x - x^*\|^2 + c \\ &= -2\frac{\delta_0\alpha_0}{\alpha_1} \|x - x^*\|^2 + c. \end{aligned}$$

Finally, if $(x_0 - x_0^*)(x_1 - x_1^*) < 0$ and $x_0 > x_0^* + 4x_1^*$, then

$$\begin{aligned} Lg(x) &\leq -2\delta_0 (x_0 - x_0^*)^2 + 4\delta_0 (x_0 - x_0^*)x_1^* - 2\delta_0 (x_1 - x_1^*)^2 + c \\ &\leq -\delta_0 (x_0 - x_0^*)^2 - 2\delta_0 (x_1 - x_1^*)^2 + c \\ &\leq -\delta_0 \|x - x^*\|^2 + c. \quad \square \end{aligned}$$

Remark. As can be seen from the proof, it is sufficient to require the existence of constants $0 < \alpha_0 < \alpha_1$ such that

$$\begin{aligned} |\tilde{a}(x_0) - \delta| &\geq \alpha_0 |x_0 - x_0^*|, & 0 \leq x_0 \leq x_0^* + 4x_1^*, \\ |\tilde{a}(x_0) - \delta| &\leq \alpha_1 |x_0 - x_0^*|, & 0 \leq x_0. \end{aligned}$$

In particular, this theorem applies to the classical chemostat with Monod uptake rate. In a biological interpretation, one expects any solution to be close to the deterministic equilibrium most of the time if the stochastic effects are small.

3.3. Extinction of microbes

Theorem 4. (a) Let $\sigma_1 > 0$, and let $X(t)$ be given by the stochastic single-species system (5) and $X(0) = x \in (0, \infty)^2$. Then

$$P_x \left\{ \liminf_{t \rightarrow \infty} X_1(t) = 0 \right\} = 1.$$

If

$$\bar{a}'(0) - s'(0) < \frac{\sigma_1^2}{2},$$

then $X(t)$ is transient and

$$P_x \left\{ \lim_{t \rightarrow \infty} X_1(t) = 0 \right\} = 1.$$

(b) Let $\sigma_1 > 0$, $\sigma_2 > 0$, and let $X(t)$ be given by the stochastic two-species system (12) and $X(0) = x \in (0, \infty)^3$. Then

$$P_x \left\{ \liminf_{t \rightarrow \infty} X_1(t) = 0 \right\} = 1, \quad P_x \left\{ \liminf_{t \rightarrow \infty} X_2(t) = 0 \right\} = 1.$$

If for $i = 1$ or $i = 2$,

$$\bar{a}'_i(0) - \bar{s}'_i(0) < \frac{\sigma_i^2}{2},$$

then $X(t)$ is transient and

$$P_x \left\{ \lim_{t \rightarrow \infty} X_i(t) = 0 \right\} = 1.$$

Proof. We will prove only part (b), and only X_1 will be considered. Fix $x \in (0, \infty)^3$. Set

$$\beta(t) = \{a_1(X_0(t), X_1(t)) - s_1(X_1(t), X_2(t))\} \frac{1}{X_1(t)} - \frac{\sigma_1^2}{2},$$

$$b(z) = \{\bar{a}_1(e^z) - \bar{s}_1(e^z)\} e^{-z} - \frac{\sigma_1^2}{2}.$$

Let $Y_i(t) = \log X_i(t)$ for $i = 0, 1, 2$, and define a one-dimensional process $Z(t)$ by

$$dZ = b(Z) dt + \sigma_1 dW_1(t), \quad Z(0) = \log x_1.$$

Note that

$$\lim_{z \rightarrow -\infty} b(z) = \lim_{u \rightarrow 0^+} \frac{\bar{a}_1(u) - \bar{s}_1(u)}{u} - \frac{\sigma_1^2}{2} = \bar{a}'_1(0) - \bar{s}'_1(0) - \frac{\sigma_1^2}{2},$$

