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On the determining equations for the nonclassical reductions of the heat and Burgers' equation

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Abstract

The determining equations for the nonclassical reductions of the heat and Burgers' equations are considered. It is shown that both systems belong to a Burgers' equation hierarchy. Each system is written in terms of the same matrix Burgers' equation that is linearized via a matrix Hopf–Cole transformation. In essence, it is shown that both systems can be solved simultaneously. Their respective solutions are then presented in a very compact form.

Keywords: Nonclassical symmetry; Determining equations; Burgers' equation; Heat equation; Hopf–Cole transformation

1. Introduction

The role of symmetry analysis has played a fundamental role in the construction of exact solutions to nonlinear partial differential equations. Based on the original work of Lie [1] on continuous groups, symmetry analysis provides a unified explanation for the seemingly diverse and ad hoc integration methods used to solve ordinary differential equations. At the present time, there is extensive litera-

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ture on the subject and we refer the reader to the books by Bluman and Kumei [2], Olver [3], and Rogers and Ames [4]. In essence, one seeks the invariance of a differential equation

$$\Delta(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots) = 0, \tag{1.1}$$

under the group of infinitesimal transformations

$$\begin{aligned} \bar{t} &= t + T(t, x, u)\epsilon + O(\epsilon^2), \\ \bar{x} &= x + X(t, x, u)\epsilon + O(\epsilon^2), \\ \bar{u} &= u + U(t, x, u)\epsilon + O(\epsilon^2). \end{aligned} \tag{1.2}$$

This leads to a set of determining equations for the infinitesimals T, X and U which, when solved, gives rise to the symmetries of (1.1). Once a symmetry is known for a differential equation, invariance of the solution leads to the invariant surface condition

$$T u_t + X u_x = U. \tag{1.3}$$

Solutions of (1.3) leads to a solution ansatz, which, when substituted into Eq. (1.1) gives a reduction of the original equation. A generalization of the so-called “classical method” of Lie was proposed by Bluman and Cole [5]. Today, it is commonly referred to as the “nonclassical method.” Their method seeks invariance of the original equation augmented with the invariant surface condition. Their original intention was to construct new exact solutions of the heat equation

$$u_t = u_{xx}; \tag{1.4}$$

however, all exact solutions obtained by their nonclassical method could also be obtained by the classical method. Subsequently, Broadbridge and Arrigo [6] showed that all solutions of linear partial differential equations can be obtained from a classical Lie symmetry. However, the nonclassical method has been very successful in obtaining exact solutions to several nonlinear partial differential equations of significant physical important such as Fishers’ equation and the Boussinesq equation (see, for example, [7–10]). Unfortunately, unlike the determining equations for the classical method, which are linear, the determining equations for the nonclassical method are usually highly nonlinear. For example, the determining equations for the heat equation obtained by Bluman and Cole [5], with $T = 1$, are

$$\begin{aligned} &X_{uu} = 0, \\ &U_{uu} - 2X_{xu} + 2XX_u = 0, \\ &U_t - U_{xx} + 2UX_x = 0, \\ &2U_{xu} + X_t - X_{xx} - 2UX_u + 2XX_x = 0. \end{aligned} \tag{1.5}$$

This system of equations can be partially integrated to give

$$X = A, \qquad U = -Bu + C, \tag{1.6}$$

where A, B and C are functions of x and t and satisfy

$$\begin{aligned} &A_t - A_{xx} + 2AA_x - 2B_x = 0, \\ &B_t - B_{xx} + 2BA_x = 0, \end{aligned} \tag{1.7}$$

and

$$C_t - C_{xx} + 2CA_x = 0. \tag{1.8}$$

Despite the various attempts to solve this system of equations, they remained unsolved for 30 years. Recently, Mansfield [11], using ideas of Arrigo et al. [12], gave both the general solution of this system of equations and the closely related system of determining equations for the nonclassical reductions of Burgers' equation. It is shown here, after partially integrating the determining equations for the heat and Burgers' equations, that both systems can be written as part of a hierarchy of matrix Burgers' equations. It is well known that the heat equation and Burgers' equation are related via the Hopf–Cole transformation. The present work reveals that there is also a close relationship between the systems of determining equations for their nonclassical reductions. This allows the determining equations for each equation to be solved easily and simultaneously and allows the solutions to be presented in compact form.

The paper is organized as follows. In Section 2, the matrix Burgers' equation is constructed for the system (1.7). Introducing a matrix Hopf–Cole transformation, this matrix Burgers' equation is linearized leading to the solution of the system. In Section 3, the process is repeated for the determining equations of Burgers' equation. Finally, generalizations are presented to higher-order nonclassical symmetries and relationships to the entire matrix Burgers' hierarchy.

