Analyzing Specific Cases of an Operator Inequality

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Dedicated to Chandler Davis

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ABSTRACT

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and suppose $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{K})$ are self-adjoint operators with $\operatorname{dist}(\sigma(A), \sigma(B)) \geqslant \delta > 0$. In 1983 Bhatia, Davis, and McIntosh showed that for any $Q \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ we must have $(\pi/2)\|AQ - QB\| \geqslant \delta\|Q\|$. In this paper we specialize their inequality to the case where A, Q, and B are 2×2 or 3×3 matrices, and give sharp estimates. Doing so, we illustrate one way that bounds on the norm of the Schur product of two matrices have applications to perturbation theory. By specializing the Fourier transform used in the proof of the theorem above, we also obtain sharp estimates in two Fourier interpolation problems.

1. INTRODUCTION

In this paper we give sharp estimates in an operator inequality, restricted to the special case when the operators involved are 2×2 or 3×3 matrices over $\mathbb{C}$. The inequality we specialize comes from perturbation theory, and is a consequence of [4, Theorem 4.1].

THEOREM 1 (Bhatia, Davis, and McIntosh). *Assume A and B are bounded self-adjoint operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, such that* $\operatorname{dist}(\sigma(A), \sigma(B)) \geqslant \delta > 0$. *Then for $C \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, the equation $AQ - QB = C$ has a unique solution $Q \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. Furthermore, there*

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exists a universal constant b_{sa} *such that*

$$b_{sa}\|C\| \geqslant \delta\|Q\|. \tag{1}$$

As explained in [4], this inequality is valid for any unitarily invariant norm, and in certain cases when A and B are unbounded. Reserving b_{sa} for the smallest constant which works in this inequality, the authors there also show $b_{sa} < 2$. In this paper we prove two special cases of (1), namely

THEOREM 2. *Assume* A *and* B *are self-adjoint* 2×2 *matrices such that* $\mathrm{dist}(\sigma(A), \sigma(B)) \geqslant \delta > 0$. *Then for any* 2×2 *matrix* C *the equation* $AQ - QB = C$ *has a unique solution* Q, *and we must have*

$$b_2\|C\| \geqslant \delta\|Q\|. \tag{2}$$

Here, $b_2 = \sqrt{6}/2 \approx 1.22474$, *and this estimate is sharp. In the case that* A, Q, *and* B *are* 3×3 *matrices, but otherwise as above, we must have*

$$b_3\|C\| \geqslant \delta\|Q\|, \tag{3}$$

where $b_3 = (8 + 5\sqrt{10})/18 \approx 1.32285$. *Furthermore, this estimate is sharp.*

The purpose of Theorem 1 is to obtain bounds on the perturbation of spectral subspaces of a self-adjoint operator. These bounds follow from the analysis in [7], where the authors discuss the geometry of pairs of subspaces and consider various measures of the distance between two subspaces. They also show how to pass from (1) to a subspace perturbation bound, and they prove this inequality in the special case when $\sigma(A)$ is in some interval of length ρ while $\sigma(B)$ is outside a centered interval of length $\rho + 2\delta$. [That is, $\sigma(A)$ and $\sigma(B)$ don't "interlace."] Theorem 1 above extends that work, applying to the case where $\sigma(A)$ and $\sigma(B)$ interlace as well, and a general subspace perturbation bound immediately follows. The authors in [4] also adapt the inequality (1) to the normal case, obtaining the expected bound on the variation of certain subspaces of a normal operator. Moreover, (1) yields an important bound (independent of the dimension n) on the variation of the spectrum of a normal matrix as well. Specifically, they prove that there is some constant k such that if S and T are normal $n \times n$ matrices then their eigenvalues (including multiplicities) can be matched one to one in such a way that each eigenvalue of T is within $k\|S - T\|$ of its corresponding eigenvalue in $\sigma(S)$. A discussion summarizing Theorem 1, its generalization to the normal case, and its role in perturbation theory is given in [2, Chapter 3]. In addition, the commentary [11] is descriptive, and includes a good bibliography.

Beyond intrinsic interest in the value of the constants b_2 and b_3, there are further motives for studying these special cases. This investigation arose from an interest in a sharp estimate for b_{sa}, which is one of six questions raised in [3, Section 1] concerning sharp estimates in spectral perturbation. As explained there, it was known that $b_{sa} \leqslant \pi/2$. To obtain lower bounds on b_{sa}, let b_n be the smallest number which works in the inequality (1), assuming A, Q, and B are $n \times n$ matrices. Clearly, the b_n increase monotonically. In this paper we'll see that the analysis in two or three dimensions is tractable, and we use these cases to illustrate certain methods in perturbation theory. Furthermore, the value b_3 obtained here is immediately an improved lower bound for b_{sa}. If $n \geqslant 4$ the analysis appears to be intractable, but the ideas developed here are used implicitly in a separate paper (see [16] or [15]) to show $\pi/2 \leqslant \lim_n b_n \leqslant b_{sa}$. Therefore, in view of the opposite inequality, it follows that $\pi/2$ is best possible in the inequality (1).

To proceed, we reformulate the question from one of a sharp estimate in an operator inequality into one concerning bounds on the Schur multiplier norm of a matrix. This adapts the strategy used in [4] to the case for finite dimensions, and allows us to use a growing body of results on the norm of a Schur multiplier. The authors in [4], given A and B meeting their hypotheses, construct the map $\mathscr{F} = \mathscr{F}_{A,B}$ as a Fourier transform sending $C \mapsto Q$. They then use standard analytic arguments to show $\|\mathscr{F}\| < 2/\delta$, and this verifies the inequality (1) for some $b_{sa} < 2$. A classical result due to Sz.-Nagy (see [21] or [22]) can be applied to show $\|\mathscr{F}\| \leqslant \pi/(2\delta)$, leading to the bound $b_{sa} \leqslant \pi/2$. In finite dimensions we represent the map $\mathscr{F}$ as Schur multiplication by a matrix $T = T_{A,B}$. [The Schur, or Hadamard, product of two $m \times n$ matrices $T = (t_{jk})$ and $X = (x_{jk})$ is $T \circ X = (t_{jk} x_{jk})$.] Bounds for b_2 and b_3 then follow from bounds on the Schur multiplier norm of T, given by $\|T\|_S = \max_{X \neq 0} \|T \circ X\|/\|X\|$. In the case with $n = 2$, basic results on the norm of a Schur multiplier lead to both an explicit candidate for b_2 and a proof that this value is correct. When $n = 3$, however, an explicit value for b_3 is not accessible in this way. Imitating the case with $n = 2$ leads to an impenetrable system of equations, and an alternate procedure is required. To obtain b_3 itself we first solve a minimization problem in Fourier analysis, specializing the transform argument used in [4] to three dimensions. This leads to what is subsequently seen to be the correct value, and with this in hand we proceed as in the 2×2 case. Thus, these cases illustrate an important connection between Schur multiplication and perturbation theory. In fact, representing $\mathscr{F}$ as a Schur multiplier leads naturally to the examples referred to above, proving $b_n \to \pi/2$ as $n \to \infty$. The 2×2 and 3×3 cases also show that the Fourier transform argument yields exact information in more cases than was previously recognized.

The outline of the remainder of this paper is as follows. In Section 2 we discuss certain preliminary ideas, giving definitions and stating a theorem we'll use concerning the Schur product of two matrices. In Section 3 we analyze the 2×2 case. In Section 4 we begin the case for $n = 3$, discussing certain special cases and introducing a method we use subsequently. In Section 5 we treat the last 3×3 case, the one of primary interest, and in Section 6 we extend the inequalities (2) and (3) to the case with an arbitrary unitarily invariant norm. In the last section we make some concluding remarks concerning the connection between Fourier analysis and matrix theory.

2. DEFINITIONS AND A PRELIMINARY RESULT

If $\mathbf{x} \in \mathbb{C}^n$, we let $\|\mathbf{x}\|$ be the usual Euclidean norm. We recall that (1) holds provided the norm $\|\cdot\|$ (on the operators in question) is unitarily invariant. (A norm is unitarily invariant, abbreviated ui, if $\|UXV\| = \|X\|$ whenever U and V are unitary operators.) The inequalities (2) and (3) also hold for any ui norm, but the extension from the usual norm to the general case is deferred to Section 6. Thus we begin by restricting $\|\cdot\|$ to the usual norm: $\|A\| = \max_{\mathbf{x} \neq \mathbf{0}} \|A\mathbf{x}\|/\|\mathbf{x}\|$.

