Preservation of the Maslov index along bifurcating branches of solutions of first order systems in $\mathbb{R}^N$☆

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Abstract

In this paper we study bifurcation from simple eigenvalues for systems of differential equations; we prove the existence of global bifurcating branches of solutions on which the Maslov index of suitable associated linear systems is preserved.
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1. Introduction

In this paper we consider a boundary value problem of the form

\[
\begin{aligned}
z' &= JS(t, z, \lambda)z, \quad z = (x, y) \in \mathbb{R}^{2N}, \quad t \in [0, \pi], \quad \lambda \in (a, b), \\
x(0) &= 0 = x(\pi),
\end{aligned}
\]

where $J$ is the standard symplectic matrix (see (1.10)) and $S(t, z, \lambda)$ is a symmetric matrix satisfying a definiteness condition. Using a bifurcation argument and the notion of Maslov index we

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shall give multiplicity results for (1.1). In particular, we shall deal with asymptotically linear first order systems.

The possibility of obtaining multiplicity results via global bifurcation starting from an abstract equation of the form

$$u = \lambda Lu + H(\lambda, u),$$

(1.2)

where $L : E \to E$ is a compact linear operator, $H : (a, b) \times E \to E$ is compact, $E$ is a Banach space and $-\infty \leq a < b \leq +\infty$, is well established since the seminal work of Rabinowitz [20]. We refer, among others, to the papers by Esteban [12] and Rynne [23]. In [20], one of the applications of the main result is the study of the Dirichlet problem associated a second order scalar equation of the form

$$-u'' = \lambda p(t)u + g(t, u, \lambda)u,$$

(1.3)

where $p \in C([0, \pi], \mathbb{R})$ and $g \in C([0, \pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In that context, two facts are crucial for the applicability of the abstract bifurcation result. The first is that the eigenvalues of the Dirichlet problem associated to the linear equation $-u'' = \lambda p(t)u$ are simple.

The second is that the integer-valued map

$$u \mapsto \text{number of zeros of } u \text{ in } (0, \pi)$$

(1.4)

(being $u$ a solution to (1.3)) is continuous.

In the present work, we take [20] and some knowledge of second order equations and systems (cf. [7]) as our starting point.

The first part of our work consists then of two aspects. First, we have to investigate linear eigenvalue problems of the form

$$z' = \lambda JS(t)z, \quad z = (x, y) \in \mathbb{R}^{2N},$$

(1.5)

with $x(0) = 0 = x(\pi)$. For this problem, we refer to the books by Atkinson [4] and by Yakubovich and Starzinski [25]. We have thus focused on a class $S$ of constant matrices for which the eigenvalues of (1.5), together with boundary conditions, are simple. In the particular case $N = 1$, a matrix belongs to this class whenever $\det S > 0$.

On the other hand, in the situation considered in this paper we have to exhibit a map which plays the role of (1.4) (cf. [6,8]). We show that this can be done for some $2N$th order equations and for some first order systems in $\mathbb{R}^n$ using the notion of Maslov index. This is an index associated to linear systems, which has been extensively studied (we refer, among others, to [1,3,13,15,22]); its main features are recalled, for the reader’s convenience, in Appendix A. We first have to develop a suitable linearization procedure; once this is done, we consider the map

$$\phi(\lambda, z) = m(S(\cdot, z(\cdot), \lambda)),$$

(1.6)

where, for every solution $(z, \lambda)$ of (1.1), $m(S(\cdot, z(\cdot), \lambda))$ denotes the Maslov index of the linear system

$$w' = JS(t, z(t), \lambda)w, \quad w = (u, v) \in \mathbb{R}^{2N}.$$

(1.7)
We then have to focus the serious problem of the continuity of this map. It turns out that a sufficient condition (suitable for our linearization purposes—cf. Lemma 3.1) for this map to be continuous is that all the (possible) moments of verticality (cf. Definition A.3) are simple. It is worth noticing that if $S \in \mathcal{S}$ then the moments of verticality are simple (cf. Definition A.3).

Our global bifurcation result is Theorem 3.2. Roughly speaking, it is proved that, when both the eigenvalues and the moments of verticality of linearized systems related to (1.1) are simple, then from every eigenvalue a closed connected branch of solutions to (1.1) bifurcates; moreover, this branch is unbounded and along the branch the Maslov index is preserved.

We point out that in the literature one can find various bifurcation results whose applicability is not restricted to the case of simple eigenvalues (we refer, among others, to [9,10,14]). As a first step in our research on this subject, we have focused on Rabinowitz bifurcation theorem since it is crucial for us to work with moments of verticality that are simple (a property strictly related to the simplicity of the eigenvalues, cf. Remark A.5). For related results for elliptic partial differential equations we refer, among others, to [16]. The applicability of [9,10,14] will be the subject of a further research.

In the second part of the paper, we focus on some problems of the form (1.1) for which the hypotheses of the global bifurcation Theorem 3.2 are satisfied. We then give conditions in order to exclude one of the two alternatives and get multiplicity results; in particular, we will consider some asymptotically linear planar systems. We observe that, by means of our technique, we are also able to reobtain the well-known results of Rabinowitz [20] for second order superlinear ordinary differential equations and also some recent results of Rynne [23] in the case of higher order equations (see Theorem 2.4).

First (cf. Section 3.1), we exhibit the class $\mathcal{S}$ of those constant matrices such that the moments of verticality of the linear system $z' = JSz$ are simple and the eigenvalues of the Dirichlet problem associated to $z' = \lambda JSz$ are simple. In the case when $S$ is constant, it is possible to compute explicitly the Maslov index $m(S)$ (cf. [7, Remark 3.9] and also [15,18]) and a global bifurcation result (Theorem 3.7) holds.

Once this is done, one tries to estimate the Maslov index along the branches in order to obtain a multiplicity result. This is accomplished (Theorem 3.10) for a nonlinear system of the form

$$\begin{cases} z' = JS(t,z)z, & z \in \mathbb{R}^2, \\ x(0) = 0 = x(\pi), \end{cases}$$

(1.8)

where we assume that $S : [0, \pi] \times \mathbb{R}^2 \to M_2^\mathbb{S}$ is continuous and satisfies the conditions

$$S(t, 0) = S_0(t), \quad \text{for every } t \in [0, \pi],$$

$$\lim_{|z| \to +\infty} S(t, z) = S_{\infty}(t), \quad \text{uniformly in } t \in [0, \pi].$$

(1.9)

Under these assumptions we will show the existence of a certain number of solutions to (1.8); the number of solutions is strictly related to the Maslov indeces of the two linear systems $z' = JS_0(t)z$ and $z' = JS_{\infty}(t)z$.

For related results, mainly related to the periodic boundary value problem, we refer to the works by [2,5,17,24].
In what follows we denote by \( M_{2N}^S \) the set of symmetric (positive or negative) definite \( 2N \times 2N \) matrices; moreover, let

\[
J = \begin{pmatrix} O & -\text{Id} \\ \text{Id} & O \end{pmatrix}.
\]

(1.10)

For every vector \((\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N\), we denote by \( \text{diag}(\lambda_1, \ldots, \lambda_N) \) the \( N \times N \) matrix whose coefficients on the diagonal are \( \lambda_1, \ldots, \lambda_N \) and whose other coefficients are all zero.

