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Asymptotic Expansions for Distributions of Latent Roots in Multivariate Analysis

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Asymptotic expansions are given for the distributions of latent roots of matrices in three multivariate situations. The distribution of the roots of the matrix $S_1(S_1 + S_2)^{-1}$, where S_1 is $W_m(n_1, \Sigma, \Omega)$ and S_2 is $W_m(n_2, \Sigma)$, is studied in detail and asymptotic series for the distribution are obtained which are valid for some or all of the roots of the noncentrality matrix Ω large. These expansions are obtained using partial-differential equations satisfied by the distribution. Asymptotic series are also obtained for the distributions of the roots of $n^{-1}S$, where S in $W_m(n, \Sigma)$, for large n, and $S_1S_2^{-1}$, where S_1 is $W_m(n_1, \Sigma)$ and S_2 is $W_m(n_2, \Sigma)$, for large $n_1 + n_2$.

1. INTRODUCTION

In this paper we derive asymptotic expansions for the distributions of latent roots of matrices in a number of multivariate situations. With the usual notations $S \sim W_m(n, \Sigma, \Omega)$ denoting that the positive definite symmetric $m \times m$ matrix S has the noncentral Wishart distribution on n degrees of freedom, and $S \sim W_m(n, \Sigma)$ that S has the central Wishart distribution, we give a method for

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deriving asymptotic *series* for the joint distributions of the roots of the following matrices:

(i) $L = n^{-1}S$, where $S \sim W_m(n, \Sigma)$, for large n, (principal components);

(ii) $F = S_1 S_2^{-1}$, where S_1 and S_2 are independent,

$$S_i \sim W_m(n_i, \Sigma_i), \qquad i = 1, 2, \text{ for large } n = n_1 + n_2 ;$$

(iii) $B = S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 are independent,

$$S_1 \sim W_m(n_1, \Sigma, \Omega), \qquad S_2 \sim W_m(n_2, \Sigma),$$

for some or all of *the latent roots of Ω large,* (multiple discriminant analysis).

Multiple integral expressions and zonal polynomial series are available for these distributions but are usually worthless for numerical work in applications to inference in multivariate analysis. However, the asymptotic expansions are obtainable in terms of elementary functions (exponentials, rational functions, etc.) and permit numerical work.

Case (i) above has been tackled by Anderson [1] and further developed by James [9, 10] and Chattopadhyay and Pillai [3]. Their method essentially is to take a multiple integral representation of the density function and derive its asymptotic behavior by a multivariate extension of Laplace's method for integrals; i.e., approximate the integrand near its maximum value and modify the domain of integration in such a way that the errors of approximation are small. A similar approach was adopted by Chang [2] and others [3, 11] for case (ii). This method, while elementary, rapidly leads to intractable algebra. For example, the asymptotic series for case (i) was given by Anderson [1] only up to the term of order n^{-1} for arbitrary m and up to terms of order n^{-2} for $m \leqslant 4$.

The method adopted here takes the same starting point to derive the first or dominant terms of the asymptotic expansions, but then uses partial-differential equations satisfied by the distributions to successively calculate the terms of order $n^{-1}, n^{-2},\ldots$. Once again, the algebra eventually becomes formidable, but we give the terms up to and including $O(n^{-3})$ for cases (i) and (ii), and $O(n^{-2})$ for case (iii).

2. Preliminaries

In terms of the hypergeometric functions ${}_pF_q$ of two matrix arguments (see James [8]), the joint density functions of the roots of the matrices L, F, and B mentioned in Section 1 are, respectively,

$$\frac{\pi^{(1/2)m^2}((1/2)n)^{(1/2)mn}\,(\det A)^{(1/2)n}}{\Gamma_m((1/2)n)\,\Gamma_m((1/2)m)}$$

$$\cdot\left(\prod_{i=1}^{m} l_i\right)^{(1/2)(n-m-1)}\prod_{i<j}^{m}(l_i - l_j)\,{}_0F_0(-(1/2)\,nL, A), \qquad (2.1)$$

(where $A = \Sigma^{-1}$),

$$\frac{\pi^{(1/2)m^2}\Gamma_m((1/2)(n_1+n_2))}{\Gamma_m((1/2)n_1)\,\Gamma_m((1/2)n_2)\,\Gamma_m((1/2)m)}$$

$$\cdot(\det \Lambda)^{(1/2)n_1}\left(\prod_{i=1}^{m} f_i\right)^{(1/2)(n_1-m-1)}\prod_{i<j}^{m}(f_i - f_j)\,{}_1F_0((1/2)n;\,-F, \Lambda) \qquad (2.2)$$

(where $n = n_1 + n_2$, $\Lambda = \Sigma_2\Sigma_1^{-1}$), and

$$\frac{\pi^{(1/2)m^2}\Gamma_m((1/2)(n_1+n_2))}{\Gamma_m((1/2)n_1)\,\Gamma_m((1/2)n_2)\,\Gamma_m((1/2)m)}$$

$$\cdot\exp(\mathrm{tr} - (1/2)\Omega)\left(\prod_{i=1}^{m} b_i\right)^{(1/2)(n_1-m-1)}\prod_{i=1}^{m}(1-b_i)^{(1/2)(n_2-m-1)}\prod_{i<j}^{m}(b_i - b_j)$$

$$\cdot\,{}_1F_1((1/2)(n_1+n_2);\,(1/2)n_1\,;\,(1/2)\Omega, B), \qquad (2.3)$$

where, in each case, it is assumed that the sample roots are ordered in decreasing order. In the following sections we derive asymptotic series for the hypergeometric functions, for large n in (2.1) and (2.2), and for "large Ω" in (2.3). Before deriving the expansions there are a few preliminaries to be disposed of.

First, we note Hsu's [6] extension of Laplace's method for obtaining the asymptotic behavior of integrals. If the function $f(x) = f(x_1, ..., x_m)$ has an absolute maximum at an interior point ξ of a closed domain D in real m-dimensional space, then under suitable conditions,

$$\int_D [f(x)]^n\,\varphi(x)\,dx \sim (2\pi/n)^{(1/2)m}\,[f(\xi)]^n\,\varphi(\xi)[\Delta(\xi)]^{-1/2}, \qquad n \to \infty, \quad (2.4)$$

where the notation "$a \sim b$, $n \to \infty$" means that $\lim_{n\to\infty}(a/b) = 1$, and Δ denotes the Hessian of $-\log f$, i.e.,

$$\Delta(\xi) = \det\left(-\frac{\partial^2 \log f(\xi)}{\partial\xi_i\,\partial\xi_j}\right).$$

The rapidity of the convergence would appear to depend on the sharpness of the peak of $f(x)$ in the neighborhood of its maximum.

In studying the distributions (2.1) and (2.3) we are led to investigate integrals of the form

$$\int_{O(m)} \exp(\operatorname{tr} \pm nRH'SH)(dH), \tag{2.5}$$

$O(m)$ being the group of orthogonal $m \times m$ matrices. The sharpness of the peaks of the integrand at its maxima depends on the "spread" of the roots of R and S. In fact, if some roots are equal the maxima are achieved not at single points but on whole submanifolds of $O(m)$. We are therefore led to separate out as special cases those instances when some roots are equal or nearly equal to give a good coverage of possible situations which could arise in practice. The following lemma and its corollaries give the asymptotic behavior of the integrals (2.5). The proof is due essentially to Anderson [1].

