Coherent-state optical qudit cluster state generation and teleportation via homodyne detection

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A B S T R A C T

Defining a computational basis of pseudo-number states, we interpret a coherent state of large amplitude, \(|\alpha| = d/2\pi\), as a qudit — a d-level quantum system — that is an even (meaning same size of amplitudes) superposition of d pseudo-number basis states. A pair of such coherent-state qudits can be maximally entangled by generalized Controlled-Z operation that is based on cross-Kerr nonlinearity, which can be weak for large d. Hence, a coherent-state optical qudit cluster state can be prepared by repetitive application of the generalized Controlled-Z operation to a set of coherent states. We thus propose an optical qudit teleportation as a simple demonstration of cluster state quantum computation.© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/3.0/).

1. Introduction

Quantum computation is expected to speed up some computational problems exponentially and some others quadratically compared to the best known digital computation [1]. Even though many experimental proposals of quantum computers are made, there seem to be many obstacles such as decoherence, scalability, inaccurate operation, and so on [2]. There are two approaches to the quantum computing — one that molds quantum state while the other sculpts it. Molding of quantum states lies in the heart of original schemes for quantum computers that are based on quantum circuits. In these schemes, one prepares an initial quantum state made of many qubits and applies quantum operations on it, which are followed by a measurement that leads to the result.

Raussendorf and Briegel proposed a special quantum entanglement called a cluster state [3] and went on proposing cluster state quantum computation with Browne [4]: you prepare a cluster state, a giant maximally entangled state of many qubits, and just measure each qubit away feedforwardly which means measurements are done based on previous measurement results — effectively sculpturing the state. To make a quantum cluster state, prepare qubits as \(|+\rangle\), even (meaning same size of amplitudes) superposition of computational basis kets \(|0\rangle\) and \(|1\rangle\) at each lattice point and apply \(CZ_c\) (c is an index for the control qubit and t is for the target qubit) operations on all neighboring qubits in the lattice.

Even though the number of required qubits is polynomially larger than quantum circuit model, cluster state quantum computation is simpler since only single qubit measurements are needed once a cluster state is prepared.

Based on Knill, Laflamme, and Milburn’s all-optical quantum computing [5] and Raussendorf, Browne, and Briegel’s cluster state quantum computing [4], Nielsen and Dawson proposed optical cluster state quantum computing [6,7]. One important demerit of the proposal might be the probabilistic nature of linear optical gating. Nonlinear optics can be used to generate quantum optical entanglements [8] and generation of optical qudit cluster states are proposed [9–11]. These schemes, however, need impractically large nonlinearities. Instead of qubits with two basis states, quantum computation using cluster states of d-state quantum systems or qudits has been proposed with the possibility of realization with high-dimensional Ising model [12].

Many proposals of using coherent states and/or nonlinear optical interactions for quantum information processing have been made [13]. In this paper, optical coherent states are interpreted as qudits of even (meaning same size of amplitudes) superposition of basis states and these qudits are deterministically entangled into qudit cluster states using cross-Kerr nonlinear interaction. Qudit cluster states might be used for quantum computation or quantum communication network.
2. Optical coherent states as qudits

Here we propose a simple deterministic optical scheme to generate a cluster state of qudits. First we notice that the infinite Taylor series of an exponential function can be decomposed into d infinite partial sums each of which asymptotically approaches $e^{\alpha/d}$ for any finite integer d as can be seen in the following:

$$
e^x = \sum_{k=0}^{d-1} f_k(x)$$

where

$$\lim_{k \to \infty} \frac{f_k(x)}{x^d} = \frac{1}{d!}$$

for $k = 0, \ldots, d-1$.

### 2.1. Qudits in pseudo-number basis and pseudo-phase basis

In a similar manner a coherent state $|\alpha\rangle$ can be interpreted as a qudit that is evenly (meaning same size of amplitudes) superposed on computational basis ket vectors when $|\alpha|\phi/2\pi$ and this condition is assumed throughout this paper:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n |n\rangle}{\sqrt{n!}} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kd\rangle$$

with orthonormalized computational basis kets

$$|kd\rangle = \sqrt{d} e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m |k-md\rangle}{\sqrt{(k-md)!}}$$

for $k = 0, \ldots, d-1$

that we call pseudo-number states since each ket is made of photon number states with definite modulo-d number of photons.

By applying a generalized Hadamard transformation $\hat{H}$

$$\hat{H} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \alpha^k |kd\rangle \langle kd|$$

on computational basis ket $|kd\rangle$'s, we can get conjugated basis kets

$$|\bar{kd}\rangle = \hat{H} |kd\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \alpha^k |kd\rangle$$

for $l = 0, \ldots, d-1$

with $\alpha = e^{i\pi/d}$.