2. Bifurcating branches preserving the Maslov index

The aim of this section is to apply the Rabinowitz bifurcation theorem in a situation where a continuous integer valued functional is defined on the set of the (possible) solutions to an abstract problem of the form

\[
u = \lambda Lu + H(\lambda, u),
\]

(2.1)

where \( L : E \to E \) is a compact linear operator, \( H : (a, b) \times E \to E \) is compact, \( E \) is a Banach space and \(-\infty \leq a < b \leq +\infty\).

For the reader’s convenience, we briefly recall the framework of the celebrated Rabinowitz bifurcation theorem [20, Theorem 2.3]; the main assumption is

\[
H(\lambda, u) = o(\|u\|), \quad u \to 0, \text{ uniformly on bounded } \lambda \text{ intervals.}
\]

(2.2)

Moreover, let \( r(L) \) be the set of characteristic values of \( L \), i.e., the set of the numbers \( \mu \neq 0 \) such that \( 1/\mu \) is an eigenvalue of \( L \). Let also \( \Sigma \) be the closure of the set of nontrivial solutions to (2.1). In this context, it is well known that [20, Theorem 2.3] guarantees that if \( \mu \in r(L) \cap (a,b) \) is a characteristic value of \( L \) of odd multiplicity, then \( \Sigma \) contains a continuum (i.e., a closed connected set) \( C \) such that \((\mu, 0) \in C \) and either:

(A1) there exists \((\lambda_n, u_n) \in C \) such that

\[
|\lambda_n| + \|u_n\| \to +\infty \text{ or } \lambda_n \to a \text{ or } \lambda_n \to b.
\]

(A2) \( C \) contains \((\hat{\mu}, 0) \) with \( \hat{\mu} \neq \mu \) and \( \hat{\mu} \in r(L) \).

Now, let us suppose that there exists a continuous functional

\[
\phi : \Sigma \to \mathbb{N}
\]

such that

\[
\phi(\mu^*, 0) \neq \phi(\hat{\mu}, 0), \quad \forall \mu^* \neq \hat{\mu}, \mu^*, \hat{\mu} \in r(L).
\]

(2.3)

It is possible to prove the following consequence of Rabinowitz theorem:
Proposition 2.1. Assume that \( \mu \in r(L) \cap (a, b) \) is a characteristic value of \( L \) of odd multiplicity and let \( C \) be the continuum obtained through Rabinowitz bifurcation theorem. Then alternative (A1) holds and

\[
\phi(\lambda, u) = \phi(\mu, 0), \quad \forall (\lambda, u) \in C. \tag{2.4}
\]

Proof. The proof directly follows from a standard connectivity and continuity argument. \( \square \)

It has to be remarked that the idea of using a topological invariant (preserved along branches of solutions of (2.1)) in order to exclude alternative (A2) can be found in [20]; indeed, in [20] the abstract bifurcation theorem has been applied to the study of the Dirichlet problem associated to a differential equation of the form

\[
Lu = \lambda a(t)u + \lambda f(t, u, u'), \quad t \in [0, \pi],
\]

where \( L \) is a suitable second order differential operator. It is proved that from every eigenvalue of the linear problem \( Lu = \lambda a(t)u, u(0) = 0 = u(\pi) \), a continuum of solutions bifurcates and every solution belonging to this branch has the same number of zeros in \([0, \pi]\). With respect to the setting of Proposition 2.1, in the situation of [20] the functional \( \phi \) is the number of zeros of every solution to the differential equation; as it can be observed from the proof of the result in [20], the continuity of \( \phi \) is a consequence of the fact that zeros of solutions to the considered equation are simple.

In the next sections, we will examine boundary value problems of the form (1.1) that are suitable for the applicability of Proposition 2.1. In these applications, we will consider the functional \( \phi \) defined by (1.6).

We give here a sufficient condition for the continuity of \( \phi \); in Section 3 we will show that this condition is satisfied when the system in (1.1) is a perturbation of an autonomous linear system. Finally, the case when a system of the form (1.1) comes from a \( 2N \)th order differential equation is treated at the end of this section.

Let now \( S : [0, \pi] \times \mathbb{R}^{2N} \times (\alpha, \beta) \rightarrow M^{2N}_S \) be continuous and let us consider a boundary value problem of the form (1.1), whose abstract formulation is (2.1). We have the following:

Proposition 2.2. Suppose that for every \( \lambda \in (a, b) \) and for every continuous function \( \alpha : [0, \pi] \rightarrow \mathbb{R}^{2N} \) the moments of verticality of

\[
w' = JS(t, \alpha(t), \lambda)w,
\]

are simple. Then, the functional \( \phi \) defined in (1.6) is continuous in \( \Sigma \).

Proof. The result follows from a general property of the abstract Maslov index (see [19]); for the reader’s convenience, we give here a direct proof based on some continuous dependence argument and the concrete definition of Maslov index in the case of differential equations.

We first show that \( \phi \) is continuous in every point \( (\lambda^*, u^*) \) with \( \lambda^* \in (a, b) \) and \( u^* \neq 0 \). To this aim, let \( (\lambda_n, u_n) \in \Sigma_\phi \) be such that \( (\lambda_n, u_n) \rightarrow (\lambda^*, u^*) \) and \( u_n \neq 0 \); let also \( z_n \) and \( z^* \) be the functions which correspond in the abstract setting to \( u_n \) and \( u^* \), respectively.
We observe that \((\lambda_n, z_n)\) is a solution of
\[
\begin{cases}
w' = J A_n(t)w, & w = (u, v) \in \mathbb{R}^{2N}, \\
u(0) = 0 = u(\pi),
\end{cases}
\] (2.5)
with \(A_n(t) = S(t, z_n(t), \lambda_n)\) and that \((\lambda^*, u^*)\) is a solution of
\[
\begin{cases}
w' = J A^*(t)w, & w = (u, v) \in \mathbb{R}^{2N}, \\
u(0) = 0 = u(\pi),
\end{cases}
\] (2.6)
with \(A^*(t) = S(t, z^*(t), \lambda_n)\). It is easy to see that
\[A_n(t) \to A^*(t),\]
uniformly in \([0, \pi]\); as a consequence (see [7, Proposition 4.4]) we obtain that
\[
\theta_{j,n}(\pi) \to \theta_{j}^*(\pi), \quad j = 1, \ldots, N,
\] (2.7)
where \(\theta_{j,n}\) and \(\theta_{j}^*\) are the phase-angles associated to (2.5) and (2.6), respectively. Now, since \((\lambda^*, u^*)\) is a nontrivial solution to (2.6), there exists at least an integer \(m\) such that
\[
\theta_{m}^*(\pi) = km\pi,
\]
for some integer \(k_m\). Moreover, the fact that the moments of verticality of (2.6) are simple ensures that there exists exactly only one integer \(m\) with this property; supposing w.l.o.g. that \(m = 1\), we have
\[
\theta_{j}^*(\pi) = k_1\pi + \alpha_j, \quad j = 2, \ldots, N,
\] (2.8)
with \(k_j \in \mathbb{Z}\) and \(\alpha_j \in (0, \pi)\). This implies that
\[
\phi(\lambda^*, u^*) = m(\lambda^*, u^*) = |k_1| + \cdots + |k_N|.
\]
Now, from condition (2.7) we deduce that, for \(n\) large, we have
\[
\theta_{j,n} = k_j \pi + \alpha'_j, \quad j = 2, \ldots, N,
\] (2.9)
with \(\alpha'_j \in (0, \pi)\). On the other hand, the fact that (2.5) has a nontrivial solution guarantees that one of the phase-angles \(\theta_{j,n}\) must be a multiple of \(\pi\) at time \(t = \pi\); from (2.9) we infer that this occurs only for \(\theta_{1,n}\). Moreover, from relations (2.7) and (2.8) we obtain that
\[
\theta_{1,n}(\pi) = k_1\pi,
\]
for large \(n\). We thus deduce that for large \(n\) we have
\[
\phi(\lambda_n, u_n) = m(\lambda_n, u_n) = |k_1| + \cdots + |k_N| = \phi(\lambda^*, u^*);
\]
this concludes the proof of the continuity of \(\phi\) in \((\lambda^*, u^*)\).
In an analogous way it is possible to prove that \( \phi \) is also continuous at every point \((\mu, 0)\), with \( \mu \in (a, b) \cap \Gamma(L) \). Indeed, the fact that \( \mu \) is a characteristic value of \( L \) implies that the linear boundary value problem