LEMMA 2.1.

$$\int_{O(m)} \exp(\operatorname{tr}(1/2)n\, RH'SH)(dH) \sim 2^m \exp\left((1/2)n \sum_{i=1}^{m} r_i s_i\right) \prod_{i<j}^{m} (2\pi/nc_{ij})^{1/2}, \tag{2.6}$$

$$\int_{O(m)} \exp(\operatorname{tr} - (1/2)n\, RH'SH)(dH) \sim 2^m \exp\left(-(1/2)n \sum_{i=1}^{m} r_i s_{m-i+1}\right) \cdot \prod_{i<j}^{m} (2\pi/nc_{ij})^{1/2}, \tag{2.7}$$

where

$$c_{ij} = (r_i - r_j)(s_i - s_j) \tag{2.8}$$

and the roots of R and S are strictly unequal and ordered in descending order. The measure (dH) is the invariant measure on O(m) defined as in James [7].

Proof. The result (2.7) was proved by Anderson [1] (with the ordering of the roots of S reversed). Then (2.6) follows by noting that the maxima of $\operatorname{tr} RH'SH$ are the minima of $\operatorname{tr}(-RH'SH)$, and the latter are achieved at the 2^m matrices of the form

$$H = \begin{bmatrix} 0 & & & \pm 1 \\ & & \pm 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \pm 1 & & & 0 \end{bmatrix}.$$

COROLLARY 2.1. *If R_1 and S are $k \times k$ and $m \times m$ diagonal matrices respectively, $k \leqslant m$, with unequal elements ordered in descending order, then*

$$\int_{V(k,m)} \exp(\operatorname{tr}(1/2)n\, R_1 H_1' S H_1)(dH_1)$$

$$\sim 2^k \exp\left((1/2)n \sum_{i=1}^{k} r_i s_i\right) \prod_{i<j}^{k} (2\pi/nc_{ij})^{1/2} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (2\pi/nd_{ij})^{1/2}, \quad (2.9)$$

where c_{ij} is given by (2.8) and $d_{ij} = r_i(s_i - s_j)$ for $i = 1,..., k$ and $j = k + 1,..., m$. $V(k, m)$ is the Stiefel manifold consisting of all $m \times k$ matrices H_1 with orthonorma. columns (see James [7]).

Proof. The maxima of the integrand are achieved at the 2^k matrices of the form

$$H_1 = \begin{bmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ 0 & & \ddots & \\ & & & \pm 1 \\ & & 0 & \end{bmatrix}$$

and Hsu's result (2.4) applies. The Hessian was calculated by James [10].

COROLLARY 2.2. *With R and S as in Lemma 2.1, but with the last $(m - k)$ roots of R being equal to r,*

$$\int_{O(m)} \exp\left(\operatorname{tr} \frac{1}{2} n\, RH'SH\right) (dH)$$

$$\sim \frac{2^m \pi^{(1/2)(m-k)^2}}{\Gamma_{m-k}((1/2)(m-k))} \exp\left(\frac{1}{2} n \sum_{i=1}^{m} r_i s_i\right) \prod_{i<j}^{k} \left(\frac{2\pi}{nc_{ij}}\right)^{1/2} \prod_{i=1}^{k} \prod_{j=k+1}^{m} \left(\frac{2\pi}{ne_{ij}}\right)^{1/2} \quad (2.10)$$

with c_{ij} as before and $e_{ij} = (r_i - r)(s_i - s_j)$ for $i = 1,..., k$ and $j = k + 1,..., m$.

Proof. Partitioning

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & rI_{m-k} \end{bmatrix}, \qquad H = [H_1 : H_2],$$

the integrand in (2.10) becomes

$$\exp(\operatorname{tr} \tfrac{1}{2} nrS) \exp(\operatorname{tr} \tfrac{1}{2} n(R_1 - rI)\, H_1' S H_1)$$

and is independent of H_2 which can be integrated out according to Lemma 2.2 below.

Note. A result similar to (2.10) was proved by James [10] (with a minus sign in the exponent). It should be noticed that James normalizes the measure (dH) so that the volume of $O(m)$ is unity. Throughout this paper we use unnormalized measures, equivalent to ordinary Lebesgue measure, so that

$$\mathrm{Vol}(V(k, m)) = \frac{2^k \pi^{(1/2)mk}}{\Gamma_k((1/2)m)}$$

(and $\mathrm{Vol}(V(m, m)) = \mathrm{Vol}(O(m))$), regarding $V(k, m)$ as a point set in $\frac{1}{2}k(2m - k - 1)$-dimensional Euclidean space (see [7]).

We conclude this section with a lemma which enables us, given a function $f(H)$ of an orthogonal matrix, to first integrate over the last $m - k$ columns of H, the first k columns being *fixed*, and then to integrate over these k columns.

LEMMA 2.2.

$$\int_{O(m)} f(H_1, H_2)(dH) = \int_{H_1 \in V(k,m)} \int_{K \in O(m-k)} f(H_1, GK)(dK)(dH_1) \tag{2.11}$$

where $H = [H_1 : H_2]$, H_1 is $m \times k$ and $G = G(H_1)$ is any $m \times (m - k)$ matrix with orthonormal columns orthogonal to H_1 (so that $GG' = I - H_1H_1'$).

Proof. For fixed H_1, the manifold $\mathscr{H}_2$, say, spanned by the columns of H_2 can be generated by orthogonal transformations of any fixed matrix G chosen so that $[H_1 : G]$ is an orthogonal matrix, i.e. any $H_2 \in \mathscr{H}_2$ can be written as $H_2 = GK$, and as H_2 runs over $\mathscr{H}_2$, K runs over $O(m - k)$ and the relationship is one-to-one. Denoting the columns of H_2 by h_{k+j}, $j = 1,\ldots, m - k$, and those of K by k_j, we have

$$dh_{k+j} = G\, dk_j$$

for fixed G. Now the invariant measure on $O(m)$ is given by (see [7])

$$\begin{aligned}(dH) = \prod_{i>j}^{m} h_i'\, dh_j &= \prod_{i>j}^{k} h_i'\, dh_j \prod_{i=1}^{m-k} \prod_{j=1}^{m} h_{k+i}'\, dh_j \prod_{i>j}^{m-k} h_{k+i}'\, dh_{k+j} \\ &= \prod_{i>j}^{k} h_i'\, dh_j \prod_{i=1}^{m-k} \prod_{j=1}^{m} k_i'G'\, dh_j \prod_{i>j}^{m-k} k_i'\, dk_j \\ &= (dH_1)(dK).\end{aligned}$$

Note that this transformation of the measure (dH) is to be interpreted as: first integrate over K for *fixed* H_1, and then integrate over H_1.

3. Asymptotic Behavior of ${}_1F_1(a; c; R, S)$

In the distribution (2.3) the function ${}_1F_1(\frac{1}{2}(n_1 + n_2); \frac{1}{2}n_1 ; \frac{1}{2}\Omega, B)$ appears, where B is a diagonal matrix with elements $1 > b_1 > b_2 > \cdots > b_m > 0$, and Ω is a matrix of noncentrality parameters with roots $\infty > \omega_1 \geqslant \omega_2 \geqslant \cdots \geqslant \omega_m \geqslant 0$. Situations arise in practice (in multiple discriminant analysis) where some or all of the ω_i are large, some equal or some zero. In this section we give a series of theorems which, in combination, describe the asymptotic behavior of ${}_1F_1$ when some, at least, of the ω_i are large.