2. Determining equations for the heat equation

If A and B satisfy (1.7) and v is any solution to the heat equation, then

$$C = Av_x + v_t + Bv \tag{2.1}$$

satisfies (1.8). This was noted by Mansfield [11] and Arrigo et al. [12]. It thus remains to solve Eqs. (1.7). By introducing the matrix

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -1 \\ B & A \end{pmatrix}, \tag{2.2}$$

the system of equations (1.7) can conveniently be written as

$$\mathbf{\Omega}_t + 2\mathbf{\Omega}_x\mathbf{\Omega} = \mathbf{\Omega}_{xx}. \tag{2.3}$$

Following Levi et al. [13], introduce the matrix Hopf–Cole transformation

$$\boldsymbol{\Omega} = -\boldsymbol{\Phi}_x \boldsymbol{\Phi}^{-1}. \tag{2.4}$$

Substitution of (2.4) into Eq. (2.3) leads to the linear matrix heat equation

$$\boldsymbol{\Phi}_t = \boldsymbol{\Phi}_{xx}, \tag{2.5}$$

where the entries ϕ_{ij} of the matrix are solutions of the heat equation. After rearranging (2.4) to give

$$\boldsymbol{\Omega}\boldsymbol{\Phi} = -\boldsymbol{\Phi}_x, \tag{2.6}$$

and using $\boldsymbol{\Omega}$ as given in (2.2), a component by component comparison of the entries of the matrices in (2.6) leads to the system

$$\phi_{21} = \partial_x \phi_{11}, \qquad \phi_{22} = \partial_x \phi_{12}, \tag{2.7}$$

$$B\phi_{11} + A\phi_{21} = -\partial_x \phi_{21}, \qquad B\phi_{12} + A\phi_{22} = -\partial_x \phi_{22}. \tag{2.8}$$

Introducing new variables ω_1 and ω_2 such that

$$\phi_{11} = \omega_1, \qquad \phi_{12} = \omega_2 \tag{2.9}$$

gives (2.7) as

$$\phi_{21} = \partial_x \omega_1, \qquad \phi_{22} = \partial_x \omega_2, \tag{2.10}$$

and (2.8) as

$$B\omega_1 + A\partial_x \omega_1 + \partial_x^2 \omega_1 = 0, \qquad B\omega_2 + A\partial_x \omega_2 + \partial_x^2 \omega_2 = 0. \tag{2.11}$$

Solving the system of equations (2.11) for A and B gives

$$A = -\frac{\begin{vmatrix} \omega_1 & \omega_2 \\ \partial_x^2 \omega_1 & \partial_x^2 \omega_2 \end{vmatrix}}{\begin{vmatrix} \omega_1 & \omega_2 \\ \partial_x \omega_1 & \partial_x \omega_2 \end{vmatrix}}, \qquad B = \frac{\begin{vmatrix} \partial_x \omega_1 & \partial_x \omega_2 \\ \partial_x^2 \omega_1 & \partial_x^2 \omega_2 \end{vmatrix}}{\begin{vmatrix} \omega_1 & \omega_2 \\ \partial_x \omega_1 & \partial_x \omega_2 \end{vmatrix}}, \tag{2.12}$$

where $|\,.\,|$ represents the determinant. The solutions (2.12) together with (2.1) give rise to the general solution of the system of determining equations for the nonclassical reductions of the heat equation as found in (1.7) and (1.8).

3. Determining equations for Burgers' equation

It is well known that the nonclassical reductions for Burgers equation

$$u_t + 2uu_x = u_{xx}, \tag{3.1}$$

with the invariant surface condition as given in (1.3) lead to the following system of determining equations (with $T = 1$):

$$\begin{aligned}
&X_{uu}=0,\\
&U_{uu}-2X_{xu}+2XX_u-4uX_u=0,\\
&U_t-U_{xx}+2UX_x+2uU_x=0,\\
&2U_{xu}+X_t-X_{xx}-2UX_u+2XX_x-2uX_x-2U=0
\end{aligned}\tag{3.2}$$

(see, for example, Clarkson and Mansfield [8] and Arrigo et al. [14]). This system of equations, when partially integrated, gives rise to three separate cases. The last case is presented here (the first two cases are easily handled; see, for example, [13] and [14]). Partial integration of (3.2) leads to X and U as

$$X=-u+A,\qquad U=-u^3+Au^2-Bu+C,\tag{3.3}$$

where A, B and C satisfy

$$\begin{aligned}
&A_t-A_{xx}+2AA_x-2B_x=0,\\
&B_t-B_{xx}+2BA_x-2C_x=0,\\
&C_t-C_{xx}+2CA_x=0.
\end{aligned}\tag{3.4}$$

Note, that the last two equations in (3.4) are the same as those in (1.7) after renaming. By introducing the matrix

$$\mathbf{\Omega}=\begin{pmatrix}0&-1&0\\0&0&-1\\C&B&A\end{pmatrix},\tag{3.5}$$

the system of equations (3.4) can also be written using the matrix Burgers' equation in (2.3). Since the analysis goes over verbatim, the analysis is resumed from (2.6) with $\mathbf{\Omega}$ now given by (3.5). This time, a component by component comparison of the entries of the matrices in (2.6) gives the system

$$\begin{aligned}
&\phi_{21}=\partial_x\phi_{11},\qquad \phi_{22}=\partial_x\phi_{12},\qquad \phi_{23}=\partial_x\phi_{13},\\
&\phi_{31}=\partial_x\phi_{21},\qquad \phi_{32}=\partial_x\phi_{22},\qquad \phi_{33}=\partial_x\phi_{23},
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
&C\phi_{11}+B\phi_{21}+A\phi_{31}=-\partial_x\phi_{31},\\
&C\phi_{12}+B\phi_{22}+A\phi_{32}=-\partial_x\phi_{32},\\
&C\phi_{13}+B\phi_{23}+A\phi_{33}=-\partial_x\phi_{33}.
\end{aligned}\tag{3.7}$$