For an $m \times n$ matrix A, let $q = \min\{m, n\}$ and define $s_i(A)$, $1 \leqslant i \leqslant q$, to be the ith singular value of A. That is, let $s_i(A) = \sqrt{\lambda_i}$, where λ_i is the ith largest eigenvalue of A^*A (including multiplicities). As noted, $\|A\|$ is the usual norm of A, and this is equivalent to defining $\|A\| = s_1(A)$. We also let $c_i(A)$ be the ith largest column norm of A, where the columns are regarded as vectors in $\mathbb{C}^n$ with the Euclidean norm. It will be convenient to write $c(A)$ for $c_1(A)$.

The following is [1, Theorem 1].

THEOREM (Ando, Horn, and Johnson). *Let T and Q be $m \times n$ matrices. Then*

$$\sum_{i=1}^{k} s_i(T \circ Q) \leqslant \sum_{i=1}^{k} c_i(X) c_i(Y) s_i(Q), \qquad k = 1, \ldots, q, \tag{4}$$

for any $r \times m$ matrix X and $r \times n$ matrix Y such that $X^*Y = T$.

In the special case $k = 1$, (4) reduces to a classical result first appearing in Schur's pioneering paper [19]. (And in this case we will refer to it as Schur's inequality.) We will use this in an important way, but we only need inequality here to get an upper bound of the form $\|T\|_S \leqslant c(X)c(Y)$; lower bounds on $\|T\|_S$ are obtained from examples. We point out, however, that a stronger result is available. It is proved by Haagerup in [10], and is shown in [17, Section 7.7] and [13], that there are X and Y for which equality holds, so that $\|T\|_S = \min_{X^*Y=T} c(X)c(Y)$. The inequality (4) is cited here, however, because we will use it with $k > 1$ later on, when we show that a bound obtained in the usual norm extends immediately to a bound for any ui norm.

The inequality above is all we need to derive a complete solution in the 2×2 case, but it is not sufficient for the 3×3 case. The missing ingredient can be obtained by reexamining the proof of Theorem 1 above, and we will discuss this in more detail in Section 4.

3. THE CASE WITH $n = 2$

Let A, Q, and B be 2×2 matrices, with A and B self-adjoint and $\operatorname{dist}(\sigma(A), \sigma(B)) \geqslant \delta > 0$. Scaling the problem if necessary, we assume $\delta = 1$. Note that A and B can be written as U^*D_AU and V^*D_BV, respectively, where D_A and D_B are diagonal and U and V are unitary. Then $\|AQ - QB\| = \|U(AQ - QB)V^*\| = \|D_A(UQV^*) - (UQV^*)D_B\|$, and so we may replace Q with UQV^* if necessary to assume A and B are diagonal. So let $A = \operatorname{diag}(\lambda_1, \lambda_2)$ and $B = \operatorname{diag}(\mu_1, \mu_2)$ with $|\lambda_j - \mu_k| \geqslant 1$ for all j and k. For any $Q = (q_{jk})$ we have

$$AQ - QB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_1 - \mu_1)q_{11} & (\lambda_1 - \mu_2)q_{12} \\ (\lambda_2 - \mu_1)q_{21} & (\lambda_2 - \mu_2)q_{22} \end{bmatrix}.$$

Thus, for $T = \big((\lambda_j - \mu_k)\big)_{j,k}$ we have $AQ - QB = T \circ Q$. Define $\mathscr{T} = \mathscr{T}_{A,B}$ to be the linear map on the set of 2×2 matrices over $\mathbb{C}$ given by $\mathscr{T}Q = AQ - QB = T \circ Q$. We know from Theorem 1 that $\mathscr{T}$ is invertible. This also follows from [18, Corollary 3.3], which refers to the equation $AQ - QB = C$ in a more general setting. Furthermore, we can simply observe that if $\mathscr{T}$ is a linear map induced by Schur multiplication by some matrix $T = (t_{jk})$, then the eigenvalues of $\mathscr{T}$ are clearly the entries t_{jk}. In our case, then, we have $\sigma(\mathscr{T}) = \{\lambda_j - \mu_k\}_{j,k} = \sigma(A) - \sigma(B)$. In particular $0 \notin \sigma(\mathscr{T})$, and so $\mathscr{T}$ is invertible. Furthermore it is immediate that $\mathscr{T}^{-1}$ is the map $Y \mapsto T' \circ Y$, where T' is the matrix $\big((\lambda_j - \mu_k)^{-1}\big)_{j,k}$.

Bounding $\|Q\|$ relative to $\|C\|$ is the same as bounding $\|\mathscr{T}^{-1}\|$, which in turn is the same as bounding $\|T'\|_S$. That is, exchanging T with T' for convenience, we have

$$b_2 = \max_{A,\,B}\left\{\max_{\|X\|=1}\|T \circ X\|\right\} = \max_{A,\,B}\|T\|_S .$$

Here $\max_{A,\,B}$ is taken over pairs A and B with $\operatorname{dist}(\sigma(A),\sigma(B)) \geqslant 1$, and if A and B are fixed, T is the matrix $\left((\lambda_j - \mu_k)^{-1}\right)_{j,\,k}$. In this formulation we see how A and B are "variables" in this problem. First, the specific values of λ_j and μ_k are not important, only the differences $\lambda_j - \mu_k$. This is consistent with the observation that for any scalar ξ we have $(A+\xi)Q - Q(B+\xi) = AQ - QB$. Furthermore, to stipulate the differences $\lambda_j - \mu_k$ we will first arrange the four values λ_1, λ_2, μ_1, and μ_2 in increasing order (including multiplicities) and then specify the gaps between successive values. To this end we introduce a *star-dot* diagram (see below) for the configuration of $\sigma(A)$ and $\sigma(B)$ as follows. Let a circle represent an element of $\sigma(A)$, and an asterisk one of $\sigma(B)$. Then denote by $\epsilon_i \geqslant 0$ a gap between two identical symbols, and by δ_i a gap between different symbols. Note that $\epsilon_i = 0$ for some i signifies a repeated eigenvalue, and we are assuming $\delta_i \geqslant 1$ for all i. We refer to consecutive symbols of the same type as a block of spectrum, and we classify the arrangements of $\sigma(A)$ and $\sigma(B)$ by the number of blocks they contain. Given this, and exchanging A with B if necessary, the possible configurations for $\sigma(A)$ and $\sigma(B)$ are those with two blocks:

$$* \overset{\epsilon_1}{\text{——}} * \overset{\delta}{\text{————}} \circ \overset{\epsilon_2}{\text{——}} \circ$$

three blocks:

$$* \overset{\delta_1}{\text{————}} \circ \overset{\epsilon}{\text{——}} \circ \overset{\delta_2}{\text{————}} *$$

or four blocks:

$$* \overset{\delta_1}{\text{————}} \circ \overset{\delta_2}{\text{————}} * \overset{\delta_3}{\text{————}} \circ$$

In the language of [4, Section 3], the first two arrangements have "favorable geometry," meaning $\sigma(A)$ and $\sigma(B)$ can be separated by an annulus of band width δ. [The authors there are assuming A and B are normal. In our language, specialized to the self-adjoint case, $\sigma(A)$ and $\sigma(B)$ don't interlace.]