We examine first the case of the function ${}_1F_1(a; c; R)$ of one argument matrix. In the original sampling situation, $\Omega = \Sigma^{-1}MM'$ and if some of the mean vectors in M are large, say

$$M = \begin{bmatrix} N^{1/2}M_1 \\ \cdots \\ M_2 \end{bmatrix},$$

then Ω would be of the form (writing R in place of Ω),

$$R = \begin{bmatrix} NR_{11} & N^{1/2}R_{12} \\ N^{1/2}R_{12}' & R_{22} \end{bmatrix}, \qquad R_{11} \text{ is } k \times k, \tag{3.1}$$

N being a "large" parameter. Note that for large N, the roots of R are approximately equal to those of NR_{11} together with those of $R_{22} - R_{12}'R_{11}^{-1}R_{12}$. Our commencing point is the integral representation (Herz [5])

$${}_1F_1(a; c; R) = \frac{\Gamma_m(c)}{\Gamma_m(a)\,\Gamma_m(c-a)} \int_{0<W<I} \exp(\operatorname{tr} RW)(\det W)^{a-(1/2)(m+1)} \cdot \det(I - W)^{c-a-(1/2)(m+1)}\, dW, \tag{3.2}$$

where we temporarily assume $\operatorname{Re}(c - a) > \frac{1}{2}(m - 1)$ so that the integral converges. This restriction may be dropped at the end. We require also the Kummer transformation (see [5])

$${}_1F_1(a; c; R) = \exp(\operatorname{tr} R)\, {}_1F_1(c - a; c; -R). \tag{3.3}$$

Then, if R is of the form (3.1) above, we have

THEOREM 3.1.

$${}_1F_1(a; c; R) \sim (\Gamma_k(c)/\Gamma_k(a)) \exp(\operatorname{tr} NR_{11}) \exp(\operatorname{tr} R_{12}'R_{11}^{-1}R_{12})(\det NR_{11})^{a-c} \cdot {}_1F_1(a - (1/2)k; c - (1/2)k; R_{22} - R_{12}'R_{11}^{-1}R_{12}) \tag{3.4}$$

as $N \to \infty$.

Proof. To avoid writing out long expressions we merely sketch the proof. The details, including evaluation and simplification of the constants, are reasonably straightforward and will be omitted. Combining (3.2) and (3.3), partition the variables of integration conformably with R; i.e., put

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & W_{22} \end{bmatrix}, \qquad W_{11} \text{ is } k \times k.$$

Transform to variables X given by

$$X_{11} = NW_{11}$$

$$X_{12} = N^{1/2} X_{11}^{-1/2} W_{12} W_{22}^{-1/2}$$

$$X_{22} = (I + X_{12}'(NX_{11}^{-1} - I)^{-1} X_{12})^{1/2}\, W_{22} (I + X_{12}'(NX_{11}^{-1} - I)^{-1} X_{12})^{1/2}$$

with Jacobian

$$N^{-(1/2)k(m+1)} (\det X_{11})^{(1/2)(m-k)}$$

$$\cdot (\det X_{22})^{(1/2)k} \det(I + X_{12}'(NX_{11}^{-1} - I)^{-1} X_{12})^{-(1/2)(m+1)}$$

The ranges of the new variables are $0 < X_{11} < NI_k$, $X_{12}'X_{12} < I_{m-k}$ and $0 < X_{22} < I_{m-k}$. As $N \to \infty$, $X_{12}'(NX_{11}^{-1} - I)^{-1} X_{12} \to 0$, and we have

$$ {}_1F_1(a; c; R) \sim \frac{\Gamma_m(c)}{\Gamma_m(a)\,\Gamma_m(c-a)} N^{k(a-c)} \exp(\operatorname{tr} R)$$

$$\cdot \int \exp[\operatorname{tr}(-R_{11}X_{11}) + \operatorname{tr}(-R_{22}X_{22}) + \operatorname{tr}(-2R_{12}'X_{11}^{1/2}X_{12}X_{22}^{1/2})]$$

$$(\det X_{11})^{c-a-(1/2)(k+1)} (\det X_{22})^{c-a-(1/2)(m-k+1)}$$

$$\det(I - X_{22})^{a-(1/2)k-(1/2)(m-k+1)} \det(I - X_{12}'X_{12})^{c-a-(1/2)(m-k)-(1/2)(k+1)}\, dX,$$

the range of integration of X_{11} now being $X_{11} > 0$. We now integrate over X_{12} and X_{11} (in that order) using formulae (3.6) and (2.5) of Herz [5] and then over X_{22} using (3.2) and (3.3) above. Q.E.D.

In principle, further terms in the asymptotic expansion of ${}_1F_1(a; c; R)$ could be obtained from a refinement of the above analysis. In practice, the integrals become intractable except in the special case $k = m$; i.e., all elements and roots of R large. We give the results for this special case.

THEOREM 3.2.

$$ {}_1F_1(a; c; NR) \sim \frac{\Gamma_m(c)}{\Gamma_m(a)} (\det NR)^{a-c} \exp(\operatorname{tr} NR) $$

$$ \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{((1/2)(m+1)-a)_\kappa (c-a)_\kappa}{k!} \frac{C_\kappa(R^{-1})}{N^k} $$

Proof. Substitute (3.3) in (3.2), change variables to $X = NW$, expand $\det(I - N^{-1}X)^{a-(1/2)(m+1)}$ formally as a series in zonal polynomials and integrate over $X > 0$. Q.E.D.

It is now relatively easy to derive the asymptotic behavior of ${}_1F_1(a; c; R, S)$ by combining Theorem 3.1 with Lemma 2.1 and its corollaries, since

$$ {}_1F_1(a; c; R, S) = 2^{-m}\pi^{-(1/2)m^2}\Gamma_m(\tfrac{1}{2}m) \int_{O(m)} {}_1F_1(a; c; RH'SH)(dH). \quad (3.5) $$

THEOREM 3.3. *Let R and S be $m \times m$ diagonal matrices, the elements of S strictly unequal and ordered in decreasing order, and let R be of the form*

$$ R = \begin{bmatrix} NR_1 & 0 \\ 0 & R_2 \end{bmatrix}, \qquad R_1 \textit{ is } k \times k, $$

the elements of R_1 strictly unequal and ordered in decreasing order. Then, as $N \to \infty$,

$$ {}_1F_1(a; c; R, S) $$

$$ \sim \frac{\Gamma_k(c)}{\Gamma_k(a)} \Gamma_k\left(\frac{1}{2}m\right) \pi^{-(1/4)k(k+1)} \exp(\operatorname{tr} NR_1S_1)(\det NR_1S_1)^{a-c} $$

$$ \cdot {}_1F_1(a-(1/2)k; c-(1/2)k; R_2, S_2) \prod_{i<j}^{k} (Nc_{ij})^{-1/2} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (Nd_{ij})^{1/2} \quad (3.6) $$

where c_{ij} and d_{ij} are as in (2.9).