\[
\begin{cases}
  w' = JS(t, 0, \mu)w, & w = (u, v) \in \mathbb{R}^{2N}, \\
  u(0) = 0 = u(\pi)
\end{cases}
\]

has a nontrivial solution. As a consequence, its phase-angles satisfy relations of the form (2.8) and this allows to repeat the previous argument. \( \square \)

We end this section by showing a first concrete case when the assumptions of Proposition 2.2 are satisfied. The remaining part of the paper (Section 3) is devoted to a detailed analysis of another significant case.

**Example (2Nth order scalar equations).** Let \( \rho_i \in C^{2N-i}([0, \pi]), i = 1, \ldots, 2N \), be strictly positive functions; for every \( u \in C^2([0, \pi]) \) define

\[
L_0 u = \rho_0 u,
\]

\[
L_i u = \rho_i (L_{i-1} u)', \quad i = 1, \ldots, 2N.
\]

Let \( g \in C([0, \pi] \times \mathbb{R}) \) and consider the scalar equation

\[
L_{2N} u = \lambda p(t) u + g(t, u) u, \quad \lambda \geq 0,
\]  
(2.10)

together with

\[
u^{(i)}(0) = u^{(i)}(\pi), \quad i = 0, \ldots, N - 1.
\]  
(2.11)

Assume that the following conditions hold true:

(H1) \((-1)^N p(t) > 0, (-1)^N g(t, u) \geq 0, \forall t \in [0, \pi], \ u \in \mathbb{R}.

(H2) \(g(t, u) = o(u)\) for \(u \to 0\), uniformly in \(t \in [0, \pi]\).

(H3) \(\rho_i = \rho_{2N-i}\) for \(i = 0, \ldots, N - 1\).

We now show that (2.10), (2.11) fit into the abstract framework of this section. First, it is well known that problem (2.10), (2.11) can be written in the abstract form (2.1) and that assumption (H2) implies that \( H \) satisfies (2.2).

Moreover, it is possible to see that, when (H3) holds true, (2.10), (2.11) is equivalent to a first order system of the form (1.1); indeed, let \( A = (a_{i,j}) \), \( j, j = 1, \ldots, N \), be such that

\[
\begin{align*}
a_{i,N-i+1} &= (-1)^{N+i}, \quad \forall i = 1, \ldots, N, \\
a_{i,j} &= 0, \quad \forall i = 1, \ldots, N, \ \forall j \neq N - i + 1.
\end{align*}
\]

Moreover, let

\[
B = \begin{pmatrix} \text{Id} & 0 \\ 0 & A \end{pmatrix}
\]
and

\[ x = (L_0 u, \ldots, L_{N-1} u), \quad \tilde{y} = (L_N u, \ldots, L_{2N-1} u), \quad v = (x, \tilde{y}). \]

If \( z = Bv \), then (2.10), (2.11) can be written in the form (1.1) with

\[
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix},
\]

where

\[
S_{11}(t, z, \lambda) = (-1)^{N+1} \frac{\lambda p(t) + g(t, x_1 / \rho_0)}{\rho_0^2(t)} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},
\]

\[
S_{12}(t, z, \lambda) = - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1/\rho_1(t) & 0 & \cdots & 0 \\ 0 & 1/\rho_2(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\rho_{N-1}(t) & 0 \end{pmatrix},
\]

and

\[
S_{22}(t, z, \lambda) = - \frac{1}{\rho_N(t)} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},
\]

for every \( t \in [0, \pi] \), \( z \in \mathbb{R}^{2N} \) and \( \lambda \geq 0 \).

We remark that, since the weight functions \( \rho_i \) never vanish, the change of variable transforms the boundary conditions (2.11) in the Dirichlet boundary conditions \( x(0) = 0 = x(\pi) \). Moreover, let us notice that in this situation moments of verticality of (1.1) correspond to the so-called conjugate points for (2.10), (2.11) (see, e.g., [11]).

We also observe that in general for every \( (t, z, \lambda) \in [0, \pi] \times \mathbb{R}^{2N} \times [0, +\infty) \) the matrix \( S(t, z, \lambda) \) does not belong to \( M_{2N}^{2N} \); however, it is still possible to define the Maslov index of the linear Eq. (1.7) (see [15]). Hence, the functional \( \phi \) is well defined.

The following result plays a crucial role; it is a straightforward consequence of [11, Lemma 3].

**Proposition 2.3.** If (H1) holds true and \( \lambda > 0 \), then for every continuous function \( z \) all the moments of verticality of (1.7) are simple.

From Propositions 2.2 and 2.3 we deduce that, along the solutions of (2.10), (2.11), \( \phi \) is continuous. Now, we recall (see [11, Theorems 1–3]) that the boundary value problem \( L_{2N} u = \lambda p(t) u \) has an infinite sequence of simple positive eigenvalues \( \mu_k \) such that \( \mu_k \to +\infty \), as \( k \to +\infty \).

Moreover, from [15, Theorem 8.2] we also plainly deduce that the Maslov index of the linear system associated to \( L_{2N} u = \mu_k p(t) u \) is exactly \( k - 1 \).

Therefore, from Proposition 2.1 we immediately obtain the following result:
Theorem 2.4. Let us consider the boundary value problem (2.10), (2.11) and let us assume that (H1)–(H3) hold true. Then, for every \( k \in \mathbb{N}, k \neq 0 \), \( \Sigma \) contains a continuum \( C_k \) such that \((\mu_k, 0) \in C_k\), alternative (A1) of Theorem 2.1 holds true and

\[
\phi(\lambda, u) = k - 1,
\]

for every \((\lambda, u) \in C_k\).