$$\limsup_{z \rightarrow \infty} b(z) \leq -\frac{\sigma_1^2}{2}.$$

This ensures that $Z(t)$ is defined for all $t \geq 0$. Moreover, $\beta(t) \leq b(Y_1(t))$. Therefore, in view of (13), the comparison theorem of Ikeda and Watanabe [21] yields that with probability one, $Y_1(t) \leq Z(t)$ for all $t \geq 0$. Since

$$\int_0^\infty \exp \left\{ - \int_0^v \frac{2b(u)}{\sigma_1^2} du \right\} dv = \infty,$$

it follows from [12, Theorem 7.1, pp. 219–220], that

$$P \left\{ \liminf_{t \rightarrow \infty} Z(t) = -\infty \right\} = 1.$$

Hence

$$P_x \left\{ \liminf_{t \rightarrow \infty} Y_1(t) = -\infty \right\} = P_x \left\{ \liminf_{t \rightarrow \infty} X_1(t) = 0 \right\} = 1.$$

If $\bar{a}'_1(0) - \bar{s}'_1(0) < \frac{\sigma_1^2}{2}$, then

$$\int_{-\infty}^0 \exp \left\{ \int_v^0 \frac{2b(u)}{\sigma_1^2} du \right\} dv < \infty,$$

and it follows that almost surely $Z(t) \rightarrow -\infty$, so that $X_1(t) \rightarrow 0$ as $t \rightarrow \infty$. \square

The first statements of parts (a) and (b), respectively, may appear less welcome from a biologist’s perspective, but they reflect a typical property of diffusions. One should keep in mind that this diffusion effect may be relevant only in very large time scales. The second statements of (a) and (b), respectively, are considerably stronger because they correspond to certain washout. Comparing the conditions to the deterministic washout

conditions $\bar{a}'_i(0) - \bar{s}'_i(0) < 0$, one sees that stochastic effects make washout more likely, and that sufficiently strong diffusion makes washout certain.

These rigorous results for the stochastic setting are, hopefully, the starting point for a continuing investigation of stochastic chemostat models. Lemma 1 provides a quick overview of possible scenarios, and in the ensuing detailed investigation of particular settings the problematic case of unbounded growth can be excluded. For the single-species case we have a persistence criterion in Theorem 2, while Theorem 3 relates the asymptotic deterministic behavior to the behavior of the stochastic system if the stochastic effects are not too strong. Finally, Theorem 4 shows that stochastic effects make it harder for the microbes to persist. It would be interesting to obtain more detailed information on persistence and exclusion for competing species, but different techniques may be necessary for this.

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Appendix

Here we show that the solutions of the stochastic differential equations (5) and (12) for one and two consumers remain within the open positive orthants. We consider slightly more general equations of the form

$$dX_i = f_i(X) dt + \sigma_i(X_i) dW_i(t), \quad i = 0, \dots, k, \quad (23)$$

with associated differential operator

$$Lg(x) = \sum_{i=0}^k f_i(x) \frac{\partial g(x)}{\partial x_i} + \frac{1}{2} \sigma_i^2(x_i) \frac{\partial^2 g(x)}{\partial x_i^2}.$$

Write $D = (0, \infty)^{k+1}$.

Proposition A.1. *Suppose f_0, \dots, f_k and $\sigma_0, \dots, \sigma_k$ are locally Lipschitz continuous, and there exists $C > 0$ such that for $i = 0, \dots, k$,*

$$x_i f_i(x) \leq C(1 + \|x\|^2), \quad \sigma_i^2(x_i) \leq Cx_i^2 \quad \text{for all } x. \quad (24)$$