Proof. Substituting (3.4) in (3.5) with

$$ R_{11} = R_1^{1/2}H_1'SH_1R_1^{1/2}, \quad R_{12} = R_1^{1/2}H_1'SH_2R_2^{1/2}, \quad R_{22} = R_2^{1/2}H_2'SH_2R_2^{1/2}, $$

we have

$$
{}_1F_1(a;\, c;\, R,\, S) \sim 2^{-m}\pi^{(1/2)m^2}\, \frac{\Gamma_k(c)\, \Gamma_m((1/2)m)}{\Gamma_k(a)} \int_{O(m)} \exp(\operatorname{tr} NR_1H_1{}'SH_1)
$$

$$
\det(NR_1H_1{}'SH_1)^{a-c} \exp(\operatorname{tr} R_2H_2{}'XH_2)
$$

$$
{}_1F_1\left(a - \frac{1}{2}k;\, c - \frac{1}{2}k;\, R_2H_2{}'(S - X)\, H_2\right) (dH) \tag{3.7}
$$

where we have put $X = SH_1(H_1{}'SH_1)^{-1}\, H_1{}'S$. From Hsu's result (2.4), the asymptotic behavior of the integrand in (3.7) depends only on the behavior of the integrand at the maxima of $\operatorname{tr} R_1H_1{}'SH_1$ and from Corollary 2.1 to Lemma 2.1, these are achieved at the 2^k matrices of the form

$$
H_1 = \begin{bmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ 0 & & \ddots & \\ & & & \pm 1 \\ & & 0 & \end{bmatrix}. \tag{3.8}
$$

Hence, in applying Lemma 2.2 to integrate over H_2 it is sufficient to evaluate the integral over H_2 at any of the values (3.8) for H_1. Therefore, putting $H_2 = GK$, $K \in O(m - k)$, we can choose

$$
G = \begin{bmatrix} 0 \\ \cdots \\ I_{m-k} \end{bmatrix}.
$$

At these values of H_1,

$$
X = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}.
$$

K is then easily integrated out of (3.7) and the integral with respect to H_1 is given by Corollary 2.1. Q.E.D.

COROLLARY 3.3. *If $k = m$ in Theorem 3.3, then*

$$
{}_1F_1(a;\, c;\, NR,\, S) \sim \frac{\Gamma_m(c)}{\Gamma_m(a)}\, \Gamma_m\left(\frac{1}{2}m\right) \pi^{-(1/4)m(m+1)} \exp(\operatorname{tr} NRS) \det(NRS)^{a-c}
$$

$$
\cdot \prod_{i<j}^{m} (Nc_{ij})^{-1/2}. \tag{3.9}
$$

THEOREM 3.4. *With S as in Theorem 3.3, but R of the form*

$$R = N\begin{bmatrix} R_1 & 0 \\ 0 & rI_{m-k} \end{bmatrix},$$

R_1 as before, then

$${}_1F_1(a;\, c;\, R,\, S) \sim \frac{\Gamma_m(c)\,\Gamma_k((1/2)m)}{\Gamma_m(a)}\, \pi^{-(1/4)k(k+1)} \exp(\operatorname{tr} NRS)(\det NRS)^{a-c}$$

$$\cdot \prod_{i<j}^{k} (Nc_{ij})^{-1/2} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (Ne_{ij})^{-1/2},$$

where c_{ij} and e_{ij} are as in (2.10).

Proof. Substitute (3.4) in (3.5) (with $k = m$) and apply Corollary 2.2.

4. ASYMPTOTIC SERIES FOR ${}_1F_1(a; c; R, S)$

In this section we derive further terms in the asymptotic expansions for ${}_1F_1(a; c; R, S)$ by means of partial-differential equations satisfied by the function. As pointed out in the discussion following Theorem 3.1, this would be possible in principle by a more careful analysis of the integrals. One could commence with an asymptotic series for ${}_1F_1(a; c; R)$ and integrate over the orthogonal group extending the results of Theorems 3.3 and 3.4. Such a method, if tractable, is seen to lead to an asymptotic series of the form

$${}_1F_1(a;\, c;\, R,\, S) \equiv F \sim F_0 + (F_1/N) + (F_2/N^2) + \cdots,$$

and, in fact, of the form $F = F_0 G$ where

$$G \sim 1 + (P_1/N) + (P_2/N^2) + \cdots \tag{4.1}$$

and the P_i are *independent of* N.

Briefly our method is as follows. From the differential equation for F, obtain the differential equation for G (F_0, of course, is given by the results in Theorems 3.3 or 3.4). The differential equation satisfied by G turns out to be of the form

$$(N\Delta_1 + \Delta_2)G = fG \tag{4.2}$$

where Δ_1 and Δ_2 are differential operators in R and S, independent of N, and $f = f(R, S)$ is also independent of N. Formally substituting (4.1) in (4.2) and equating coefficients of like powers of N^{-1} on both sides gives a recursive system of differential equations for the P_k:

$$\Delta_1 P_1 = f$$
$$\Delta_1 P_k + (\Delta_2 - f)\, P_{k-1} = 0, \qquad k = 2, 3, \ldots . \tag{4.3}$$

We can solve the system (4.3) for the P_k successively. The method is validated by the uniqueness of asymptotic power series; i.e., if

$$0 \sim \sum_{k=1}^{\infty} (\Delta_1 P_k + (\Delta_2 - f)\, P_{k-1})\, N^{-k} \tag{4.4}$$

then $\Delta_1 P_k + (\Delta_2 - f) P_{k-1} \equiv 0$.

In our applications the operator Δ_1 will be a ***first-order linear*** differential operator, and for convenience we quote the general form of solution of first order linear partial-differential equations in a form suitable for our purposes. Given the differential equation

$$\sum_{i=1}^{n} f_i(\partial P/\partial x_i) = R \tag{4.5}$$

where the f_i are independent of P, and given any particular solution, Q, of (4.5), then the general solution of (4.5) is

$$P = Q + \Psi(u_1, \ldots, u_{n-1}) \tag{4.6}$$

where the arbitrary function Ψ is the general solution of the homogeneous-differential equation $\sum_{i=1}^{n} f_i(\partial P/\partial x_i) = 0$. $u_i = c_i$ are any $n - 1$ independent solutions of the system of ordinary-differential equations

$$\frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \cdots = \frac{dx_n}{f_n}. \tag{4.7}$$

We consider first the case of Corollary 3.3; i.e., all elements of S unequal and all elements of R large and unequal. Introducing again the dummy variable N, the function ${}_1F_1(a; c; NR, S)$ satisfies the partial-differential equation (see Constantine and Muirhead [4])

$$\sum_{i=1}^{m} s_i \frac{\partial^2 F}{\partial s_i^2} + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{s_i}{s_i - s_j} \frac{\partial F}{\partial s_i} + \left(c - \frac{1}{2}(m-1)\right) \sum_{i=1}^{m} \frac{\partial F}{\partial s_i}$$
$$- N \sum_{i=1}^{m} r_i^2 \frac{\partial F}{\partial r_i} - Na \sum_{i=1}^{m} r_i F = 0.$$