In the case of this scalar equation, a classical result of Elias [11] enables us to establish a precise relation between the number of moments of verticality and the number of zeros \( n(u, \lambda) \) \((u \neq 0)\) belonging to the branch \( C_k \). Indeed, we have

Proposition 2.5. For every \((\lambda, u) \in C_k\), with \( u \neq 0 \), \( u \) has only simple zeros. Moreover,

\[
\phi(\lambda, u) = n(\lambda, u).
\]

Proof. For the first assertion it is sufficient to observe that \((\lambda, u)\) is a solution of the linear equation \( L_{2N}u = (\lambda p(t) + g(t, u(t)))y \) and to use [11, Theorem 1]. By Theorem 2.4, we know that \( \phi(\lambda, u) = k - 1 \). Since the moments of verticality are simple, we deduce that the number of moments of verticality in \((0, \pi)\) is exactly \( k - 1 \).

In the framework of [11] this is equivalent to say that \( t = \pi \) is the \( k \)th conjugate point of \( t = 0 \) for \( L_{2N}u = (\lambda p(t) + g(t, u(t)))y \); from [11, Theorem 3] we infer that \( u \) has \( k - 1 \) simple zeros in \((0, \pi)\). This concludes the proof. \( \square \)

As a consequence of the above proposition, we can observe that Theorem 2.4 is the same as [23, Theorem 3.1]. We also remark that, in the same spirit of [23], conditions at infinity on the nonlinearity \( g \) give rise to multiplicity results for (2.10), (2.11).

3. Application to first order systems of differential equations

In this section we present some results for systems of differential equations of the form

\[
\begin{cases}
    z' = \lambda JS(t)z + JF(t, z, \lambda), & z = (x, y) \in \mathbb{R}^{2N}, \ t \in [0, \pi], \ \lambda \in (a, b),
    \\
x(0) = 0 = x(\pi),
\end{cases}
\]

where \( S \in C([0, \pi], M^{2N}_S) \) and \( F(t, z, \lambda) \) is a symmetric matrix. We assume that the application \((t, z, \lambda) \mapsto F(t, z, \lambda)\) is continuous and that \( F(t, 0, \lambda) = 0 \), for every \( t \in [0, \pi], \ \lambda \in (a, b) \). We then have the following:

Lemma 3.1. Let us suppose that, for every \( \lambda \in (a, b) \), the moments of verticality of

\[
z' = \lambda JS(t)z
\]

are simple. Then there exists \( \xi : (a, b) \to (0, +\infty) \) such that if

\[
\|F(t, z, \lambda)\| \leq \xi(\lambda), \quad \forall (t, z, \lambda) \in [0, \pi] \times \mathbb{R}^{2N} \times (a, b),
\]

(3.3)
it follows that for every continuous function $\alpha : [0, \pi] \to \mathbb{R}^{2N}$ and for every $\lambda \in (a, b)$ the number of moments of verticality of

$$z' = \lambda J S(t) z + F(t, \alpha(t), \lambda) z$$

(3.4)

is well defined and all the moments of verticality are simple.

**Proof.** Let us fix $\lambda \in (a, b)$ and let us consider the system (3.2); let $X(t)$, $D(t)$ and $R(t)$ be defined as in Appendix A. For every continuous function $\alpha : [0, \pi] \to \mathbb{R}^{2N}$, let

$$S^\alpha(t) = \lambda S(t) + F(t, \alpha(t), \lambda), \quad \forall t \in [0, \pi];$$

moreover, for the system

$$z' = J S^\alpha(t) z,$$

let $X_\alpha(t)$, $D_\alpha(t)$ and $R_\alpha(t)$ be again as in Appendix A.

Now, assume w.l.o.g. that $S(t)$ is positive definite, for every $t \in [0, \pi]$. We denote by $t_1, \ldots, t_K$ the simple moments of verticality of (3.2). By Lemma A.10 and the continuity of $R$ there exist $\delta > 0$ and disjoint intervals $I_j = (t_j - \delta_j, t_j + \delta_j)$, $j = 1, \ldots, K$, such that

$$R(t) \geq \delta, \quad \forall t \in I := \bigcup_{j=1}^K I_j. \quad (3.5)$$

Moreover, since $D(t) \neq 0$, for every $t \in A := [0, \pi] \setminus I$, there exists $\mu > 0$ such that

$$D(t) \geq \mu, \quad \forall t \in A. \quad (3.6)$$

A continuous dependence argument shows that there exists $\xi(\lambda) > 0$ such that if $\|F(t, z, \lambda)\| \leq \xi(\lambda)$, for every $t \in [0, \pi]$ and $z \in \mathbb{R}^{2N}$, then

$$S^\alpha(t) \text{ is positive definite for every } t \in [0, \pi] \text{ and } \alpha \in C([0, \pi]). \quad (3.7)$$

Moreover,

$$R_\alpha(t) \geq \frac{\delta}{2}, \quad \forall t \in I, \quad (3.8)$$

and

$$|D(t) - D_\alpha(t)| \leq \frac{\mu}{2}, \quad \forall t \in [0, \pi]. \quad (3.9)$$

From (3.7) we deduce that the number of moments of verticality of (3.4) is well defined; moreover, if $\bar{t}$ is a moment of verticality of (3.4), then

$$D_\alpha(\bar{t}) = 0.$$
From (3.9) we deduce that

$$|D(\bar{t})| \leq \frac{\mu}{2};$$

condition (3.6) implies that $\bar{t} \in I_j$, for some $j = 1, \ldots, K$. As a consequence, from (3.8) we conclude that $R(\bar{t}) \geq \delta/2 > 0$. From Lemma A.10 we deduce that $\bar{t}$ is simple.

We are ready to prove the following bifurcation result:

**Theorem 3.2.** Let us consider the boundary value problem (3.1) and let us assume that $F(t, 0, \lambda) = 0$, for every $t \in [0, \pi]$, $\lambda \in (a, b)$. Moreover, suppose that for every $\lambda \in (a, b)$ the moments of verticality of (3.2) are simple and that $F$ satisfies (3.3).

Finally, assume that for every pair $(\hat{\mu}, \mu^*)$ of eigenvalues of (3.2), with $\hat{\mu} \neq \mu^*$, we have

$$m(\hat{\mu}S) \neq m(\mu^*S).$$

(3.10)

Then, if $\mu \in (a, b)$ is an eigenvalue of the Dirichlet problem associated to (3.2), $\Sigma$ contains a continuum $C$ such that $(\mu, 0) \in C$, alternative (A1) of Theorem 2.1 holds true and

$$\phi(\lambda, u) = m(\mu S)$$

for every $(\lambda, u) \in C$.

**Proof.** We show that all the assumptions of Proposition 2.1 are satisfied.

Indeed, let us first observe that (3.1) can be written in a standard way in the form (2.1); moreover, the condition $F(t, 0, \lambda) = 0$ implies that (2.2) holds true. It is also well known that eigenvalues of the Dirichlet problem associated to (3.2) correspond to characteristic values of the operator $L$; moreover, according to Remark A.5, the fact that the moments of verticality of (3.2) are simple implies that the eigenvalues of (3.2) are simple.