Suppose also that for every $i = 0, \dots, k$, there exist $\varepsilon_i, c_i, C_i > 0$ such that at least one of the following conditions holds:

$$f_i(x) \geq c_i - C_i x_i (1 + x_0 + \dots + x_k), \quad \text{for all } x \text{ with } x_i \leq \varepsilon_i \quad (25)$$

or

$$f_i(x) \geq -C_i x_i, \quad \text{for all } x \text{ with } x_i \leq \varepsilon_i. \tag{26}$$

Then for every $x \in D$, (23) has a strong solution with $X(0) = x$, pathwise uniqueness holds and

$$P_x \{X(t) \in D \text{ for all } t \geq 0\} = 1.$$

Proof. The local Lipschitz continuity and the growth conditions (24) ensure the existence of a strong solution of (23) in \mathbb{R}^{k+1} and pathwise uniqueness, provided the f_i and σ_i are suitably extended to \mathbb{R}^{k+1} and \mathbb{R} , respectively, see [7, p. 190].

To prove that $X(t)$ stays in D if $X(0) \in D$, consider

$$\phi(x) = 1 + \sum_{i=0}^k x_i^{-\frac{1}{2}} + K x_i^2,$$

where $K = \max \left\{ 1, \varepsilon_0^{-\frac{5}{2}}, \dots, \varepsilon_k^{-\frac{5}{2}} \right\}$. We have

$$L\phi(x) = \sum_{i=0}^k f_i(x) \left(2K x_i - \frac{1}{2} x_i^{-\frac{3}{2}} \right) + \sigma_i^2(x_i) \left(K + \frac{3}{8} x_i^{-\frac{5}{2}} \right).$$

By (24), for all $x \in D$,

$$\sum_{i=0}^k \sigma_i^2(x_i) \left(K + \frac{3}{8} x_i^{-\frac{5}{2}} \right) \leq C \phi(x).$$

If f_i satisfies (25), then for every $x \in D$ with $x_i \leq \varepsilon_i$,

$$\begin{aligned} -f_i(x) x_i^{-\frac{3}{2}} &\leq -c_i x_i^{-\frac{3}{2}} + C_i x_i^{-\frac{1}{2}} (1 + x_0 + \dots + x_k) \\ &\leq -c_i x_i^{-\frac{3}{2}} + \frac{1}{2} C_i^2 x_i^{-1} + \frac{1}{2} (1 + x_0 + \dots + x_k)^2 \\ &\leq K_i \phi(x), \end{aligned}$$

for some $K_i > 0$. Otherwise, f_i must satisfy (26), so that for every $x \in D$ with $x_i \leq \varepsilon_i$,

$$-f_i(x) x_i^{-\frac{3}{2}} \leq C_i x_i^{-\frac{1}{2}} \leq C_i \phi(x).$$

This and (24) imply that there exists $K' > 0$ such that for every i ,

$$f_i(x) \left(2Kx_i - \frac{1}{2}x_i^{-\frac{3}{2}} \right) \leq K'\phi(x),$$

provided $x_i \leq \varepsilon_i$. But if $x_i > \varepsilon_i$, then $2Kx_i > \frac{1}{2}x_i^{-\frac{3}{2}}$, and so

$$\begin{aligned} f_i(x) \left(2Kx_i - \frac{1}{2}x_i^{-\frac{3}{2}} \right) &\leq \max \{0, 2Kx_i f_i(x)\} \leq 2KC \left(1 + \|x\|^2 \right) \\ &\leq 2KC\phi(x). \end{aligned}$$

It now follows that there exists $K'' > 0$ such that $L\phi(x) \leq K''\phi(x)$ for all $x \in D$. An argument similar to that in [7, pp. 191–192], shows then that

$$P_x \{ \tau_n \leq T \} \leq \frac{\phi(x)e^{K''T}}{n}, \quad T > 0, \quad n \in \mathbb{N},$$

where $\tau_n = \inf \{ t > 0 : \phi(X(t)) \geq n \}$. Hence $P_x \{ \lim_{n \rightarrow \infty} \tau_n = \infty \} = 1$, which completes the proof. \square

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