From (3.9) we put (ignoring the constant)

$$F = \exp\left(\sum_{i=1}^{m} Nr_i s_i\right)\left(\prod_{i=1}^{m} Nr_i s_i\right)^{a-c} \prod_{i<j}^{m} (Nc_{ij})^{-1/2}\, G \tag{4.8}$$

and obtain for G the partial-differential equation

$$\begin{aligned}\sum_{i=1}^{m} s_i \frac{\partial^2 G}{\partial s_i^2} &+ \left(2a - c - \frac{1}{2}(m-1)\right) \sum_{i=1}^{m} \frac{\partial G}{\partial s_i} \\ &+ N\left(2 \sum_{i=1}^{m} r_i s_i \frac{\partial G}{\partial s_i} - \sum_{i=1}^{m} r_i^2 \frac{\partial G}{\partial r_i}\right) \\ &= -\left[(a-c)\left(a - \frac{1}{2}(m+1)\right) \sum_{i=1}^{m} \frac{1}{s_i} + \frac{1}{4} \sum_{i<j}^{m} \frac{s_i + s_j}{(s_i - s_j)^2}\right] G. \end{aligned}\tag{4.9}$$

This is of the form (4.2) with

$$\Delta_1 = 2 \sum_{i=1}^{m} r_i s_i \frac{\partial}{\partial s_i} - \sum_{i=1}^{m} r_i^2 \frac{\partial}{\partial r_i}$$

and

$$f = -\left[(a-c)\left(a - \frac{1}{2}(m+1)\right) \sum_{i=1}^{m} \frac{1}{s_i} + \frac{1}{4} \sum_{i<j}^{m} \frac{s_i + s_j}{(s_i - s_j)^2}\right]. \tag{4.10}$$

Substituting $G = 1 + \sum_{k=1}^{\infty} N^{-k} P_k$ gives the system (4.3). The general solution is of the form (4.6),

$$P_k = Q_k + \Psi_k(u_1, \ldots, u_{2m-1})$$

where Q_k is any particular solution and the u_i are any $(2m-1)$ independent solutions of the system

$$\frac{ds_1}{2r_1 s_1} = \frac{ds_2}{2r_2 s_2} = \cdots = \frac{ds_m}{2r_m s_m} = \frac{dr_1}{-r_1^2} = \cdots = \frac{dr_m}{-r_m^2}. \tag{4.11}$$

Such solutions are easily found to be

$$\begin{aligned} u_i &= r_i^2 s_i^2 = c_i, & i &= 1, \ldots, m \\ u_{m+i} &= \frac{1}{r_i} - \frac{1}{r_m} = c_{m+i}, & i &= 1, \ldots, m-1. \end{aligned}\tag{4.12}$$

We now apply boundary conditions to evaluate Ψ_k. Looking at (4.8) we see that F, F_0 and therefore G must satisfy the boundary conditions

$$G(N, R, S) = G(N, S, R) \tag{4.13}$$

and

$$G(N, R, S) = G(NR, S). \tag{4.14}$$

Each $N^{-k}P_k$ must satisfy these conditions also. Therefore, if we can find a Q_k such that $N^{-k}Q_k$ satisfies them, so must $N^{-k}\Psi_k$. Examination of (4.12) shows that the function Ψ_k satisfying (4.13) must be *identically constant.* The constant can then be evaluated from the form of Q_k and condition (4.14).

A particular solution of the first equation in (4.3), $\Delta_1 P_1 = f$, where Δ_1 and f are given by (4.10), is easily found to be

$$P_1 = Q_1 = (a-c)\left(a - \frac{1}{2}(m+1)\right)\sum_{i=1}^{m}\frac{1}{r_i s_i} + \frac{1}{4}\sum_{i<j}^{m} c_{ij}^{-1}, \tag{4.15}$$

where, as usual, $c_{ij} = (r_i - r_j)(s_i - s_j)$. Q_1 satisfies (4.13) and therefore $P_1 = Q_1 + \alpha$, α constant. Since $N^{-1}P_1$ satisfies (4.14) it follows that $\alpha = 0$, so that (4.15) gives the complete solution. Similarly, we find

$$\begin{aligned} P_2 = Q_2 &= \frac{1}{2}(a-c)^2\left(a - \frac{1}{2}(m+1)\right)^2\left(\sum_{i=1}^{m}\frac{1}{r_i s_i}\right)^2 \\ &\quad - \frac{1}{2}(a-c)\left(a - \frac{1}{2}(m+1)\right)\left(2a - c - \frac{1}{2}(m+3)\right)\sum_{i=1}^{m}\frac{1}{r_i^2 s_i^2} \\ &\quad + \frac{1}{4}\sum_{i<j}^{m}\frac{1}{c_{ij}^2} + \frac{1}{32}\left(\sum_{i<j}^{m}\frac{1}{c_{ij}}\right)^2 \\ &\quad + \frac{1}{4}(a-c)\left(a - \frac{1}{2}(m+1)\right)\sum_{i=1}^{m}\frac{1}{r_i s_i}\sum_{i<j}^{m}\frac{1}{c_{ij}}. \end{aligned} \tag{4.16}$$

Further P_k could be calculated if required, the form of the terms in it being clear from (4.16). We have, therefore

$$\begin{aligned} {}_1F_1(a; c; NR, S) = K_1 \exp\left(\sum_{i=1}^{m} N r_i s_i\right)\prod_{i=1}^{m}(N r_i s_i)^{a-c}\prod_{i<j}^{m}(N c_{ij})^{-1/2} \\ \{1 + (P_1/N) + (P_2/N^2) + O(N^{-3})\} \end{aligned} \tag{4.17}$$

where K_1 is given in (3.9) and P_1 and P_2 are given by (4.15) and (4.16).

Using (4.17) we can derive the asymptotic expansion for the case of Theorem 3.3, i.e., some roots of R small. We define G_1 by

$$ {}_1F_1(a; c; R, S) = K_2 \exp\left(\sum_{i=1}^{k} Nr_i s_i\right)\left(\prod_{i=1}^{k} Nr_i s_i\right)^{a-c} \prod_{i<j}^{k} (Nc_{ij})^{-1/2} $$

$$ \cdot \prod_{i=1}^{k} \prod_{j=k+1}^{m} (Nc_{ij}^{*})^{-1/2} \, {}_1F_1(a - \tfrac{1}{2}k; c - \tfrac{1}{2}k; R_2, S_2) \cdot G_1 \quad (4.18) $$

(from 3.6), where for $j = k + 1,\ldots, m$ we have "symmetrized" the d_{ij} by replacing $r_i(s_i - s_j)$ with $c_{ij}^{*} = (r_i - N^{-1}r_j)(s_i - s_j)$. Since $c_{ij}^{*}/d_{ij} \to 1$ as $N \to \infty$, we have not disturbed the asymptotic formula (3.6). Now, if we replace R_2 by NR_2 and substitute the asymptotic expansion

$$ {}_1F_1(a - \tfrac{1}{2}k; c - \tfrac{1}{2}k; NR_2, S_2) = K_3 \exp\left(\sum_{i=k+1}^{m} Nr_i s_i\right)\left(\prod_{i=k+1}^{m} Nr_i s_i\right)^{a-c} $$

$$ \cdot \prod_{\substack{i<j \\ k+1}}^{m} (Nc_{ij})^{-1/2} G_2 $$

in (4.18) and equate to (4.17), it follows that

$$ G = G_1 G_2 ; $$

that is, G_1 contains those terms which when multiplied by those in G_2 give G. We thus obtain

$$ G_1 = 1 + (1/N) \left\{ (a - c)(a - (1/2)(m + 1)) \sum_{i=1}^{k} (1/r_i s_i) + (1/4) \sum_{i<j}^{k} c_{ij}^{-1} \right. $$

$$ \left. + (1/4) \sum_{i=1}^{k} \sum_{j=k+1}^{m} c_{ij}^{*-1} \right\} + O(N^{-2}). \quad (4.19) $$

The term of order N^{-2} is rather lengthy and will not be given here. Note that G_1 does not contain terms in $(s_i - s_j)^{-1}$ for $i, j > k$ so that the expansion is valid for arbitrary R_2, including zero or equal elements.