From (3.3) and the assumptions on $S$, using Lemma 3.1, we deduce that the functional $\phi$ given in (1.6) is well defined. Lemma 3.1 and Proposition 2.2 show that $\phi$ is continuous in $\Sigma_\phi$; finally, from (3.10) we infer that also (2.3) holds true. This concludes the proof.

**Remark 3.3.** In the next sections, we will show some situations in which the assumptions on the eigenvalues of the Dirichlet problem associated to (3.2) are fulfilled; in particular, we will prove that this is true for some autonomous equations of the form $z' = JSz$ (see Section 3.1) or in the case of some planar systems (see Section 3.2).

### 3.1. Systems in $\mathbb{R}^{2N}$ with $N \geq 2$

In this subsection we consider again the boundary value problem (3.1), with $N \geq 2$; the results are valid also for $N = 1$, but this case will be considered, under weaker assumptions, in the next subsection. Our aim is to give conditions on $S$ ensuring the validity of the assumptions of Theorem 3.2. In particular, we will present a class of constant matrices $S$ for which we are able to compute the eigenvalues and the moments of verticality of (3.2) and to prove that they are simple.
In what follows we denote by $\lambda_1, \ldots, \lambda_{2N}$ the (real) eigenvalues of a matrix $S \in M_{2N}^N$ and let $D = \text{diag}(\lambda_1, \ldots, \lambda_N)$; moreover, let $P$ be the orthogonal matrix such that

$$P^T S P = D.$$ 

We denote by $S$ the class of constant matrices $S \in M_{2N}^N$ such that:

1. For every $i \neq j$, $i, j = 1, \ldots, N$, we have

$$\frac{\sqrt{\lambda_i \lambda_{N+i}}}{\sqrt{\lambda_j \lambda_{N+j}}} \notin \mathbb{Q}.$$ 

2. The matrix $P$ has the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{pmatrix}$$

and $P_{11}$ is invertible.

3. The matrix $Q = -P_{12} P_{11}^{-1}$ is diagonal.

We remark that it would be possible to define a different class $S$, suitable for the proof of our results, by requiring that $P_{12}$ is invertible and by replacing the matrix $Q$ with the matrix $Q' = -P_{11} P_{12}^{-1}$.

We recall that every symplectic orthogonal matrix $P$ can be written in the form (3.11); we also observe that it is possible to find matrices $S$ belonging to the class $S$. Indeed, let us take $P_{11} \in M_N^N$ such that $P_{11}^2 = \text{Id}$ and set $P_{12} = P_{11}$. Then, for every diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_N)$ whose eigenvalues satisfy condition 1, the matrix $S = PDP^T$ belongs to the class $S$.

The next results show that, for every $\lambda \in \mathbb{R}, \lambda \neq 0$, the moments of verticality of

$$z' = \lambda JSz,$$  

(3.12)

when $S \in S$, are simple.

**Proposition 3.4.** Assume that $S \in S$. Then for every $\lambda \neq 0$ the moments of verticality of (3.12) are simple.

**Proof.** We first make a change of variables in order to transform the linear system (3.12) in an analogous system where $S$ is replaced by its diagonal form $D$. Let $w = P^T z$; it is easy to see that $z$ is a solution to (3.12) if and only if $w$ is a solution of

$$w' = \lambda J Dw.$$ 

(3.13)

Indeed, we observe that assumption 2 on the matrix $P$ implies that $JP^T = P^T J$; hence, we have

$$w' = P^T z' = \lambda P^T JSz = \lambda J P^T S z = \lambda J D P^T z = \lambda J Dw.$$ 

Letting $w_0 = w(0)$, we obtain

$$w(t) = \exp(\lambda JDt)w_0;$$
a simple computation shows that we have

\[
\exp(\lambda J Dt) = \begin{pmatrix} C_1(t) & S_1(t) \\ S_2(t) & C_1(t) \end{pmatrix},
\]

(3.14)

where

\[
C_1(t) = \text{diag}(\cos \lambda R_1 t, \ldots, \cos \lambda R_N t),
\]

\[
S_1(t) = \text{diag}\left(-\sqrt{\frac{\lambda_1}{\lambda_{N+1}}} \sin \lambda R_1 t, \ldots, -\sqrt{\frac{\lambda_N}{\lambda_{N+1}}} \sin \lambda R_N t\right),
\]

\[
S_2(t) = \text{diag}\left(\sqrt{\frac{\lambda_1}{\lambda_{N+1}}} \sin \lambda R_1 t, \ldots, \sqrt{\frac{\lambda_N}{\lambda_{N+1}}} \sin \lambda R_N t\right),
\]

for every \( t \in [0, \pi] \), and \( R_j = \sqrt{\lambda_j \lambda_{N+j}}, \ j = 1, \ldots, N \).

As far as boundary conditions are concerned, it is easy to see that \( x(t) = 0 \) if and only if \( u(t) = Qv(t) \); therefore, the boundary conditions satisfied by \( w \) are

\[
\begin{cases}
  u(0) = Qv(0), \\
  u(\pi) = Qv(\pi).
\end{cases}
\]

(3.15)

Therefore, \( \bar{t} \) is a moment of verticality of (3.12) if and only if there exists a nontrivial solution \( w = (u, v) \) of (3.13) such that \( u(0) = Qv(0) \) and \( u(\bar{t}) = Qv(\bar{t}) \); moreover, since \( P^T \) is invertible, the multiplicity of \( \bar{t} \) is the number of linearly independent solutions \( w \) satisfying these boundary conditions.

According to the procedure described in Appendix A, let us consider the solutions \( w^1, \ldots, w^N \) to (3.13) such that \( w^j(0) = (Qe_j, e_j), \ j = 1, \ldots, N, \ e_j \in \mathbb{R}^N \). If we set \( w^j = (u^j, v^j) \), then it is easy to see that

\[
u^j(t) = \begin{pmatrix} 0, \ldots, q_j \sqrt{\frac{\lambda_j}{\lambda_{N+j}}} \sin \lambda R_j t + \cos \lambda R_j t, \ldots, 0 \end{pmatrix},
\]

for every \( t \in [0, \pi] \) and \( j = 1, \ldots, N \). Therefore, \( \bar{t} \) is a moment of verticality if and only if there exists \( j = 1, \ldots, N \) such that

\[
q_j \cos \lambda R_j \bar{t} - \sqrt{\frac{\lambda_{N+j}}{\lambda_j}} \sin \lambda R_j \bar{t} = q_j \left( q_j \sqrt{\frac{\lambda_j}{\lambda_{N+j}}} \sin \lambda R_j \bar{t} + \cos \lambda R_j \bar{t} \right),
\]

i.e.,

\[
\sin \lambda R_j \bar{t} = 0.
\]

(3.16)
From assumption 1 on the eigenvalues of $S$ there exists exactly one index $j$ such that this condition holds true; this implies that $\bar{t}$ is simple. \hfill \Box

According to Remark A.5, from Proposition 3.4 we plainly deduce the following:

**Proposition 3.5.** Assume that $S \in \mathcal{S}$. Then the eigenvalues of

$$
\begin{aligned}
z' &= \lambda J Sz, \ z = (x, y), \\
x(0) &= 0 = x(\pi),
\end{aligned}
$$

are given by

$$
\lambda_{j,k} = \frac{k}{\sqrt{\lambda_j \lambda_{N+j}}}, \quad \forall j = 1, \ldots, N, \ k \in \mathbb{Z}.
$$

Moreover, for every $k \neq 0$ and for every $j = 1, \ldots, N$, $\lambda_{j,k}$ is simple.