Then by [4, Theorem 3.1] we must have $\|C\| \geqslant \|Q\|$, and obviously $b_2\|C\| \geqslant \|Q\|$ as well. However, examples in [7, p. 24] and [4, p. 57] show that $\|C\| \geqslant \|Q\|$ is not generally true, and so it's essential in our analysis that $\sigma(A)$ and $\sigma(B)$ interlace. Thus we restrict our attention to the case above with four blocks. Here and in the 3×3 case it will be convenient to index the $\lambda_j \in \sigma(A)$ in decreasing order and the $\mu_k \in \sigma(B)$ in increasing order. Doing so, we obtain

$$T = \begin{bmatrix} 1/(\delta_1 + \delta_2 + \delta_3) & 1/\delta_3 \\ 1/\delta_1 & -1/\delta_2 \end{bmatrix}. \tag{5}$$

Consider first the special case where $\delta_i = 1$ for all i. We will refer to this as the equally spaced case, and this leads to

$$T = \begin{bmatrix} \frac{1}{3} & 1 \\ 1 & -1 \end{bmatrix}. \tag{6}$$

Define X and Y by

$$X = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 2\sqrt{2}/3 \end{bmatrix}, \qquad Y = \begin{bmatrix} \frac{1}{3} & 1 \\ 5/(3\sqrt{2}) & -1/\sqrt{2} \end{bmatrix}.$$

Then $X^*Y = T$, and $c(X) = 1$ while $c(Y) = \sqrt{6}/2$. Therefore $\|T\|_S \leqslant \sqrt{6}/2$. For the nonequally spaced case, let T be as in (5), take X as above, and define $Y = Y(\delta_1, \delta_2, \delta_3)$ to be $(X^*)^{-1}T$. As defined, then,

$$Y = \frac{1}{2\sqrt{2}} \begin{bmatrix} \dfrac{2\sqrt{2}}{\delta_1 + \delta_2 + \delta_3} & \dfrac{2\sqrt{2}}{\delta_3} \\ \dfrac{1}{\delta_1 + \delta_2 + \delta_3} + \dfrac{3}{\delta_1} & \dfrac{1}{\delta_3} - \dfrac{3}{\delta_2} \end{bmatrix}.$$

We know $T = X^*Y$, and $c(X) = 1$. To bound $c(Y)$, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be its columns. Routine calculations show that both $\|\mathbf{y}_1\|$ and $\|\mathbf{y}_2\|$ are maximal when $\delta_1 = \delta_2 = \delta_3 = 1$, and the maximum value both times is $\sqrt{6}/2$. Thus $c(Y) \leqslant \sqrt{6}/2$ in the general interlacing case, and it follows that $\|T\|_S \leqslant b_2$ no matter how $\sigma(A)$ and $\sigma(B)$ are configured. That is, we must have $b_2\|C\| \geqslant \|Q\|$ as claimed, and the inequality (2) is verified.

To see that b_2 is best possible, we give examples which are essentially [4, Example 4.3]. Let $A = \operatorname{diag}(3, 1)$, $B = \operatorname{diag}(0, 2)$, and set

$$Q = \begin{bmatrix} \sqrt{3}/3 & \sqrt{5} \\ \sqrt{5} & \sqrt{3} \end{bmatrix}.$$

Then one may check that $\|C\| = \|AQ - QB\| = 2\sqrt{2}$ while $\|Q\| = 2\sqrt{3}$. Thus $b_2\|C\| = \|Q\|$ for these A, Q, and B, and the bound in (2) is sharp.

Before proceeding to the 3×3 case, we wish to record certain observations. First, with T as in (6), there are several ways to determine $\|T\|_S$. Not all procedures extend easily to (5), however, and still fewer to the 3×3 case. In general, if M is an arbitrary matrix it is difficult to compute $\|M\|_S$ explicitly. Given T as in (6), however, and one natural assumption, we can use Schur's inequality to produce X and Y such that $c(X)c(Y) = \|T\|_S$.

Note that $X^*Y = X^*U^*UY$ for any unitary U, and so we may assume X is upper triangular. We now add the assumption that if X and Y are extremal, in the sense that $c(X)c(Y) = \|T\|_S$, then $c_1(X) = c_2(X)$ and $c_1(Y) = c_2(Y)$. (We will comment on this assumption below.) Given this, we may assume

$$X = \begin{bmatrix} 1 & c \\ 0 & s \end{bmatrix}, \qquad Y = \begin{bmatrix} \frac{1}{3} & 1 \\ \pm\sqrt{l^2 - \frac{1}{9}} & \pm\sqrt{l^2 - 1} \end{bmatrix},$$

where $c^2 + s^2 = 1$ and $l = \|T\|_S$. Setting $X^*Y = T$ and choosing the signs on the radicals by trial and error, one obtains the appropriate system

$$\begin{aligned} \frac{c}{3} + s\sqrt{l^2 - \tfrac{1}{9}} &= 1, \\ c - s\sqrt{l^2 - 1} &= -1, \\ c^2 + s^2 &= 1. \end{aligned} \tag{7}$$

This system has solution $c = -\frac{1}{3}$, $s = 2\sqrt{2}/3$, and $l = \sqrt{6}/2$, and for this reason we say that Schur's inequality is all we need to give a sharp estimate for b_2. In general, however, we cannot assume both that X and Y are square and that they have columns of equal length. For example, if

$$T = \begin{bmatrix} 1 & \epsilon_1 \\ \epsilon_2 & \epsilon_3 \end{bmatrix}$$

then $\|T\|_S = 1$ provided the ϵ_i are sufficiently small. But when $\|T\|_S$ is the magnitude of its largest entry, in this case 1, it can be true that either X and Y have columns of unequal length, or they must be 3×2 matrices.

It is also interesting to note that if $M = (m_{jk})$ is a 2×2 matrix with real entries, there is a formula for $\|M\|_S$ which is a function of the m_{jk} alone. This is given in [5], and shows that $\|T\|_S = \sqrt{6}/2$ in the equally spaced case. Thus, we could have chosen this formula as our starting point. It would be difficult to use it when T is as in (5), however, and it doesn't extend to the 3×3 case at all. (The Schur multiplier norm of a 3×3 matrix can depend on the arrangement of its entries as well as their values.) The authors in [5] also consider the question of when $\|M\|_S$ is the magnitude of its largest entry, and given an answer in some special cases.

Finally, we note which configurations of $\sigma(A)$ and $\sigma(B)$ permit extremal A, Q, and B, so that $b_2\|C\| = \delta\|Q\|$. Obviously we need only consider the fully interlacing case. If $\delta_3 > 1$, then $\|\mathbf{y}_1\| < \sqrt{6}/2$ by inspection, and with a little extra work one can determine that $\|\mathbf{y}_2\|$ is, too. Thus, $c(Y) < \sqrt{6}/2$, and it follows that for equality to hold in (2) we must have $\delta_3 = 1$. By symmetry we must also have $\delta_1 = 1$. (Exchanging A with B and multiplying both by -1 is the same as exchanging δ_3 with δ_1.) Given this, with a little more work one can similarly show that we must have $\delta_2 = 1$ as well. Thus, equality in (2) can occur only if the eigenvalues of A and B are fully interlacing and equally spaced.

4. PRELIMINARIES TO THE CASE WITH $n = 3$

The analysis with $n = 3$ is more complicated than that for $n = 2$. For instance, we know that $\sigma(A)$ and $\sigma(B)$ must interlace, but now there are several ways this can happen. This forces us to examine five distinct spectral configurations, instead of just one as before. Also, if one attempts to produce a candidate for b_3 in the same way we did before, one obtains a system of equations much more complicated than (7). In this section we address both of these issues in order to facilitate our subsequent work. To begin we examine the possible configurations of $\sigma(A)$ and $\sigma(B)$, and indicate that $1.25\|C\| \geqslant \delta\|Q\|$ in all but one of the resulting cases. Of course, we then have $b_3\|C\| \geqslant \delta\|Q\|$ as well, and this leaves only the fully interlacing case to be treated in the next section. As mentioned, however, we need more than Schur's inequality to derive b_3, and for this reason we also adapt the proof of the Bhatia-Davis-McIntosh inequality (1) to the finite dimensional case here. We will specialize this proof even further in the next section, and doing so we are led to a minimization problem from Fourier analysis. Solving this yields the (conjectured) value $b_3 = (8 + 5\sqrt{10})/18$, and given this we can then proceed as in the 2×2 case to verify the inequality (3) for every configuration of $\sigma(A)$ and $\sigma(B)$.