The asymptotic expansion in the remaining case of Theorem 3.4 is similarly obtained. We merely substitute NrI for R_2 in (4.18) and use the result in

Theorem 3.2. It is of interest, however, to note that this expansion could be obtained from the modified differential equation

$$\sum_{i=1}^{m} s_i \frac{\partial^2 F}{\partial s_i^2} + \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{s_i}{s_i - s_j} \frac{\partial F}{\partial s_i} + \left(c - \frac{1}{2}(m-1)\right) \sum_{i=1}^{m} \frac{\partial F}{\partial s_i}$$

$$- N\left(\sum_{i=1}^{k} r_i^2 \frac{\partial F}{\partial r_i} + r^2 \frac{\partial F}{\partial r}\right) - Na\left(\sum_{i=1}^{k} r_i + (m-k)r\right) F = 0$$

for ${}_1F_1$ when the last $m - k$ elements of R are identically equal to Nr. Up to order N^{-2} we have

$$F = F_0 \left\{1 + \frac{1}{N}\left(\sum_{i=1}^{m} \frac{1}{r_i s_i} + \frac{1}{4}\sum_{i<j}^{k} c_{ij}^{-1} + \frac{1}{4}\sum_{i=1}^{k}\sum_{j=k+1}^{m} c_{ij}^{-1}\right) + O(N^{-2})\right\} \tag{4.20}$$

where F_0 is given in Theorem 3.4.

The results from this section and Section 3 can be substituted in (2.3) to give asymptotic forms for the distribution. We give just one simple example. Substituting (3.6) (with $a = \frac{1}{2}(n_1 + n_2)$, $c = \frac{1}{2}n_1$, $R = \frac{1}{2}\Omega$, $S = B$) in (2.3) (and dropping the dummy variable N) gives the following:

THEOREM 4.1. *The asymptotic joint distribution of the latent roots of the matrix B, when the first k latent roots $\omega_1, \ldots, \omega_k$ of the noncentrality matrix Ω are large, is*

$$\text{const}\left(\prod_{i=1}^{k} \omega_i\right)^{(1/2)(k-m)} \prod_{i<j}^{k} (\omega_i - \omega_j)^{-1/2} \exp\left(-\tfrac{1}{2}\sum_{i=1}^{k} \omega_i\right) \exp\left(\tfrac{1}{2}\sum_{i=1}^{k} \omega_i b_i\right)$$

$$\times \left(\prod_{i=1}^{k} b_i\right)^{(1/2)(n_1-m-1)} \prod_{i=1}^{k} (1 - b_i)^{(1/2)(n_2-m-1)} \prod_{i<j}^{k} (b_i - b_j)^{1/2} \left(\prod_{i=1}^{k} \omega_i b_i\right)^{(1/2)n_2}$$

$$\times \prod_{i=1}^{k} \prod_{j=k+1}^{m} (b_i - b_j)^{1/2}$$

$$\times \exp\left(-\tfrac{1}{2}\sum_{i=k+1}^{m} \omega_i\right)\left(\prod_{i=k+1}^{m} b_i\right)^{(1/2)(n_1-m-1)} \prod_{i=k+1}^{m} (1 - b_i)^{(1/2)(n_2-m-1)} \prod_{\substack{k+1\\ i<j}}^{m} (b_i - b_j)$$

$${}_1F_1(\tfrac{1}{2}(n_1 + n_2 - k);\ \tfrac{1}{2}(n_1 - k);\ \tfrac{1}{2}\Omega_2, B_2). \tag{4.21}$$

The value of the constant in (4.21) is easily obtained from (3.6) and (2.3). From (4.21) we easily obtain

COROLLARY 4.1. *The asymptotic conditional distribution of the last roots* $b_{k+1},\ldots, b_m$ *given the first roots* $b_1,\ldots, b_k$ *is*

$$\text{const} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (b_i - b_j)^{1/2} \exp\left(-\tfrac{1}{2} \sum_{i=k+1}^{m} \omega_i\right)\left(\prod_{i=k+1}^{m} b_i\right)^{(1/2)(n_1-m-1)}$$

$$\times \prod_{i=k+1}^{m} (1 - b_i)^{(1/2)(n_2-m-1)}$$

$$\times \prod_{\substack{k+1\\ i<j}}^{m} (b_i - b_j)\, {}_1F_1(\tfrac{1}{2}(n_1 + n_2 - k);\ \tfrac{1}{2}(n_1 - k);\ \tfrac{1}{2}\Omega_2, B_2), \qquad (4.22)$$

which does not depend on the population parameters $\omega_1,\ldots, \omega_k$, *and depends on* $b_1,\ldots, b_k$ *via the "linkage" factor*

$$\prod_{i=1}^{k} \prod_{j=k+1}^{m} (b_i - b_j)^{1/2}.$$

The most important practical case is when $\omega_{k+1} = \cdots = \omega_m = 0$ (i.e., $\Omega_2 = 0$), in which case the ${}_1F_1$ function in (4.22) is identically equal to one.

5. ASYMPTOTIC SERIES FOR ${}_0F_0(-\frac{1}{2}nR, S)$

The derivation of asymptotic series for ${}_0F_0(-\frac{1}{2}nR, S)$ for large n is very similar to that for ${}_1F_1$, and, in fact, the algebra is simpler and it is possible to derive further terms without much extra labor. As with ${}_1F_1$, we are led to consider a number of different cases:

(i) all roots of R and S different and widely spaced;

(ii) some roots not widely spaced; and

(iii) some roots of R equal.