These conjugated basis kets are nothing but the coherent states

$$|\bar{kd}\rangle = |\alpha^l\rangle$$

that we call pseudo-phase states since each ket of this basis is a coherent state centered at a definite optical phase.

A generalized $\bar{Z}$ operator for qudits is defined as

$$\bar{Z} = \sum_{k=0}^{d-1} \alpha^k |kd\rangle \langle kd|$$

with a photon number operator $\hat{n}$ and a generalized Controlled-Z operator, $\bar{Z}_{ct}$, is defined as

$$\bar{Z}_{ct} = \sum_{k=0}^{d-1} |kd\rangle \langle kd| \otimes \bar{Z}_t = \alpha^{kl} \hat{n}_t$$

with c and t for control and target qudits respectively.

### 2.2. Qudit cluster state generation through optical operations

A generalized $\bar{Z}$ operator can be easily implemented by a phase shifter $e^{i\pi/d} \hat{n}$ with photon number operator $\hat{n}$, and a generalized Controlled-Z operator, $\bar{Z}_{ct}$, can be realized by cross-Kerr medium. If the cross-Kerr interaction with Hamiltonian $\hat{H} = -\hbar \chi \hat{n}_1 \hat{n}_2$ is applied to two-coherent-state input $|\alpha_1\rangle |\alpha_2\rangle$ for time $t = 2\pi/\hbar \chi$, we can get

$$e^{i2\chi \phi \hat{n}_1 \hat{n}_2 |\alpha_1\rangle |\alpha_2\rangle}$$

$$= \frac{1}{\sqrt{d}} \left( \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kd\rangle \langle kd| \right)$$

$$= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \alpha^k |kd\rangle \langle kd|$$

which is a maximally entangled state of two qudits, that is, we can generate a maximal entanglement of pseudo-phase and pseudo-number states by simply applying cross-Kerr interaction on two coherent beams. The larger the d, the easier the implementation of $\bar{Z}_{ct}$ of qudits is since it can be achieved with smaller $\chi t = 2\pi/d\hbar$. Van Enk's entangled state is the type of pseudo-number/pseudo-number or pseudo-phase/pseudo-phase, i.e.,

$$|\alpha_1\rangle \sum_{l=0}^{d-1} |l\rangle \langle l|$$

Cheong and Lee proposed the use of cross-Kerr interaction for making d-dimensional entangled coherent states [15] as in this paper. Van Enk's proposal cannot be extended beyond two qudit entanglement using only self-Kerr interaction and Cheong and Lee's proposal was not extended beyond two qudit entanglement.

If we apply $\bar{Z}_{ct}$ to all neighboring coherent states as illustrated in Fig. 1, we can get a cluster state of qudits

$$\prod_{(p,q) \in \text{lattice}} \prod_{r \in \text{neighboring}} |\alpha_r\rangle$$

where (p, q) represents neighbors in the lattice. Since all the Controlled-Z's are commuting with each other, the order of the operations is not important.

It used to be believed that two-qubit operations are the most difficult part and single qubit operations are relatively easier in quantum information processing. Now contrary to this conventional wisdom of qubit processing, Controlled-Z of two qudits and preparation of cluster states of optical qudits gets easier as the dimension d gets larger. A generalized $\bar{X}$ operator can be defined as

$$\bar{X} = \sum_{k=0}^{d-1} |(k-1)d\rangle \langle kd|$$

with $|1d\rangle = |d-1\rangle$

which is similar to Pegg–Barnett phase operator [16] and could be called pseudo-phase operator. In pseudo-phase basis it can be written as

$$\bar{X} = \sum_{l=0}^{d-1} \alpha^l |\bar{kd}\rangle \langle \bar{kd}|$$

Fig. 1. Generating a cluster state of coherent-state optical qudits. $\alpha^{l\phi} = e^{i\pi/d} \hat{n}_1 \hat{n}_2 = \bar{Z}_{12}$ and so on.
and \( \hat{Z} \) can be written as
\[
\hat{Z} = \sum_{k=0}^{d-1} |k+1\rangle_d \langle k|_d \quad \text{with} \quad |d\rangle_d = |0\rangle_d
\]
and can be called a pseudo-number operator \[16\]. The two operators are related to each other through generalized Hadamard operation and the followings can be readily shown:
\[
\hat{H}\hat{Z}\hat{H} = \hat{X}, \quad \hat{H}\hat{Z}\hat{H} = \hat{X}^{-1},
\]
and
\[
\hat{H}\hat{Z} = \hat{R},
\]
where \( \hat{R} \) operation reverses the order of the computational basis with 0 to 0, 1 to \( d-1 \), 2 to \( d-2 \), and so on.