As before, we classify the arrangements of $\sigma(A)$ and $\sigma(B)$ by the number of blocks they contain. Cases with favorable geometry, where $\sigma(A)$ and $\sigma(B)$ can be separated by an annulus of band width δ, include case 0 (two blocks):

$$* \overset{\epsilon_1}{\overline{\quad}} * \overset{\epsilon_2}{\overline{\quad}} * \overset{\delta}{\overline{\qquad\qquad}} \circ \overset{\epsilon_3}{\overline{\quad}} \circ \overset{\epsilon_4}{\overline{\quad}} \circ$$

and case 00 (three blocks):

$$* \overset{\epsilon_1}{\overline{\quad}} * \overset{\delta_1}{\overline{\qquad\qquad}} \circ \overset{\epsilon_2}{\overline{\quad}} \circ \overset{\epsilon_3}{\overline{\quad}} \circ \overset{\delta_2}{\overline{\qquad\qquad}} *$$

Any arrangement with two or three blocks can be reduced to one of these by exchanging A with B, or multiplying A and B by -1, if necessary. Furthermore, as indicated before, we must have $\|C\| \geqslant \delta\|Q\|$ in these cases.

Configurations with four or five blocks recall the 2×2 case, and these are case 1 (four blocks, 1-1-2-2):

$$* \overset{\delta_1}{\overline{\qquad\qquad}} \circ \overset{\delta_2}{\overline{\qquad\qquad}} * \overset{\epsilon_1}{\overline{\quad}} * \overset{\delta_3}{\overline{\qquad\qquad}} \circ \overset{\epsilon_2}{\overline{\quad}} \circ$$

case 2 (four blocks, 1-2-2-1):

$$* \overset{\delta_1}{\overline{\qquad\qquad}} \circ \overset{\epsilon_1}{\overline{\quad}} \circ \overset{\delta_2}{\overline{\qquad\qquad}} * \overset{\epsilon_2}{\overline{\quad}} * \overset{\delta_3}{\overline{\qquad\qquad}} \circ$$

case 3 (four blocks, 2-1-1-2):

$$* \overset{\epsilon_1}{\overline{\quad}} * \overset{\delta_1}{\overline{\qquad\qquad}} \circ \overset{\delta_2}{\overline{\qquad\qquad}} * \overset{\delta_3}{\overline{\qquad\qquad}} \circ \overset{\epsilon_2}{\overline{\quad}} \circ$$

and case 4 (five blocks):

$$* \overset{\delta_1}{\overline{\qquad\qquad}} \circ \overset{\epsilon}{\overline{\quad}} \circ \overset{\delta_2}{\overline{\qquad\qquad}} * \overset{\delta_3}{\overline{\qquad\qquad}} \circ \overset{\delta_4}{\overline{\qquad\qquad}} *$$

Just as before, note that any configuration with four or five blocks is reducible to one of these arrangements. Cases 1–4 are treated in detail in [15, Section IV.2], and in each instance it is shown that $1.25\|C\| \geqslant \delta\|Q\|$. We remark that the estimates there are clearly not best possible, and it seems likely that $b_2 \approx 1.22474$ would work if simpler arguments could be found.

We will not actually prove $1.25\|C\| \geqslant \delta\|Q\|$ in any of the cases here, but to illustrate the arguments used in [15] we will look at case 1 in some detail. To begin, take $\delta_i \geqslant 1$ for each i. With $\sigma(A)$ and $\sigma(B)$ arranged as indicated, it is convenient to index the $\lambda_j \in \sigma(A)$ in decreasing order and the $\mu_k \in \sigma(B)$ in increasing order. Doing this, and with $T = \left((\lambda_j - \mu_k)^{-1}\right)_{j,k}$ as usual, we obtain

$$T = \begin{bmatrix} 1/(\delta_{123}+\epsilon_{12}) & 1/(\delta_3+\epsilon_{12}) & 1/(\delta_3+\epsilon_2) \\ 1/(\delta_{123}+\epsilon_1) & 1/(\delta_3+\epsilon_1) & 1/\delta_3 \\ 1/\delta_1 & -1/\delta_2 & -1/(\delta_2+\epsilon_1) \end{bmatrix}.$$

Here we let $\delta_{123} = \delta_1 + \delta_2 + \delta_3$, $\epsilon_{12} = \epsilon_1 + \epsilon_2$, etc.

Note that the 2×2 southwest corner of T, call it T_1, is obtained from the 2×2 case by replacing δ_3 in (5) with $\delta_3 + \epsilon_1$. Furthermore, if ϵ_1 or ϵ_2 is zero, then T has a repeated column or row. But if M is any matrix and N is obtained by repeating a row or column of M, then clearly $\|N\|_S = \|M\|_S$.

Suppose first $\epsilon_2 = 0$, so that the top two rows of T are identical. From Section 3, we know that for

$$X = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 2\sqrt{2}/3 \end{bmatrix}$$

there is some Y_1 such that $X^*Y_1 = T_1$ and $c(Y_1) \leqslant \sqrt{6}/2$. Similarly, if T_2 is the 2×2 submatrix formed by deleting the top row and middle column of T, then T_2 can be obtained from (5) by replacing δ_2 with $\delta_2 + \epsilon_1$. Thus there is some Y_2 such that $X^*Y_2 = T_2$ and $c(Y_2) \leqslant \sqrt{6}/2$. Now let the columns of Y_1 be $\mathbf{y}_1$ and $\mathbf{y}_2$, and the columns of Y_2 be $\mathbf{y}_1$ and $\mathbf{y}_3$. Then for

$$X = \begin{bmatrix} 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 2\sqrt{2}/3 \end{bmatrix} \quad \text{and} \quad Y = [\mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{y}_3],$$

it follows that $X^*Y = T$ while $c(X)c(Y) \leqslant \sqrt{6}/2$. Then with T as above, if $\epsilon_2 = 0$ we must have $\|T\|_S \leqslant \sqrt{6}/2 < 1.25$.

Suppose now $\epsilon_2 \neq 0$. For simplicity let $\epsilon = \epsilon_2$ and set

$$X = \begin{bmatrix} \dfrac{\sqrt{\epsilon^2 + 2\epsilon}}{1+\epsilon} & 0 & 0 \\ \dfrac{1}{1+\epsilon} & 1 & -\dfrac{1}{3} \\ 0 & 0 & \dfrac{2\sqrt{2}}{3} \end{bmatrix}.$$

Then X^* is invertible, so define $Y = Y(\delta_1, \delta_2, \delta_3, \epsilon_1, \epsilon_2)$ to be $(X^*)^{-1}T$. Clearly $c(X) = 1$, so we want to show $c(Y) \leqslant \frac{5}{4}$. If we let $Y = (y_{jk})$, $T = (t_{jk})$ and write $X^*Y = T$ in matrix form, one can note the following. First, in light of our arguments above, we must have $y_{2k}^2 + y_{3k}^2 \leqslant \frac{3}{2}$ for $k = 1$, 2, and 3. Also, $y_{2k} = t_{2k}$ for $k = 1$, 2, and 3, and in particular $y_{21} = 1/\delta$ if we let $\delta = \delta_1 + \delta_2 + \delta_3 + \epsilon$ for convenience. To bound $|y_{11}|$, and hence the norm of the first column, we observe that

$$\frac{\sqrt{\epsilon^2 + 2\epsilon}}{1+\epsilon} y_{11} + \frac{1}{1+\epsilon}\frac{1}{\delta} = \frac{1}{\delta + \epsilon}.$$

Solving for y_{11}, we obtain

$$y_{11} = \frac{\delta - 1}{\delta(\delta+\epsilon)}\sqrt{\frac{\epsilon}{\epsilon + 2}} < \frac{\delta - 1}{\delta(\delta + \epsilon)} \leqslant \frac{\delta - 1}{\delta^2}.$$

Since $\delta \geqslant 1$, y_{11} is nonnegative. (In fact $\delta \geqslant 3$ here, but in considering $|y_{12}|$ and $|y_{13}|$, only the weaker bound applies.) Furthermore, $(\delta - 1)/\delta^2$ is maximal on $[1, \infty)$ when $\delta = 2$, and this leads to the bound $|y_{11}| \leqslant \frac{1}{4}$. Thus, the norm of the first column of Y cannot exceed $(\frac{1}{16} + \frac{3}{2})^{1/2} = \frac{5}{4}$, as desired. Similarly, $|y_{12}|, |y_{13}| \leqslant \frac{1}{4}$, and it follows that $c(Y) \leqslant \frac{5}{4}$. This implies $1.25\|Q\| \geqslant \delta\|C\|$ in case 1.