Consider first (ii), a case which has not been widely studied. We assume for convenience that the roots of R and S are ordered $0 < r_1 < r_2 < \cdots < r_m$, $s_1 > s_2 > \cdots > s_m > 0$, that the first k roots of R are "widely spaced" but that the last $m - k$ roots are not. More precisely, we assume that

$$r_i - r_j = O(n^{-1}) \qquad \text{for} \quad i, j = k + 1,\ldots, m. \qquad (5.1)$$

Note that this includes case (iii) also, i.e., $r_i = r_j$ for $i, j \geqslant k+1$. Then, partitioning R and H as before,

$$\begin{aligned} {}_0F_0(-\tfrac{1}{2}nR, S) &= 2^{-m}\pi^{-(1/2)m^2}\Gamma_m(\tfrac{1}{2}m) \int_{O(m)} \exp(\operatorname{tr} - \tfrac{1}{2}nRH'SH)(dH) \\ &= 2^{-m}\pi^{-(1/2)m^2}\Gamma_m(\tfrac{1}{2}m) \int_{O(m)} \exp(\operatorname{tr} - \tfrac{1}{2}nR_1H_1'SH_1) \\ &\quad \times \exp(\operatorname{tr} - \tfrac{1}{2}nR_2H_2'SH_2)(dH) \\ &= 2^{-k}\pi^{-(1/2)km}\Gamma_k(\tfrac{1}{2}m) \int_{V(k,m)} \exp(\operatorname{tr} - \tfrac{1}{2}nR_1H_1'SH_1) \\ &\quad \times {}_0F_0(-\tfrac{1}{2}nR_2, G'SG)(dH_1) \end{aligned} \tag{5.2}$$

(using (2.11)), where $GG' = I - H_1H_1'$. Putting

$$\bar{r}_2 = \frac{1}{m-k} \sum_{j=k+1}^{m} r_j,$$

the mean of the roots of R_2, we have

$$\begin{aligned} {}_0F_0(-\tfrac{1}{2}nR_2, G'SG) &= \exp(-\tfrac{1}{2}n\bar{r}_2 \operatorname{tr} S(I - H_1H_1')) \\ &\quad \times {}_0F_0(-\tfrac{1}{2}n(R_2 - \bar{r}_2I), G'SG). \end{aligned} \tag{5.3}$$

From the assumptions (5.1), the elements of $R_2 - \bar{r}_2I$ are $O(n^{-1})$ and hence the ${}_0F_0$ function on the right-hand side of (5.3) is bounded above and below. Substituting (5.3) in (5.2) we can apply Hsu's result (2.4) and Corollary 2.1 to obtain

$$\begin{aligned} {}_0F_0(-(1/2)\, nR, S) &\sim \pi^{-(1/2)km}\Gamma_k((1/2)m) \exp\left(-(1/2)n \sum_{i=1}^{k} r_is_i\right) \\ &\quad \times \prod_{i<j}^{k} (2\pi/nc_{ij})^{1/2} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (2\pi/nd_{ij}^*)^{1/2}\, {}_0F_0(-(1/2)\, nR_2, S_2) \end{aligned} \tag{5.4}$$

where $c_{ij} = (r_i - r_j)(s_j - s_i)$ for $i, j = 1,\ldots, k$ and $d_{ij}^* = (r_i - \bar{r}_2)(s_j - s_i)$ for $i = 1,\ldots, k; j = k+1,\ldots, m$.

For $j = k+1,\ldots, m$, $\bar{r}_2 = r_j + O(n^{-1})$ so that it is valid to replace $r_i - \bar{r}_2$ by $r_i - r_j$ for symmetry. The asymptotic result (5.4) verifies a conjecture by James [9].

As for ${}_1F_1$, it is seen that it is sufficient to derive the asymptotic series for ${}_0F_0$ for case (i) above only. The series for cases (ii) and (iii) are readily obtainable by the technique described in Section 4. The differential equation for $F \equiv {}_0F_0$ is (see [4])

$$\sum_{i=1}^{m} s_i \frac{\partial^2 F}{\partial s_i^2} + \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{s_i}{s_i - s_j} \frac{\partial F}{\partial s_i} + \frac{1}{2} n \sum_{i=1}^{m} r_i^2 \frac{\partial F}{\partial r_i}$$

$$+ \frac{1}{4} n(m-1) \sum_{i=1}^{m} r_i F = 0.$$

Putting $F = F_0 G$, where

$$F_0 = \pi^{-(1/4)m(m+1)} 2^{(1/4)m(m-1)} \Gamma_m(\tfrac{1}{2}m) \exp\left(-\tfrac{1}{2}n \sum_{i=1}^{m} r_i s_i\right) \prod_{i<j}^{m} (nc_{ij})^{-1/2}$$

(from (5.4), with $k = m$), and $c_{ij} = (r_i - r_j)(s_j - s_i)$, we obtain for G the differential equation

$$\sum_{i=1}^{m} s_i \frac{\partial^2 G}{\partial s_i^2} - \frac{1}{2} n \left(2 \sum_{i=1}^{m} r_i s_i \frac{\partial G}{\partial s_i} - \sum_{i=1}^{m} r_i^2 \frac{\partial G}{\partial r_i}\right)$$

$$+ \frac{1}{4} \sum_{i<j}^{m} \frac{s_i + s_j}{(s_i - s_j)^2} G = 0.$$

The operator Δ_1 is the same as in Section 4 so that it is sufficient to find particular solutions for P_k, $G = 1 + \sum_{k=1}^{\infty} n^{-k} P_k$, satisfying

$$P_k(R, S) = P_k(S, R)$$

and

$$n^{-k} P_k(R, S) = P_k(nR, S).$$

The first three P_k are

$$P_1 = \frac{1}{2} \sum_{i<j}^{m} \frac{1}{c_{ij}},$$

$$P_2 = \sum_{i<j}^{m} \frac{1}{c_{ij}^2} + \frac{1}{8} \left(\sum_{i<j}^{m} \frac{1}{c_{ij}}\right)^2, \tag{5.5}$$

$$P_3 = 4 \sum_{i<j}^{m} \frac{1}{c_{ij}^3} + \frac{3}{4} \sum_{i<j}^{m} \frac{1}{c_{ij}} \sum_{i<j}^{m} \frac{1}{c_{ij}^2} - \frac{1}{16} \left(\sum_{i<j}^{m} \frac{1}{c_{ij}}\right)^3$$

where $c_{ij} = (r_i - r_j)(s_j - s_i)$. The form of P_2 verifies a conjecture of Anderson [1]. Substituting (5.4) and (5.5) with their special cases in (2.1) gives the joint asymptotic distributions of the roots of the Wishart matrix. For example, substituting (5.4) (with $S = L$, $R = \Sigma^{-1} = A$) in (2.1) gives the following:

THEOREM 5.1. *The asymptotic joint distribution of the latent roots* $l_1 > l_2 > \cdots l_m > 0$ *of the matrix L for large degrees of freedom n, when the roots* $\alpha_1, \ldots, \alpha_m$ *of the information matrix* $A = \Sigma^{-1}$ *satisfy*

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} \leqslant \alpha_{k+2} \leqslant \cdots \leqslant \alpha_m$$

with $\alpha_i - \alpha_j = O(n^{-1})$ *for* $i, j = k+1, \ldots, m$, *is*

$$\text{const} \left(\prod_{i=1}^{k} \alpha_i\right)^{(1/2)n} \exp\left(-\frac{1}{2}\sum_{i=1}^{k} \alpha_i l_i\right)\left(\prod_{i=1}^{k} l_i\right)^{(1/2)(n-m-1)} \prod_{i<j}^{k} \left(\frac{l_i - l_j}{\alpha_j - \alpha_i}\right)^{1/2}$$

$$\times \prod_{i=1}^{k} \prod_{j=k+1}^{m} \left(\frac{l_i - l_j}{\alpha_j - \alpha_i}\right)^{1/2}$$

$$\times \left(\prod_{i=k+1}^{m} \alpha_i\right)^{(1/2)n} \left(\prod_{i=k+1}^{m} l_i\right)^{(1/2)(n-m-1)} \prod_{\substack{k+1 \\ i<j}}^{m} (l_i - l_j)\, {}_0F_0\left(-\frac{1}{2} nL_2, A_2\right), \quad (5.6)$$

where $L_2 = \operatorname{diag}(l_{k+1}, \ldots, l_m)$ *and* $A_2 = \operatorname{diag}(\alpha_{k+2}, \ldots, \alpha_m)$.