**Proof.** The fact that the eigenvalues of (3.17) are simple follows from Remark A.5 and Proposition 3.4.

In order to compute the eigenvalues, we use again the change of variable $w = P^Tz$; hence, according to the computations given in the proof of Proposition 3.4, $\lambda$ is an eigenvalue of (3.17) if and only if there exists a nontrivial solution $w = (u, v)$ of (3.13) such that $u(0) = Qv(0)$ and $u(\pi) = Qv(\pi)$. Recalling (3.16), this is equivalent to say that

$$
\sin \lambda R_j \pi = 0,
$$

for some $j = 1, \ldots, N$. Solving this equation, we obtain that $\lambda$ is an eigenvalue if and only if $\lambda = \lambda_{j,k}$, for some $j = 1, \ldots, N$ and $k \in \mathbb{Z}$. \hfill \Box

Now, let us order the eigenvalues of (3.17) and let us denote by

$$
\{\mu_k\}_{k \in \mathbb{Z}}
$$

the double sequence of eigenvalues; we set $\mu_0 = 0$. By slightly modifying the proof of [15, Theorem 8.2], it is possible to prove the following:

**Proposition 3.6.** For every $k \in \mathbb{Z}$, $k \neq 0$, we have

$$
m(\mu_k S) = |k| - 1.
$$

We are now in position to state a bifurcation result for system (3.1) for $\lambda > 0$.

**Theorem 3.7.** Let us consider the boundary value problem (3.1) and let us assume that $S(t) \equiv S \in \mathcal{S}$ and $F(t, 0, \lambda) = 0$, for every $t \in [0, \pi]$, $\lambda > 0$. Moreover, suppose that $F$ satisfies (3.3).

Then, for every $k \in \mathbb{N}$, $k \neq 0$, $\Sigma$ contains a continuum $C_k$ such that $(\mu_k, 0) \in C_k$, there exists $(\lambda_n, u_n) \in C_k$ such that

$$
\lambda_n + \|u_n\| \to +\infty \quad \text{or} \quad \lambda_n \to 0^+,
$$

and $\phi(\lambda, u) = k - 1$, for every $(\lambda, u) \in C_k$. 
Theorem 3.7 plainly follows from an application of Theorem 3.2 to (3.1). We remark that an analogous result holds true for (3.1) when \( \lambda < 0 \).

3.2. Systems in \( \mathbb{R}^2 \)

In this subsection we are still concerned with a system of the form (3.1) but we assume that \( N = 1 \).

In this situation some of the assumptions of Theorem 3.2 are obviously fulfilled. Indeed, for every linear system in \( \mathbb{R}^2 \) the moments of verticality are simple and so we need not require any hypothesis on \( F \) of the form (3.3).

Analogously, for every matrix \( S = S(t) \) the eigenvalues of the Dirichlet problem associated to (3.2) (when they exist) are necessarily simple.

Therefore, in order to get a bifurcation result on the lines of Theorem 3.7 we only have to concentrate on the study of conditions on \( S \) which ensure that the Dirichlet problem associated to (3.2) has eigenvalues.

This is true, of course, when \( S \in \mathcal{S} \); we observe that, for \( N = 1 \), the class \( \mathcal{S} \) reduces to the set

\[
\mathcal{S} = \{ S \in M_2^2 : \text{det} S > 0 \}.
\]

More generally, from [4, Chapter 10] we have the following:

**Lemma 3.8.** Assume that \( S \in C([0, \pi], \mathcal{S}) \). Then the Dirichlet problem associated to (3.2) has a double sequence of eigenvalues \( \mu_k \) such that

\[
\mu_k \longrightarrow \pm \infty, \quad \text{as} \quad k \longrightarrow \pm \infty.
\]

From the above discussion and Lemma 3.8, an application of Theorem 3.2 gives the following:

**Theorem 3.9.** Consider the boundary value problem (3.1) and let \( N = 1 \).

Assume that \( S(t) \equiv S \in \mathcal{S} \) and \( F(t, 0, \lambda) = 0 \), for every \( t \in [0, \pi], \lambda > 0 \). Moreover, suppose that

\[
\lambda S(t) + F(t, z, \lambda) \in M_2^2 \mathcal{S}, \quad \forall (t, z, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times (0, +\infty).
\]

Then, for every \( k \in \mathbb{N}, k \neq 0 \), \( \Sigma \) contains a continuum \( C_k \) such that \( (\mu_k, 0) \in C \), (3.18) holds and \( \phi(\lambda, u) = k - 1 \), for every \( (\lambda, u) \in C_k \).

We remark that an analogous result holds true for (3.1) when \( \lambda < 0 \).

We now give an application of Theorem 3.9 to the study of the multiplicity of solutions to the nonlinear problem (1.8). We assume that \( S : [0, \pi] \times \mathbb{R}^2 \to \mathcal{S} \) is continuous and that

\[
S(t, 0) = S_0(t), \quad \text{for every} \quad t \in [0, \pi],
\]

\[
\lim_{{|z| \to +\infty}} S(t, z) = S_\infty(t), \quad \text{uniformly in} \quad t \in [0, \pi]. \quad (3.19)
\]

We again assume, without loss of generality, that \( S(t, z) \) is positive definite for every \( (t, z) \in [0, \pi] \times \mathbb{R} \).
The case when $S$ is negative definite can be faced with minor changes; indeed, we observe that the Maslov indeces of the matrices $A$ and $-A$, for $A \in S$, satisfy the relation $m(A) = m(-A)$.

Hence, the case of negative definite matrices can be reduced to the case of definite positive ones by a change of sign.

Let $m_0 = m(S_0)$ and $m_\infty = m(S_\infty)$; we will prove the following:

**Theorem 3.10.** Assume (3.19). Let us suppose that $S_0 \in S$, $S_\infty \in S$ and $m_0 + 1 \neq m_\infty$ or $m_0 \neq m_\infty + 1$.

Then, for every integer $k \in (m_0 + 1, m_\infty)$ (or $k \in (m_\infty + 1, m_0)$) there exists a solution $z_k$ of (1.8) such that the Maslov index of $z' = JS(t, z_k(t))z$ is $k$.

In order to prove the result we use a bifurcation argument. We give the proof in the case when $m_0 + 1 < m_\infty$; the other case can be proved in an analogous way.

Let us consider the boundary value problem

\[
\begin{cases}
z' = \lambda S_0(t)z + JF(t, z)z, \\
x(0) = 0 = x(\pi),
\end{cases}
\]

where $F(t, z) = S(t, z) - S_0(t)$. We are interested in finding solutions of (3.20) with $\lambda = 1$.

Let us fix $k \in (m_0 + 1, m_\infty)$; since $\lambda S_0(t) + F(t, z)$ is positive definite, for every $(t, z, \lambda) \in [0, \pi] \times \mathbb{R}^2 \times (1, +\infty)$, and $F(t, 0) = 0$, for every $t \in [0, \pi]$, we can apply Theorem 3.9 to get a branch $C_k$ of solutions to (3.20) bifurcating from $(\mu_k, 0)$, where $\mu_k$ is the $k$th positive eigenvalue of the Dirichlet problem associated to $z' = \lambda S_0(t)z$.