This bound similarly holds for cases 2–4, and so we are left with the fully interlacing case, where the configuration of $\sigma(A)$ and $\sigma(B)$ contains six blocks. This is discussed in the next section, and our work there will show that this is the "true" 3×3 case. Before proceeding, however, we outline a proof of Theorem 1.

Although the Bhatia-Davis-McIntosh inequality applies to the infinite dimensional case, we specialize it to finite dimensions in order to highlight the connection between Fourier analysis and the Schur product of two matrices. As when $n = 2$, we may assume A and B are diagonal, and we continue to take $\delta = 1$. As before, then, the map $Q \mapsto AQ - QB$ is the same as Schur multiplication by the matrix $(\lambda_j - \mu_k)_{j,k}$, and $\mathscr{T}^{-1}$ is Schur multiplication by $\left((\lambda_j - \mu_k)^{-1}\right)_{j,k}$.

Proof of Theorem 1. Let $A = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $B = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n)$. For any $X = (x_{jk})$ set $\mathscr{F}(X) = \int_{-\infty}^{\infty} e^{-itA} X e^{itB} f(t)\, dt$, where $f \in L_1(\mathbb{R})$ is a function to be specified below. Then e^{-itA} and e^{itB} are diagonal unitaries, and $e^{-itA}Xe^{itB}$ is the Schur product of the matrices X and E, where $E = \left(e_{jk}(t)\right)_{j,k}$ and $e_{jk}(t) = \exp[-i(\lambda_j - \mu_k)t]$. Then

$$\mathscr{F}(X) = \int_{-\infty}^{\infty} \left(x_{jk} e_{jk}(t)\right)_{j,k} f(t)\, dt = \left(\int_{-\infty}^{\infty} x_{jk} e_{jk}(t) f(t)\, dt\right)_{j,k}$$

$$= \left(x_{jk} \int_{-\infty}^{\infty} \exp\left[-i(\lambda_j - \mu_k)t\right] f(t)\, dt\right)_{j,k} = \left(x_{jk} \hat{f}(\lambda_j - \mu_k)\right)_{j,k}.$$

Since $|\lambda_j - \mu_k| \geqslant 1$, we choose f so that $\hat{f}(s) = 1/s$ whenever $|s| \geqslant 1$. This gives $\mathscr{F}(X) = T \circ X$, where $T = \left((\lambda_j - \mu_k)^{-1}\right)_{j,k}$ as before, which is to say $\mathscr{F}$ is the map $AQ - QB \mapsto Q$. Clearly, then, we seek bounds on $\|\mathscr{F}\|$. We have

$$\|\mathscr{F}(X)\| = \left\| \int_{-\infty}^{\infty} e^{-itA} X e^{itB} f(t)\, dt \right\| \leqslant \int_{-\infty}^{\infty} \|e^{-itA} X e^{itB}\| \, |f(t)|\, dt$$

$$= \|X\| \int_{-\infty}^{\infty} |f(t)|\, dt = \|X\|\, \|f\|_1. \tag{8}$$

Thus $\|\mathscr{F}\| \leqslant \|f\|_1$. In [4] the authors continue by producing a function $f \in L_1(\mathbb{R})$ such that $\hat{f}(s) = 1/s$ whenever $|s| \geqslant 1$. Furthermore, $\|f\|_1 < 2$ for this function, and since the general case is not essentially different from the finite dimensional case, this implies $b_{\mathrm{sa}} < 2$. ■

One looks to sharpen this estimate by finding other L_1 functions f with smaller norm whose transform $\hat{f}$ takes on the appropriate values. To this end, define

$$\mathscr{S} = \left\{ f \in L_1(\mathbb{R}) : \hat{f}(s) = 1/s \text{ whenever } |s| \geqslant 1 \right\}.$$

Then for $m = \inf_{f \in \mathscr{S}} \|f\|_1$, we have $b_{sa} \leqslant m$. The example cited above proves $m < 2$, and Sz.-Nagy later noted (from earlier work; see [21] or [22]) that $m = \pi/2$, and this bound is achieved for certain $f \in \mathscr{S}$. This implies $b_{sa} \leqslant \pi/2$. In the 3×3 case ahead, we require $\hat{f}(s) = 1/s$ on only a few data points of the form $\lambda_j - \mu_k$. Thus $\mathscr{S}$ becomes a larger class, and permits a lower min for $\|f\|_1$.

5. THE CASE FOR $n = 3$

As indicated in the last section, we need only consider the situation when $\sigma(A)$ and $\sigma(B)$ are fully interlacing. This is when their configuration has six blocks, and given this, we may assume they have star-dot diagram

$$* \overset{\delta_1}{\text{———}} \circ \overset{\delta_2}{\text{———}} * \overset{\delta_3}{\text{———}} \circ \overset{\delta_4}{\text{———}} * \overset{\delta_5}{\text{———}} \circ$$

With $\sigma(A)$ denumerated in decreasing order and $\sigma(B)$ in increasing order, the matrix $T = \left((\lambda_j - \mu_k)^{-1} \right)$ is given by

$$T = \begin{bmatrix} 1/\delta_{12345} & 1/\delta_{345} & 1/\delta_5 \\ 1/\delta_{123} & 1/\delta_3 & -1/\delta_4 \\ 1/\delta_1 & -1/\delta_2 & -1/\delta_{234} \end{bmatrix}. \tag{9}$$

As before, we let $\delta_{12345} = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$, and so on. Then we must show that $\|T\|_S \leqslant (8 + 5\sqrt{10})/18$ provided only that $\delta_i \geqslant 1$ for all i, and that equality holds in some case. The arguments we give here follow the outline of the 2×2 case, but are more involved.

To begin, we make the natural assumption that $\|T\|_S$ is maximal in the equally spaced case. This is when $\delta_1 = \delta_2 = \cdots = \delta_5 = 1$, and in this case

$$T = \begin{bmatrix} \frac{1}{5} & \frac{1}{3} & 1 \\ \frac{1}{3} & 1 & -1 \\ 1 & -1 & -\frac{1}{3} \end{bmatrix}. \tag{10}$$

We then solve three computational problems in order. First, what is the exact value of $\|T\|_S$ when T is as in (10)? Numerical evidence shows that there is some U such that $\|U\| = 1$ while $\|T \circ U\| = 1.322\,854\,906$, and this is best possible. Thus $\|T\|_S$ seems to have this value, and we seek an explicit expression for it. Second, given this number [which turns out to be $b_3 = (8 + 5\sqrt{10})/18$], we seek X and Y such that $X^*Y = T$ while $c(X)c(Y) = b_3$. Given these, we can then keep X fixed and let Y vary to show $\|T\|_S \leqslant b_3$ in the general case, when T is as in (9). Finally, we seek some matrix U on which T achieves its Schur multiplier norm, to prove $\|T\|_S \geqslant b_3$ as well.

Suppose now T is as in (10). The differences $\lambda_j - \mu_k$ are $\{\pm 1, \pm 3, 5\}$, and so we define

$$\mathscr{S} = \left\{f \in L_1(\mathbb{R}) : \hat{f}(s) = 1/s \text{ for } s = \pm 1, \pm 3, 5\right\}.$$

It is clear from our earlier discussion that for any $f \in \mathscr{S}$, $\|f\|_1 \geqslant \|T\|_S$. That is, if $m = \min_{f \in \mathscr{S}} \|f\|_1$ then $m \geqslant \|T\|_S$. In fact, the integral transform $\mathscr{F}$: $AQ - QB \mapsto Q$ described in the last section works with distributions such as Dirac δ-functions as well, so we broaden our definition of $\mathscr{S}$ to include these. It turns out that equality holds here, so that $m = \|T\|_S$, and this fact is useful in calculating $\|T\|_S$ explicitly. Note that the only inequality among the relations (8) is

$$\left\| \int_{-\infty}^{\infty} e^{-itA} X e^{itB} f(t)\, dt \right\| \leqslant \int_{-\infty}^{\infty} \|e^{-itA} X e^{itB}\| \, |f(t)| \, dt.$$