COROLLARY 5.1. *The asymptotic conditional distribution of the last roots* $l_{k+1}, \ldots, l_m$ *given the first roots* $l_1, \ldots, l_k$ *is*

$$\text{const} \prod_{i=1}^{k} \prod_{j=k+1}^{m} (l_i - l_j)^{1/2} \left(\prod_{i=k+1}^{m} \alpha_i\right)^{(1/2)n} \left(\prod_{i=k+1}^{m} l_i\right)^{(1/2)(n-m-1)} \prod_{\substack{k+1 \\ i<j}}^{m} (l_i - l_j)$$

$$\times\, {}_0F_0(-\tfrac{1}{2} nL_2, A_2), \quad (5.7)$$

which does not depend on the population parameters $\alpha_1, \ldots, \alpha_k$ *and depends on* $l_1, \ldots, l_k$ *via the "linkage" factor*

$$\prod_{i=1}^{k} \prod_{j=k+1}^{m} (l_i - l_j)^{1/2}.$$

In the special case when $\alpha_{k+1} = \cdots = \alpha_m = \alpha$ (i.e., $A_2 = \alpha I$),

$${}_0F_0(-\tfrac{1}{2} nL_2, A_2) = \exp\left(-\tfrac{1}{2} n\alpha \sum_{i=k+1}^{m} l_i\right)$$

and (5.6) and (5.7) reduce to [10, (3.11) and (3.13)] of James.

6. Asymptotic Series for ${}_1F_1(\frac{1}{2}n; -R, S)$

Asymptotic series for ${}_1F_0(\frac{1}{2}n; -R, S)$ can be derived as above from the partial-differential equation given by Constantine and Muirhead [4]. For the case of all roots of R and S unequal, Chang [2] proved that

$${}_1F_0\left(\frac{1}{2}n; -R, S\right) \sim \frac{\Gamma_m((1/2)m)}{\pi(1/2)m^2} \prod_{i=1}^{m} (1 + r_i s_i)^{-(1/2)(n-m-1)} \prod_{i<j}^{m} \left(\frac{2\pi}{nc_{ij}}\right)^{1/2} \tag{6.1}$$

for $r_1 > r_2 > \cdots > r_m > 0, 0 < s_1 < s_2 < \cdots < s_m$, and $c_{ij} = (r_i - r_j)(s_j - s_i)$. Putting $F = F_0 G$, where F_0 is given by (6.1), we obtain for G the partial-differential equation

$$(N\Delta_1 + \Delta_2)G = fG$$

where $N = \frac{1}{2}(n - m - 1)$, and

$$\Delta_1 = 2\sum_{i=1}^{m} \frac{r_i s_i}{1 + r_i s_i} \frac{\partial G}{\partial s_i} + 2\sum_{i=1}^{m} \frac{r_i^3 s_i}{1 + r_i s_i} \frac{\partial G}{\partial r_i} - \sum_{i=1}^{m} r_i^2 \frac{\partial G}{\partial r_i},$$

$$\Delta_2 = \Delta_1 - \sum_{i=1}^{m} s_i \frac{\partial^2 G}{\partial s_i^2} - \sum_{i=1}^{m} r_i^3 \frac{\partial^2 G}{\partial r_i^2} - \sum_{i=1}^{m} r_i^2 \frac{\partial G}{\partial r_i},$$

$$f = \frac{1}{4}\sum_{i<j}^{m} \frac{r_i r_j (r_i + r_j)}{(r_i - r_j)^2} + \frac{1}{4}\sum_{i<j}^{m} \frac{s_i + s_j}{(s_i - s_j)^2}. \tag{6.2}$$

The use of $n - m - 1$ instead of n as the large parameter simplifies the form of the P_k. A boundary condition for G is

$$G(N, R, S) = G(N, S, R). \tag{6.3}$$

Unfortunately we do not have the homogeneity condition (4.14) so that our asymptotic series is incompletely determined. However, as before, (6.3) is sufficient to determine the P_k up to an additive constant, and the constants can be factored out of the series as follows.

Put $G = 1 + \sum_{k=1}^{\infty} N^{-k} P_k$ in the differential equation and equate coefficients of like powers of N^{-1} on both sides. The first equation in the system is

$$\Delta_1 P_1 = f.$$

With the aid of (6.3) we solve for P_1 up to an additive constant,

$$P_1 = P_1^* + a_1.$$

The next equation is

$$\Delta_2 P_1 + \Delta_1 P_2 = P_1 f.$$

Putting $P_2 = Q_2 + a_1 P_1{}^*$, we obtain

$$\Delta_2 P_1{}^* + \Delta_1 Q_2 = P_1{}^* f$$

which solves for Q_2 and hence P_2 up to a constant:

$$P_2 = P_2{}^* + a_1 P_1{}^* + a_2 .$$

Proceeding in this way, we solve for P_k in terms of the particular solutions $P_{k-1}^*, \ldots, P_1{}^*$ as

$$P_k = P_k{}^* + a_1 P_{k-1}^* + a_2 P_{k-2}^* + \cdots + a_{k-1} P_1{}^* + a_k .$$

Hence, we obtain G in the form

$$G = \left(1 + \sum_{k=1}^{\infty} N^{-k} P_k{}^*\right)\left(1 + \sum_{k=1}^{\infty} N^{-k} a_k\right) \tag{6.4}$$

with constants a_k that so far we have been unable to evaluate. However, for some purposes this form of G may be sufficient; for example, for numerical work on the likelihood function of Λ in (2.2) the constant multiplier in (6.4) is irrelevant as only functions of Λ are appropriate. The first three $P_k{}^*$ are

$$\begin{aligned}
P_1{}^* &= \frac{1}{2} \sum_{i<j}^{m} \frac{1}{d_{ij}}, \\
P_2{}^* &= \sum_{i<j}^{m} \frac{1}{d_{ij}^2} + \frac{1}{8}\left(\sum_{i<j}^{m} \frac{1}{d_{ij}}\right)^2, \\
P_3{}^* &= 4 \sum_{i<j}^{m} \frac{1}{d_{ij}^3} + \frac{3}{4} \sum_{i<j}^{m} \frac{1}{d_{ij}} \sum_{i<j}^{m} \frac{1}{d_{ij}^2} - \frac{1}{16}\left(\sum_{i<j}^{m} \frac{1}{d_{ij}}\right)^3 + \frac{9}{4} \sum_{i<j}^{m} \frac{1}{d_{ij}^2},
\end{aligned} \tag{6.5}$$

where

$$d_{ij} = \frac{(r_i - r_j)(s_j - s_i)}{(1 + r_i s_i)(1 + r_j s_j)} .$$

Asymptotic series for other cases of interest, e.g., some roots of R equal or nearly equal, are easily obtainable using the techniques given in Sections 4 and 5.

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