### 2.3. Qudit teleportation via homodyne detection

Now as a simple demonstration of cluster state quantum computation of optical qudits, we propose a qudit teleportation via homodyne detection. Let us first consider a one-step teleportation as in Fig. 2. If a qudit state \( |\psi\rangle_1 = \sum_{i,m} \frac{1}{\sqrt{d}} a_{i,m} |i\rangle_1 |m\rangle_2 \) is entangled with a coherent state \( |\alpha\rangle_2 \) by \( \hat{Z}_{12} \) and the first qudit is measured in pseudo-phase basis into \( |\hat{k}\rangle_1 \), then the second qudit becomes \( \hat{H}\hat{Z}^{-1} |\psi\rangle_2 \) as can be seen in the following:

\[
\hat{Z}_{12} |\psi\rangle_1 |\alpha\rangle_2 = \sum_{i,m} a_i a_m^{\dagger} |i\rangle_1 |m\rangle_2 \frac{|m\rangle_2}{\sqrt{d}} = \sum_{i} a_i |i\rangle_1 \langle i|_2
\]

\[
\text{measured into } |\hat{k}\rangle = \sum_{i} a_i \hat{Z}^{-\hat{k}} |i\rangle_2 = \hat{H}\hat{Z}^{-\hat{k}} |\psi\rangle_2.
\]

A qudit teleportation is a repetition of one-step teleportation. Alice has a qudit state \( |\psi\rangle_1 = \sum_{k,l} a_{k,l} |k\rangle_1 |l\rangle_2 \), and Bob has \( |\alpha\rangle_2 |\beta\rangle_3 \). Bob applies \( Z_{23} \) to prepare a maximally entangled state of qudits and sends the second qudit to Alice. Now Alice applies \( Z_{12} \) on qudits 1 and 2 and measures the first and the second qudits in conjugated basis (pseudo-phase states) and gets \( |\hat{k}\rangle_1 |\hat{s}\rangle_2 \) and informs Bob of the values \( k \) and \( s \):

\[
\hat{Z}_{12} \hat{Z}_{23} |\psi\rangle_1 |\alpha\rangle_2 |\beta\rangle_3 = \sum_{i,m,n} a_i a_m a_n \frac{|m\rangle_2}{\sqrt{d}} \frac{|n\rangle_3}{\sqrt{d}} \hat{Z}_{23}^{-\hat{k}} \hat{Z}_{12}^{-\hat{s}} |\psi\rangle_2 |\alpha\rangle_3 |\beta\rangle_2
\]

\[
\text{measured into } |\hat{k}\rangle = \hat{H}\hat{Z}^{-\hat{k}} |\psi\rangle_2 |\alpha\rangle_3 |\beta\rangle_2
\]

Even though Bob can recover \( |\psi\rangle \) by applying \( \hat{X}, \hat{R} \) and \( \hat{Z} \) operations in order as in Fig. 3, just knowing Alice’s measurement results \( k \) and \( s \) might be enough to complete the teleportation without actually applying the operations.

Alice’s projective measurement of qudits in pseudo-phase basis, which is essential part of the above qudit teleportation, can be done by a double-arm homodyne detection. The qudit whose pseudo-phase is to be measured is split by a 50/50 beam splitter an d quadrature \( X_1 \) is measured in one arm and \( X_2 \) in the other by controlling local oscillators for each arm. \( X_1 \) and \( X_2 \) will fix the pseudo-phase of the measured qudit as in Fig. 4. If we entangle an optical qudit with a coherent state by Controlled-\( Z \), we can measure the qudit in pseudo-number basis by measuring the entangled coherent state in pseudo-phase state as in Fig. 5.

Even though new proposals of giant Kerr effects have been made, the present limit of cross-Kerr nonlinearity \( \chi' \) is the order of \( 10^{-4} \), the dimension \( d \) of qudit is the order of \( 10^{5} \), which means the average photon number \( |\alpha|^2 = \langle \alpha |\alpha \rangle \) should be the order of \( 10^{10} \). Even though \( \chi' \) is of order \( 10^{-5} \), much stronger Kerr nonlinearity of ion strings \[17\] might be exploited for qudit cluster state quantum computation. A recent experiment \[18\] demonstrates that large and noiseless cross Kerr nonlinearity is being developed for quantum information processing.

### 3. Summary

We have proposed a simple way of generating a cluster state of optical qudits from coherent states. This cluster state could provide a platform for practical large scale quantum computation. As a simple demonstration of qudit cluster state quantum computation, a qudit teleportation scheme is proposed.

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