Then $m = \|T\|_S$ if and only if some f and X are found for which equality holds above. Such an f exists, and producing it (via guesswork) leads to an appropriate matrix X as well. The inequality above is an integral version of the triangle inequality, and to increase the likelihood that equality holds we assume f is a linear combination of a few δ-functions. We must have at least three in our combination, because two is not enough. That is, there are no α_1, α_2, θ_1, and θ_2 such that $f(t) = \alpha_1 d(t - \theta_1) + \alpha_2 d(t - \theta_2)$ while $\hat{f}(s) = 1/s$ for $s = \pm 1$, ± 3, and 5. [Here, let $d(t)$ be a unit point mass measure at 0.] So assume $f(t) = \sum_{i=1}^{3} \alpha_i d(t - \theta_i)$ while $\hat{f}$ takes on the stipulated values at the indicated points. Minimizing $|\alpha_1| + |\alpha_2| + |\alpha_3|$ as we vary θ_1, θ_2, and θ_3, we are led to

$$f(t) = -\frac{5\sqrt{10}\,i}{36} d(t + \theta) + \frac{5\sqrt{10}\,i}{36} d(t - \theta) + \frac{4i}{9} d\left(t - \frac{\pi}{2}\right), \quad (11)$$

where $\theta = \arcsin\sqrt{\frac{2}{5}}$. Then $\hat{f}(s) = 1/s$ for $s = \pm 1, \pm 3$, and ± 5, while $\|f\|_1 = b_3 = (8 + 5\sqrt{10})/18$. This implies $b_3 \geqslant \|T\|_S$. We will give an example presently showing $b_3 \leqslant \|T\|_S$ as well, and so equality holds here. Before citing our example, however, we continue by extending our upper bound on $\|T\|_S$ to the general case, where T is as in (9).

With our value b_3 in hand, we seek X and Y to work with as in the 2×2 case. That is, we want to factor T as X^*Y so that $c(X)c(Y) = (8 + 5\sqrt{10})/18$. We assume each column of X has length one and each column of Y has length $(8 + 5\sqrt{10})/18$. (This assumption is justified after the fact.) As indicated in the 2×2 case, we may also assume X is upper triangular. Thus we set

$$X = \begin{bmatrix} 1 & c_1 & c_2 c_3 \\ 0 & s_1 & c_2 s_3 \\ 0 & 0 & s_2 \end{bmatrix},$$

$$Y = \begin{bmatrix} \frac{1}{5} & \alpha_1 & \sqrt{l^2 - \alpha_1^2 - \frac{1}{25}} \\ \frac{1}{3} & \alpha_2 & -\sqrt{l^2 - \alpha_2^2 - \frac{1}{9}} \\ 1 & \alpha_3 & -\sqrt{l^2 - \alpha_3^2 - 1} \end{bmatrix}^T.$$

Here, $l = (8 + 5\sqrt{10})/18$; c_i and s_i are the cosine and sine, respectively, of some angle θ_i; and the signs on the radicals are chosen by trial and error. Then setting $X^*Y = T$ yields six equations in six unknowns, and the unique solution is

$$X = \begin{bmatrix} 1 & (19 - 8\sqrt{10})/31 & (-245 + 64\sqrt{10})/155 \\ 0 & 2a_1/31 & (-28 + 8\sqrt{10})a_1/155 \\ 0 & 0 & 8a_2/155 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \frac{1}{5} & \frac{1}{3} & 1 \\ (49 + 12\sqrt{10})/15a_1 & (37 + 4\sqrt{10})/3a_1 & (-25 + 4\sqrt{10})/a_1 \\ (434 - 31\sqrt{10})/15a_2 & (62 - 31\sqrt{10})/3a_2 & (-310 + 93\sqrt{10})/3a_2 \end{bmatrix}$$

Here, $a_1 = (-10 + 76\sqrt{10})^{1/2}$ and $a_2 = (4340 - 1271\sqrt{10})^{1/2}$.

To extend our bound to the general case, where T is as in (9), keep X fixed and define $Y = Y(\delta_1, \delta_2, \dots, \delta_5)$ to be $(X^*)^{-1}T$. Clearly $X^*Y = T$ and $c(X) = 1$, so to bound $\|T\|_S$ above we need only show $c(Y) \leqslant b_3$. To do this we bound the size of the columns of Y one at a time. Let $\mathbf{y}$ and $\mathbf{t}$ be the first columns of Y and T, respectively, and note that $\mathbf{y} = (X^*)^{-1}\mathbf{t}$. Note also that the entries of $\mathbf{t}$, denoted t_{11}, t_{21}, and t_{31}, are such that $0 \leqslant t_{11} \leqslant \frac{1}{5}$, $0 \leqslant t_{21} \leqslant \frac{1}{3}$, and $0 \leqslant t_{31} \leqslant 1$. So let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_8$ be the eight vectors whose first entry is either 0 or $\frac{1}{5}$, second entry is either 0 or $\frac{1}{3}$, and third entry is either 0 or 1. These are clearly the extreme points of their convex hull $\mathscr{K}$, and furthermore, $\mathbf{t} \in \mathscr{K}$. Thus, $\|\mathbf{y}\|$ is a convex function of $\mathbf{t} = \mathbf{t}(\delta_1, \delta_2, \dots, \delta_5)$, and we may consider the domain of this function to be the convex set $\mathscr{K}$. It is well known that a convex function on a compact convex set achieves its maximum on an extreme point, so in our case we need to check $\|\mathbf{y}\|$ at each of the $\mathbf{e}_i$. Doing so, we see that $\|\mathbf{y}\|$ is maximal when $\mathbf{t} = [\frac{1}{5}, \frac{1}{3}, 1]^T$, and this maximum is unique. Therefore $\|\mathbf{y}\| \leqslant b_3$ provided by $\delta_i \geqslant 1$ for all i. Treating the second and third columns in essentially the same way (details are given in [15, Section IV.3]), we obtain the same bound both times. Therefore, $c(Y) \leqslant b_3$ in the nonequally spaced case as well, and this implies $b_3\|C\| \geqslant \delta\|Q\|$ in case 5. Combining this with our analyses in the last section, we see that $b_3\|C\| \geqslant \delta\|Q\|$ regardless of the configuration of $\sigma(A)$ and $\sigma(B)$, verifying the inequality (3).

To prove this bound is best possible, we seek a matrix U such that $\|U\| = 1$ while $\|T \circ U\| = (8 + 5\sqrt{10})/18$. To find it we make use of the following observations. First, if $\mathscr{B}_n$ is the set of $n \times n$ matrices with norm no greater than 1, the extreme points of $\mathscr{B}_n$ are the unitaries. Thus to find $\|T\|_S$ we may assume U is an extreme point of $\mathscr{B}_3$, i.e., U is unitary. Also, since T is self-adjoint, has only real entries, and has inertia (2, 0, 1), we may assume U is self-adjoint, has only real entries, and has inertia (2, 0, 1) as well. (The symmetry of U is justified by [13, Corollary 3.3], and we may assume U has only real entries in light of [15, Section II.4]. Also, U cannot have all eigenvalues of the same sign, or it would be a multiple of I, and $\|T \circ I\|$ is too small. This justifies our assumption on the inertia of U.) Thus $U = I - 2\mathbf{x}\mathbf{x}^T$ for some unit vector $\mathbf{x} \in \mathbb{R}^3$.