Using Proposition 3.6 it is easy to see that condition $k > m_0 + 1$ implies that $\mu_k > 1$. Moreover, we have

\[
\phi(\lambda, u) = k - 1, \quad \forall (\lambda, u) \in C_k.
\]

Theorem 3.9 ensures that one of the following conditions holds true:

(i) there exists $(\lambda_n, u_n) \in C_k$ such that $\lambda_n \to +\infty$;
(ii) there exists $(\lambda_n, u_n) \in C_k$ such that $\|u_n\| \to +\infty$;
(iii) there exists $(\lambda_n, u_n) \in C_k$ such that $\lambda_n \to 0^+$.

We will show that (i) and (ii) cannot occur; this implies that the bifurcating branch $C_k$ must intersect the line $\lambda = 1$, giving rise to a solution to (1.8).

To prove that (i) and (ii) cannot hold, we will use some estimates on the Maslov index of linearized equations associated to

\[
z' = \lambda S_0(t)z + JF(t, z)z;
\]

for every continuous function $\alpha : [0, \pi] \to \mathbb{R}^2$, we define

\[
R_{\lambda, \alpha}(t) = \lambda S_0(t) + F(t, \alpha(t)) = (\lambda - 1)S_0(t) + S(t, \alpha(t)),
\]

for every $t \in [0, \pi]$. The first property of the Maslov index we will use is the following:
Proposition 3.11. (See [19, Theorem 5.7].) Let $A_1$ and $A_2$ be positive definite and assume that $A_1 - A_2$ is positive definite. Then we have

$$m(A_1) \geq m(A_2).$$

(3.22)

From Proposition 3.11 we immediately deduce the following:

Proposition 3.12. There exists $M_k > 0$ such that for every $(\lambda, u) \in C_k$ we have $\lambda \leq M_k$.

Proof. For every $(\lambda, u) \in C_k$ we have

$$m(R_{\lambda,u}) = k - 1.$$ (3.23)

Since $S$ is positive definite, from (3.22) we deduce that

$$m(R_{\lambda,u}) = m((\lambda - 1)S_0 + S) \geq m((\lambda - 1)S_0).$$ (3.24)

The fact that $S_0$ is positive definite implies that there exists $\lambda_0 > 0$ such that $S_0 - \lambda_0 I_d$ is positive definite; therefore we obtain

$$m((\lambda - 1)S_0) \geq m((\lambda - 1)\lambda_0 I_d) = (\lambda - 1)([\lambda_0] - 1).$$ (3.25)

From (3.23)–(3.25) we infer that

$$\lambda \leq 1 + \frac{k - 1}{\lambda_0}.$$ (3.26)

This concludes the proof. □

Using now the assumption on the behaviour of $S$ at infinity, we can exclude also alternative (ii). To this aim, we need to state more properties of the solutions to (3.20) and of the Maslov index; for the proofs we refer to [7,17].

Proposition 3.13. Assume $|\lambda| \leq M_k$. Then, for every $R_1 > 0$ there exists $R_2 > 0$ such that for every solution $z$ of (3.20) we have

$$|z(0)| \leq R_1 \quad \Rightarrow \quad \|z(t)\| \leq R_2, \quad \forall t \in [0, \pi].$$

Proposition 3.14. Assume $|\lambda| \leq M_k$. Then, there exists $Z > 0$ such that for every solution $z$ of (3.20) we have

$$|z(0)| > Z \quad \Rightarrow \quad \left| m(R_{\lambda,z}) - m((\lambda - 1)S_0 + S_\infty) \right| \leq 1.$$ (3.26)

Using these results we can prove the following:

Proposition 3.15. There exists $R_k > 0$ such that for every $(\lambda, u) \in C_k$ we have $\|u\| \leq R_k$. 
Proof. By contradiction, assume that for every $R > R_2(Z)$ (with $Z$ given in Proposition 3.14 and $R_2(Z)$ as in Proposition 3.13) there exists $(\lambda, u) \in C_k$ such that $\|u\| \geq R$. For the solution $u$ we necessarily have $|u(0)| \geq Z$; hence, from (3.26) we infer that

$$m(R_{\lambda,u}) \geq m((\lambda - 1)S_0 + S_\infty) - 1.$$  

(3.27)

On the other hand, from Proposition 3.11 we deduce that

$$m((\lambda - 1)S_0 + S_\infty) \geq m(S_\infty) = m_\infty.$$  

(3.28)

From (3.27) and (3.28) we get

$$k - 1 = m(R_{\lambda,u}) \geq m_\infty - 1,$$

which is impossible by the choice of $k$. □

From Propositions 3.12 and 3.15 we are able to exclude conditions (i) and (ii); as already observed, this proves Theorem 3.10.

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Appendix A. Some results on the moments of verticality

In this appendix we consider a linear system of the form

$$z' = JA(t)z,$$  

(A.1)

where $A : [0, \pi] \to M^{2N}_S$ is continuous; we are interested in the study of (A.1) with the boundary conditions $x(0) = 0 = x(\pi)$, being $z = (x, y)$. It is possible to associate to (A.1) an integer index (which is strictly related to the Maslov index) which is called the “number of moments of verticality;” we only give a brief sketch of the definition of this index and of its properties. We refer to [7,15] for a complete description.

We observe that we are going to define the Maslov index for (positive or negative) definite symmetric matrices $A$ (see the definition of $M^{2N}_S$ at the end of the Introduction). This is only a sufficient condition for the existence of the index; in particular, for systems equivalent to second order or higher order equations this condition might not be satisfied. Nevertheless, in these situations the index can be defined as well; we refer to [15] for a more complete discussion.

In what follows we first review some definitions and results of symplectic geometry. Recall first that $\mathbb{R}^{2N}$ can be endowed with the symplectic form defined by

$$\{z, \tilde{z}\} = x \cdot \tilde{y} - y \cdot \tilde{x},$$  

(A.2)

being $z = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \times \mathbb{R}^2$. 

Definition A.1. An $N$-dimensional subspace $P \subset \mathbb{R}^{2N}$ is called a “lagrangian plane” if $[z, \tilde{z}] = 0$, for every $z, \tilde{z} \in P$.

It is easy to check (cf. [15, Theorem 5.2]) that the subspace $Y = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2: x = 0\}$ is a lagrangian plane of $\mathbb{R}^{2N}$.

In order to describe the solutions $z = (x, y)$ to (A.1) satisfying $x(0) = 0 = x(\pi)$, it is possible to consider the evolution, under the action of the flow associated to (A.1), of the lagrangian plane $Y$; indeed, for every $t \in [0, \pi]$, let us consider the map $\phi_t : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ defined by

$$\phi_t(z_0) = z(t; z_0), \quad \forall z_0 \in \mathbb{R}^{2N},$$

where $z(\cdot; z_0)$ denotes the (unique) solution to (A.1) satisfying $z(0) = z_0$. Setting $Y_t = \phi_t(Y)$, for every $t \in [0, \pi]$, we have the following:

Proposition A.2. For every $t \in [0, \pi]$, the subspace $Y_t$ is a lagrangian plane of $\mathbb{R}^{2N}$. Moreover, $z = (x, x')$ is a nontrivial solution to (A.1) such that $x(0) = 0 = x(t)$ if and only if $Y_t \cap Y \neq \{0\}$.