Introducing the new variables ω_1, ω_2 and ω_3, where

$$\phi_{11}=\omega_1,\qquad \phi_{12}=\omega_2,\qquad \phi_{13}=\omega_3,\tag{3.8}$$

alters (3.6) to

$$\begin{aligned}
&\phi_{21}=\partial_x\omega_1,\qquad \phi_{22}=\partial_x\omega_2,\qquad \phi_{23}=\partial_x\omega_3,\\
&\phi_{31}=\partial_x^2\omega_1,\qquad \phi_{32}=\partial_x^2\omega_2,\qquad \phi_{33}=\partial_x^2\omega_3,
\end{aligned}\tag{3.9}$$

while (3.7) becomes

$$\begin{aligned}
&C\omega_1 + B\partial_x\omega_1 + A\partial_x^2\omega_1 + \partial_x^3\omega_1 = 0,\\
&C\omega_2 + B\partial_x\omega_2 + A\partial_x^2\omega_2 + \partial_x^3\omega_2 = 0,\\
&C\omega_3 + B\partial_x\omega_3 + A\partial_x^2\omega_3 + \partial_x^3\omega_3 = 0.
\end{aligned} \tag{3.10}$$

Solving Eqs. (3.10) for A, B and C gives

$$A = -\frac{\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \partial_x\omega_1 & \partial_x\omega_2 & \partial_x\omega_3 \\ \partial_x^3\omega_1 & \partial_x^3\omega_2 & \partial_x^3\omega_3 \end{vmatrix}}{\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \partial_x\omega_1 & \partial_x\omega_2 & \partial_x\omega_3 \\ \partial_x^2\omega_1 & \partial_x^2\omega_2 & \partial_x^2\omega_3 \end{vmatrix}}, \qquad B = \frac{\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \partial_x^2\omega_1 & \partial_x^2\omega_2 & \partial_x^2\omega_3 \\ \partial_x^3\omega_1 & \partial_x^3\omega_2 & \partial_x^3\omega_3 \end{vmatrix}}{\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \partial_x\omega_1 & \partial_x\omega_2 & \partial_x\omega_3 \\ \partial_x^2\omega_1 & \partial_x^2\omega_2 & \partial_x^2\omega_3 \end{vmatrix}},$$

$$C = -\frac{\begin{vmatrix} \partial_x\omega_1 & \partial_x\omega_2 & \partial_x\omega_3 \\ \partial_x^2\omega_1 & \partial_x^2\omega_2 & \partial_x^2\omega_3 \\ \partial_x^3\omega_1 & \partial_x^3\omega_2 & \partial_x^3\omega_3 \end{vmatrix}}{\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \partial_x\omega_1 & \partial_x\omega_2 & \partial_x\omega_3 \\ \partial_x^2\omega_1 & \partial_x^2\omega_2 & \partial_x^2\omega_3 \end{vmatrix}}. \tag{3.11}$$

The solutions given in (3.11) constitute the general solution of the system of determining equations for the nonclassical reductions of Burgers' equation as found in (3.4).

4. Conclusion

Having considered the determining equations for the nonclassical reductions of the heat equation and Burgers' equations, it has been shown that, after a partial integration, each system of reduced determining equations falls within the same class of matrix Burgers' equation. The general solutions are then presented for each. The fact that the same matrix equation occurs should not really come as a surprise. It is well known that the Hopf–Cole transformation

$$u = -\frac{U_x}{U} \tag{4.1}$$

transforms Burgers' equation (3.1) to the linear heat equation $U_t = U_{xx}$. The invariant surface condition (1.3), with the infinitesimals as given by (3.3), is

$$u_t + (-u + A)u_x + u^3 - Au^2 + Bu - C = 0, \tag{4.2}$$

or using the original equation to eliminate u_t gives

$$u_{xx} - 2uu_x + (-u + A)u_x + u^3 - Au^2 + Bu - C = 0. \tag{4.3}$$

Substitution of (4.1) into (4.3) leads to

$$U_{xxx} + AU_{xx} + BU_x + CU = 0, \tag{4.4}$$

which is seen as a higher-order version of the invariance condition of the heat equation,

$$U_{xx} + AU_x + BU = 0. \tag{4.5}$$

This easily generalizes to the invariant surface condition

$$\partial_x^{n+1} U + \sum_{i=0}^{n} A_i \partial_x^i U = 0, \tag{4.6}$$

which would lead to a hierarchy of Burgers' equations. The second and third members of the hierarchy have been considered here.

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