Such matrices form a two-parameter family in $\mathbb{R}^{3\times 3}$, and it is convenient to choose these parameters to be angles in the spherical coordinatization of $\mathbf{x}$, say ϕ and ψ. To determine their values we seek two equations simultaneously satisfied by U. The first one is obvious: $\|T \circ U\| = (8 + 5\sqrt{10})/18$. To obtain this in algebraic form compute the characteristic polynomial of $T \circ U$, regarding it as a polynomial $p(\lambda)$ whose coefficients are themselves polynomials in $\cos^2 \phi$ and $\cos^2 \psi$. Then we seek ϕ and ψ so that $p((8 + 5\sqrt{10})/18) = 0$. To obtain the second equation we return to our familiar transform argument. With A, B, and θ as in Section 4, let $f_1(t) =$

$(-5\sqrt{10}\,i/36)d(t+\theta)+(5\sqrt{10}\,i/36)d(t-\theta)$ and $f_2(t)=(4i/9)d(t-\pi/2)$. Then for $j=1$ or 2 set

$$\mathscr{T}_j(U)=\int_{-\infty}^{\infty}e^{-itA}Ue^{itB}f_j(t)\,dt.$$

As before, $\mathscr{T}_j(U)$ is Schur multiplication by a matrix, say T_j, and we have

$$T_1=\frac{1}{45}\begin{bmatrix}-11 & 35 & 25\\ 35 & 25 & -25\\ 25 & -25 & -35\end{bmatrix},\qquad T_2=\frac{4}{9}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}.$$

As before, it follows that $\|T_j\|_S\leqslant\|f_j\|_1$, and this implies $\|T_1\|_S\leqslant 5\sqrt{10}/18$ and $\|T_2\|_S\leqslant\frac{4}{9}$. Note that if U is optimal we must have equality both times, since $\|T\circ U\|=\|(T_1+T_2)\circ U\|\leqslant\|T_1\circ U\|+\|T_2\circ U\|$. So we set $\|T_1\circ U\|=5\sqrt{10}/18$, and this yields our second algebraic equation in $\cos^2\phi$ and $\cos^2\psi$. (Note that $\|T_2\circ U\|=\frac{4}{9}$ holds for all U, and thus is not helpful.) Our two equations in two unknowns have a unique solution, and this leads to an optimal U, which in turn leads to optimal A, Q, and B in our original formulation.

So let $A=\operatorname{diag}(5,3,1)$, $B=\operatorname{diag}(0,2,4)$, and note that $\operatorname{dist}(\sigma(A),\sigma(B))=1$. Then set

$$Q=\frac{1}{360}\begin{bmatrix}30+3\sqrt{10} & 10\alpha_1 & 15\alpha_2\\ 10\alpha_1 & 60+60\sqrt{10} & 30\alpha_3\\ 15\alpha_2 & 30\alpha_3 & -50+25\sqrt{10}\end{bmatrix},$$

where we let $\alpha_1=(80-19\sqrt{10})^{1/2}$, $\alpha_2=(146+56\sqrt{10})^{1/2}$, and $\alpha_3=(20+11\sqrt{10})^{1/2}$. Computing $C=AQ-QB$, one obtains

$$C=\frac{1}{24}\begin{bmatrix}10+\sqrt{10} & 2\alpha_1 & \alpha_2\\ 2\alpha_1 & 4+4\sqrt{10} & -2\alpha_3\\ \alpha_2 & -2\alpha_3 & 10-5\sqrt{10}\end{bmatrix}.$$

To verify that $b_3\|C\|=\|Q\|$, note first that $\|C\|=1$. (The C here is the U

referred to above.) In particular, with

$$\mathbf{x} = \frac{1}{4\sqrt{3}}\left[-\left(14 - \sqrt{10}\right)^{1/2}, \left(20 - 4\sqrt{10}\right)^{1/2}, \left(14 + 5\sqrt{10}\right)^{1/2}\right]^T$$

we have $C = I - 2\mathbf{x}\mathbf{x}^T$, proving C is unitary and so $\|C\| = 1$. Combining this with our established upper bound, then, we have $b_3 \geqslant \|Q\|$. To show equality holds here, observe that

$$\mathbf{y} = \left[\left(25 - 5\sqrt{10}\right)^{1/2}, \left(28 - 2\sqrt{10}\right)^{1/2}, \left(-5 + 7\sqrt{10}\right)^{1/2}\right]^T$$

is an eigenvector for Q with eigenvalue $(8 + 5\sqrt{10})/18$. Thus, $\|Q\| \geqslant b_3$, and it follows that equality holds here. Therefore $b_3\|C\| = \delta\|Q\|$ for A, Q, and B as defined, and b_3 is sharp in the inequality (3).

Before proceeding we remark on the calculations, alluded to above, needed to determine X and Y such that $X^*Y = T$ while $c(X)c(Y) = \|T\|_S$. As formulated, setting $X^*Y = T$ yields six equations in six unknowns, and this has a unique solution. In practice, however, additional information can be used to ease the work. For example, if Y is partitioned as $[\mathbf{y}_1 \ \mathbf{y}_2 \ \mathbf{y}_3]$, then for reasons not entirely clear to this author, we have $\langle \mathbf{y}_1, \mathbf{y}_2\rangle = \langle \mathbf{y}_2, \mathbf{y}_3\rangle$. Furthermore, for reasons made perfectly clear in [6], it follows that if X is similarly partitioned then $\langle \mathbf{x}_1, \mathbf{x}_2\rangle = \langle \mathbf{x}_2, \mathbf{x}_3\rangle$ and $\langle \mathbf{y}_1, \mathbf{y}_2\rangle = l^2\langle \mathbf{x}_1, \mathbf{x}_2\rangle$ as well. These equations are helpful, and the resulting (overdetermined) system can be solved without too much trouble.

6. EXTENSIONS FOR ARBITRARY UI NORMS

Let A, Q, and B be $n \times n$ matrices meeting the hypotheses of Theorem 1, and as before let b_n be the smallest number such that $b_n\|C\| \geqslant \delta\|Q\|$ provided $\|\cdot\|$ is the usual norm. In this section we apply the Ando-Horn-Johnson inequality (4) to note that $b_n \| C \| \geqslant \delta \| Q \|$ as well, provided $\| \cdot \|$ is any ui norm. This essentially follows from [1] (see remark I on pp. 361–362), but we include the details here in order to be self-contained.

On the set of $n \times n$ matrices over $\mathbb{C}$ there are n distinguished ui norms known as the Ky Fan norms. These are defined for $k = 1, 2, \dots, n$ by $\|A\|_k = \sum_{i=1}^{k} s_i(A)$, and it is well known that $\| A \| \geqslant \| B \|$ for all ui norms if and only if $\|A\|_k \geqslant \|B\|_k$ for $1 \leqslant k \leqslant n$. (See, for example, [8] or [9].)

We begin by defining a family of Schur multiplier norms for a matrix T. Specifically, let $\mathrm{n}_k(T) = \max_{X \neq 0} \|T \circ X\|_k / \|X\|_k$. Note that $\mathrm{n}_1(T) = \|T\|_S$ in the sense in which we have already defined it, since $\|X\|_1$ is the usual norm. Suppose now A and B are fixed, and $T = \left((\lambda_j - \mu_k)^{-1} \right)$ as usual. Let R and S be such that $R^*S = T$ and $\mathrm{c}(R)\mathrm{c}(S) = \|T\|_S$. Then for any X, by (4) we have

$$\begin{aligned}\|T \circ X\|_k &= \sum_{i=1}^{k} \mathrm{s}_i(T \circ X) \leqslant \sum_{i=1}^{k} \mathrm{c}_i(R)\mathrm{c}_i(S)\mathrm{s}_i(X) \\ &\leqslant \mathrm{c}(R)\mathrm{c}(S) \sum_{i=1}^{k} \mathrm{s}_i(X) = \mathrm{c}(R)\mathrm{c}(S)\|X\|_k.\end{aligned}$$

Thus $\mathrm{n}_k(T) \leqslant \mathrm{c}(R)\mathrm{c}(S) = \mathrm{n}_1(T)$, and it follows that $b_n\|C\|_1 \geqslant \delta\|Q\|_1$ implies $b_n\|C\|_k \geqslant \delta\|Q\|_k$ for $k = 2, 3, \ldots, n$. Therefore $b_n \,|||\, C \,|||\, \geqslant \delta \,|||\, Q \,|||$ for any ui norm $|||\cdot|||$, as claimed.

7. ON THE CONNECTION BETWEEN FOURIER ANALYSIS AND MATRIX THEORY

The fact that b_3, the sharp perturbation bound in the 3×3 case, has value $(8 + 5\sqrt{10})/18$ derives from the fact that $b_3 = \|T\|_S$, where T is as in (10). Thus, much of our work has been devoted to calculating this norm. There are certain matrices for which computing their Schur multiplier norm is easy. For example, if M is positive semidefinite then $\|M\|_S$ is its largest diagonal entry. If M is unitary then $\|M\|_S = 1$. These facts are well known, and the survey paper [12], for example, is a good reference for results of this type. More recently, R. Mathias [14] has shown that if M is a generalized circulant then $\|M\|_S$ is $1/n$ times the trace norm of M. And, as mentioned, a formula for $\|M\|_S$ is given in [5] when M is a 2×2 matrix. When M is not in such a class, however, even computing $\|M\|_S$ numerically can be challenging. For a numerical solution there are various approaches possible, but the one outlined in [6] is probably best. When one seeks explicit values, however, one must work in an ad hoc fashion. And as we've seen, sometimes the Fourier transform argument introduced in [4] can be applied to obtain sharp estimates in the Schur multiplier norm of a matrix.