Proposition A.2 suggests the following definition:

Definition A.3. 1. The instant $t_0 \in [0, \pi]$ is called a “moment of verticality” for (A.1) (or, equivalently, for (A.1)) if $Y_{t_0} \cap Y \neq \{0\}$.

2. Let $t_0 \in [0, \pi]$ be a moment of verticality for (A.1). The “multiplicity” of $t_0$ is the dimension of the subspace $Y_{t_0} \cap Y$.

Remark A.4. We observe that, according to Definition A.3 and Proposition A.2, if $t_0$ is a moment of verticality of (A.1), then there exists at least one nontrivial solution $z = (x, y)$ to (A.1) such that $x(0) = 0 = x(t_0)$. The multiplicity of $t_0$ as a moment of verticality of (A.1) is then the number of linearly independent solutions to (A.1) such that $x(0) = 0 = x(t_0)$.

Remark A.5. Let us consider the parameter dependent equation

$$z' = \lambda A(t)z, \quad \lambda \in \mathbb{R};$$

(A.3)

it is immediate to see that $\lambda$ is an eigenvalue of (A.3), together with the boundary condition $x(0) = 0 = x(\pi)$, if and only if $t = \pi$ is a moment of verticality of (A.3). Moreover, the multiplicity of the eigenvalue $\lambda$ is exactly the multiplicity of the moment of verticality $t = \pi$. In particular, the eigenvalue is simple if and only if $t = \pi$ has multiplicity one.

We denote by $\mathcal{M}_A$ the set of moments of verticality of (A.1) in $(0, \pi)$; using the fact that $A(t)$ is definite, for every $t \in [0, \pi]$, it can be proved that the cardinality of $\mathcal{M}_A$ is finite (see also the proof of [15, Theorem 8.2]). For every $t \in \mathcal{M}_A$, we indicate by $\mu(t)$ the multiplicity of $t$, according to Definition A.3.

Definition A.6. We call “number of moments of verticality of the matrix $A$” (or “Maslov index” of the system (A.1)) the number

$$m(A) = \sum_{t \in \mathcal{M}_A} \mu(t).$$
We observe that when $N = 1$ the number $m(A)$ is the number of zeros in $(0, \pi)$ of every solution to (A.1) satisfying $x(0) = 0$.

The computation of the number of the moments of verticality through its definition is not easy; however, it is possible to give another useful expression for this number. To this aim, we introduce some definitions. Let us consider $N$ independent solutions to (A.1)

\[
\begin{align*}
z^1(t) &= \begin{bmatrix} x^1(t) \\ y^1(t) \end{bmatrix}, & \ldots & & z^N(t) &= \begin{bmatrix} x^N(t) \\ y^N(t) \end{bmatrix}, & t \in [0, \pi],
\end{align*}
\]

satisfying $x^1(0) = \cdots = x^N(0) = 0$. We then construct the matrices

\[
X(t) = \begin{bmatrix} x^1(t) & \ldots & x^N(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} y^1(t) & \ldots & y^N(t) \end{bmatrix},
\]

for every $t \in [0, \pi]$. Since the vectors $z^1(t), \ldots, z^N(t)$ are linearly independent, for every $t \in [0, \pi]$, the matrix

\[
Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}
\]

has rank $N$; this implies that the matrix $Y(t) - iX(t)$ is invertible, for every $t \in [0, \pi]$. Therefore, we can consider the matrix

\[
\Theta(t) = (Y(t) + iX(t))(Y(t) - iX(t))^{-1},
\]

for every $t \in [0, \pi]$. It can be shown that $\Theta(t)$ is unitary, for every $t \in [0, \pi]$ (cf. [15] and [21, Section V.10]).

Let $\lambda_1(t), \ldots, \lambda_N(t)$ be the eigenvalues of $\Theta(t)$. In particular we have that each $\lambda_j$ continuously maps the interval $[0, \pi]$ in $S^1 = \{ \lambda \in \mathbb{C}: |\lambda| = 1 \}$ and satisfies $\lambda_j(0) = 1$. By the unique path lifting theorem, for every $j \in \{1, \ldots, N\}$ there is a unique continuous function $\vartheta_j : [0, \pi] \to \mathbb{R}$ such that $e^{2i\vartheta_j(t)} = \lambda_j(t)$ for all $t \in [0, \pi]$ and $\vartheta_j(0) = 0$ (see also [21, Lemma V.10.1]).

**Definition A.7.** We call “phase angles” of the system (A.1) the continuous functions $\theta_1, \ldots, \theta_N : [0, \pi] \to \mathbb{R}$ obtained by arranging the values $\vartheta_1(t), \ldots, \vartheta_N(t)$ in increasing order, i.e.,

\[
\theta_1(t) \leq \theta_2(t) \leq \cdots \leq \theta_N(t), \quad \forall t \in [0, \pi].
\]

By means of the phase-angles we can characterize the moments of verticality of (A.1):

**Proposition A.8.** The following facts are equivalent:

1. The number $t_0 \in [0, \pi]$ is a moment of verticality for (A.1) of multiplicity $\mu$, with $1 \leq \mu \leq N$.
2. $1$ is an eigenvalue of algebraic multiplicity $\mu$ for the matrix $\Theta(t_0)$.
3. There exist exactly $\mu$ different integers $j_1, \ldots, j_\mu \in \{1, \ldots, N\}$ and there exist $h_1, \ldots, h_\mu \in \mathbb{N}$ such that $\theta_{j_1}(t_0) = h_1 \pi, \ldots, \theta_{j_\mu}(t_0) = h_\mu \pi$. 


Now, for every $t \in [0, \pi]$, we write the phase angles in the form
\[ \theta_j(t) = k_j(t)\pi + \alpha_j(t), \quad j = 1, \ldots, N, \]
where $k_j(t)$ are integers and $0 < \alpha_j(t) \leq \pi$, if $\theta_j(t) > 0$, and $-\pi \leq \alpha_j(t) < 0$, if $\theta_j(t) < 0$. Then, we have the following result:

**Proposition A.9.** (See [15, Theorem 8.4].) For every continuous map $A : [0, \pi] \to \mathbb{R}^2$ we have
\[ m(A) = \left| k_1(\pi) \right| + \cdots + \left| k_N(\pi) \right|. \]

Finally, we give a useful sufficient condition to guarantee that a moment of verticality is simple (i.e., of multiplicity one); to this aim, we denote by $D(t)$ the modulus of the determinant of the matrix $X(t)$ and by $R(t)$ the sum of the squares of the determinants of all the minors of order $N - 1$ of $X(t)$. Then, we have the following:

**Lemma A.10.** The instant $t_0$ is a simple moment of verticality of (A.1) if and only if
\[ D(t_0) = 0, \quad R(t_0) > 0. \]

**References**