In the 3×3 case we sought an L_1 function f such that $\hat{f}$ took on prescribed values at the points -3, -1, 1, 3, and 5. These points are equally spaced, and it is equivalent to assume that we seek an f such that $\hat{f}$ has prescribed values at, say, -2, -1, 0, 1, and 2. So let $\phi = (\phi_k)_{k=-2}^{2} = (\frac{1}{5}, \frac{1}{3}, 1, -1, -\frac{1}{3})$, let

$$\mathscr{S} = \left\{ f \in L_1(\mathbb{R}) : \hat{f}(k) = \phi_k \text{ for } k = -2, -1, 0, 1, \text{and } 2 \right\},$$

and let $m = \inf_{f \in \mathscr{S}} \|f\|_1$. Then with T as in (10), we know $m \geqslant \|T\|_S$, and one can easily adapt the function (11) to see that equality holds here. Looking back to the 2×2 case, suppose T is as in (6). Define $\phi = (\phi_{-1}, \phi_0, \phi_1) = (\frac{1}{3}, 1, -1)$, set $\mathscr{S}$ as usual, and let $m = \min_{f \in \mathscr{S}} \|f\|_1$. On one hand, we know $m \geqslant \|T\|_S = \sqrt{6}/2$. On the other hand, let $\alpha = \frac{1}{2} - i/(2\sqrt{2})$, $\theta = \arccos(-\frac{1}{3})$, and set $f(t) = \alpha d(t - \theta) + \bar{\alpha} d(t + \theta)$. Then one many check that $\hat{f}(k) = \phi_k$ for $k = -1$, 0, and 1, and $\|f\|_1 = \sqrt{6}/2$. Thus $m = \|T\|_S$ again.

In general we define the Fourier interpolation problem in this way. Let $\mathscr{A}$ be an index set, and let $\mathscr{S} = \{f \in L_1(\mathbb{R}) : \hat{f}(s_\alpha) = \phi_\alpha \text{ for } \alpha \in \mathscr{A}\}$. Then we seek $\min_{f \in \mathscr{S}} \|f\|_1$, as well as extremal functions $f^\star$ when the infimum here is a minimum. This problem is defined and discussed in [20, Chapter 7], and the author there also generalizes Sz.-Nagy's result. Our work in two and three dimensions leads to the obvious question whether or not the Schur multiplier norm of a Hankel matrix always equals the infimum in the corresponding Fourier interpolation problem. Specifically, for $\phi = (\phi_k)_{k=-n+1}^{n-1}$ let

$$T = \begin{bmatrix} \phi_{-n+1} & \phi_{-n+2} & \cdots & \phi_{-1} & \phi_0 \\ \phi_{-n+2} & \cdots & \phi_{-1} & \phi_0 & \phi_1 \\ \vdots & \iddots & \iddots & \phi_1 & \vdots \\ \phi_{-1} & \phi_0 & \iddots & \vdots & \phi_{n-2} \\ \phi_0 & \phi_1 & \cdots & \phi_{n-2} & \phi_{n-1} \end{bmatrix},$$

and set $\mathscr{S} = \{f \in L_1(\mathbb{R}) : \hat{f}(k) = \phi_k \text{ for } -n+1 \leqslant k \leqslant n-1\}$. Also, set $m = \inf_{f \in \mathscr{S}} \|f\|_1$. Then one might ask whether $\|T\|_S = m$ for all choices of ϕ, at least if ϕ has an odd number of components. In separate calculations this author has shown that if ϕ is any triple of real numbers then the answer is yes. Of course, this includes the special case $\phi = (\frac{1}{3}, 1, -1)$ relevant to the 2×2 case. And, as we have seen, equality holds in our 3×3 case, when $\phi = (\frac{1}{5}, \frac{1}{3}, 1, -1, -\frac{1}{3})$ and T is as in (10). In general, however, the answer is no. Equality holds at least sometimes, however, and when it does, one can use that fact to assist in calculating $\|T\|_S$ explicitly.

Note added in proof: Prof. C. C. Cowen is circulating the preprint, "Finding norms of Hadamard Multipliers," jointly written with P. A. Ferguson, D. K. Jackman, E. A. Sexauer, C. Vogt, and H. J. Woolf. The authors there include a Matlab algorithm for finding the Schur multiplier norm of a matrix numerically. ("Hadamard" and "Schur" multiplication are different names for the same thing.)

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REFERENCES

1 T. Ando, R. A. Horn, and C. R. Johnson, The singular values of a Hadamard product: A basic inequality, *Linear and Multilinear Algebra* 21:345–365 (1987).

2 R. Bhatia, *Perturbation Bounds for Matrix Eigenvalues*, Pitman Res. Notes 162, Longman Scientific and Technical, Essex, U.K., 1987.

3 R. Bhatia, C. Davis, and P. Koosis, An extremal problem in Fourier analysis with applications to operator theory, *J. Funct. Anal.* 82:138–150 (1989).

4 R. Bhatia, C. Davis, and A. McIntosh, Perturbation of spectral subspaces and solution of linear operator equations, *Linear Algebra Appl.* 52–53:45–67 (1983).

5 C. C. Cowen, K. E. Debro, and P. D. Sepanski, Geometry and the norms of Hadamard multipliers, *Linear Algebra Appl.*, to appear.

6 C. C. Cowen, M. A. Dritschel, and R. C. Penney, Norms of Hadamard multipliers, *SIAM J. Matrix Anal. Appl.*, to appear.

7 C. Davis and W. M. Kahan, The rotation of eigenvectors by a perturbation. III, *SIAM J. Numer. Anal.* 7:1–46 (1970).

8 K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Nat. Acad. Sci. U.S.A.* 37:760–766 (1951).

9 I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Nonselfadjoint Operators*, Transl. Math. Monographs 18, Amer. Math. Soc., Providence, 1969.

10 U. Haagerup, Decompositions of completely bounded maps on operator algebras, preprint.

11 J. A. R. Holbrook, Commentary, preprint.

12 R. Horn, The Hadamard product, in *Matrix Theory and Applications*, Proc. Symp. Appl. Math. 40 (C. R. Johnson, Ed.), Amer. Math. Soc., Providence, 1990.

13 R. Mathias, Matrix completions, norms and Hadamard products, *Proc. Amer. Math. Soc.* 117:905–917 (1993).

14 R. Mathias, The Hadamard operator norm of a circulant and applications, *SIAM J. Matrix Anal. Appl.*, 14:1152–1167 (1993).

15 R. McEachin, Analysis of an Inequality Concerning Perturbation of Self-adjoint Operators, Ph.D. Thesis, Univ. of Illinois, Urbana, 1990.

16 R. McEachin, A sharp estimate in an operator inequality, *Proc. Amer. Math. Soc.* 115:161–165 (1992).

17 V. I. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Res. Notes 146, Longman Scientific and Technical, New York, 1986.

18 M. Rosenblum, On the operator equation $BX - XA = Q$, *Duke Math. J.* 23: 263–270 (1956).

19 I. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, *J. Reine Angew Math.* 140:1–128 (1911).

20 H. S. Shapiro, *Topics in Approximation Theory*, Lecture Notes in Math. 187, Springer-Verlag, New York, 1971.

21 B. Sz.-Nagy, Über die ungleichung von H. Bohr, *Math. Nachr.* 9:225–259 (1953).

22 B. Sz.-Nagy, Bohr inequality and an operator equation, in *Operators in Indefinite Metric Spaces, Scattering Theory, and Other Topics (Bucharest, 1985)*, Oper. Theory Adv. Appl. 24, Birkhäuser, Boston, 1987, pp. 321–327.

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