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Images of Connected Sets by Semicontinuous Multifunctions*

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The image of a connected set by an upper-semicontinuous (or a lower-semicontinuous) multifunction whose values are nonempty and connected is connected. We prove this theorem in its most general setting and show its usefulness in various examples from optimization and nonlinear analysis.

INTRODUCTION

This paper is an outgrowth of a technical note we wrote for graduate students some years ago [17]. It is concerned with the generalization of two well-known theorems in general topology, namely:

(i) the continuous image of a compact space is compact;

(ii) the continuous image of a connected space is connected.

The concept of continuity can be generalized to multifunctions in various ways; we consider here what is customarily called the upper-semicontinuity and the lower-semicontinuity of multifunctions. The generalizations of the two aforementioned results are:

(i′) the image of a compact space by an upper-semicontinuous compact-valued multifunction is compact;

(ii′) the image of a connected space by an upper-semicontinuous (or a lower-semicontinuous) multifunction whose values are nonempty and connected is connected.

The theorem (i′) is known in the context of multifunctions and used in applications. In Section II we state the result in its detailed form along with a corollary on the compactness of the graph of an upper-semicontinuous multifunction. An example illustrating the usefulness of such a result is given from Convex Optimization.

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The theorem (ii′) seems much less known, although it has been noticed incidentally from time to time in the literature on multifunctions. The complete and correct statement of the theorem is given in Section III (Theorem 3.1). Actually, the proof is not difficult to derive and we have found it a suitable exercise in general topology. However, more interesting than the theorem itself are the applications. Although it appears to be less important than its "compact" relative, it turns out that a result on the connectedness of the image of a set by semicontinuous multifunctions may be useful in applications. We have collected in Section IV several examples from various areas where such a result is shown "at work": mean value theorems in Nonsmooth Analysis, existence theorems in problems of Calculus of Variations, connectedness of the set of nondominated outcomes in Multicriteria Optimization, qualitative properties of the reachable set through the trajectories of a differential inclusion, and structure of the set of solutions of nonlinear equations.

Section I devoted to preliminaries on multifunctions; in particular we fix the notations we use since some of them are of our own.

I. Preliminaries on Multifunctions

Let X and Y be two spaces. If with each element $x \in X$ one associates a (possibly empty) subset $\Gamma(x)$ of Y, one say that the correspondence $x \to \Gamma(x)$ is a *multifunction* (or a *set-valued mapping*) of X into Y. Throughout this paper, such mappings will be denoted by "$\Gamma: X \rightrightarrows Y$." The set $\operatorname{dom} \Gamma = \{x \in X \mid \Gamma(x) \neq \phi\}$ is called the *domain* of Γ and the set $\operatorname{gr} \Gamma = \{(x, y) \in X \times Y \mid y \in \Gamma(x)\}$ the *graph* of Γ. Given a subset S of X, Γ_S will denote the restriction of Γ to S and $\Gamma(S)$ the *image* of S by Γ, i.e., $\bigcup_{x \in S} \Gamma(x)$.

If Γ is such that $\Gamma(x)$ consists of a single element whenever $x \in S$, i.e., $\Gamma(x) = \{\gamma(x)\}$ for $x \in S$, we say that Γ is single-valued on S.

Given $\Gamma: X \rightrightarrows Y$, two kinds of inverse multifunctions may be defined:

(1.1) the $\cap$-inverse multifunction $\overset{\cap}{\Gamma}{}^{-1}: Y \rightrightarrows X$ which associates with $T \subset Y$ the *$\cap$-inverse image* of T

$$\overset{\cap}{\Gamma}{}^{-1}(T) = \{x \in X \mid \Gamma(x) \cap T \neq \phi\};$$

(1.2) the $\subset$-inverse multifunction $\overset{\subset}{\Gamma}{}^{-1}: Y \rightrightarrows X$ which associates with $T \subset Y$ the *$\subset$-inverse image* of T

$$\overset{\subset}{\Gamma}{}^{-1}(T) = \{x \in X \mid \Gamma(x) \subset T\}.$$

Note that $\overset{\cap}{\Gamma}{}^{-1}(T)$ is included in $\operatorname{dom} \Gamma$ while $\overset{\subset}{\Gamma}{}^{-1}(T)$ always contains the complementary set of dom Γ in X. Γ^{-1} is actually the true inverse multifunction and we set

$$\Gamma^{-1}(y) = \Gamma^{-1}(\{y\}) \qquad \text{for all} \quad y \in Y.$$

Of course, when Γ is single-valued on X, $\Gamma(x) = \{\gamma(x)\}$, both inverse images of $T \subset Y$ reduce to the usual inverse $\gamma^{-1}(T)$.

Algebraic properties of multifunctions (i.e., those with respect to operations on sets) are easy to get; they can be found, for example, in Berge's classic reference [4, Chap. II]. As for the topological properties of multifunctions (only the topological viewpoint is considered in this paper), we confine ourselves to what we believe are the most familiar properties of "continuity" of a multifunction, namely, the upper-semicontinuity and the lower-semicontinuity. Their definitions will be recalled below. For an historical sketch and further variants of the notion of continuity for multifunctions as also various examples of their use in mathematical programming, we refer the reader to the booklet [24]; this reference together with lecture notes like [2] or [25] should be a good starting point for anyone desiring to have a good command of multifunctions and their applications in Optimization or Nonlinear Analysis. Those who approach multifunctions for the first time may wonder why one does not consider Γ as a mere mapping from X into $\mathscr{P}(Y)$. People from Analysis are more inclined to use this approach, by equipping $\mathscr{P}(Y)$ with adequate topologies or pseudo-topologies. However, the structures on $\mathscr{P}(Y)$ are too poor for the properties we intend to derive, as will be conspicuous in the next results. Let X and Y be two Hausdorff topological spaces and $\Gamma: X \rightrightarrows Y$. We recall that Γ is said to be *upper-semicontinuous* at $x_0 \in X$ if, for every open subset $\mathcal{O}$ containing $\Gamma(x_0)$, there exists a neighbourhood N of x_0 such that $\mathcal{O}$ contains $\Gamma(x)$ for all $x \in N$. Γ will be said to be upper-semicontinuous on X (or simply upper-semicontinuous) if it is upper-semicontinuous at each point of X. As for continuous functions, an upper-semicontinuous multifunction can be characterized in terms of inverse image as follows:

(1.1) *$\Gamma: X \rightrightarrows Y$ is upper-semicontinuous if and only if for each open subset $\mathcal{O}$ in Y, $\Gamma^{-1}(\mathcal{O})$ is an open subset in X (resp. for each closed subset F in Y, $\Gamma^{-1}(F)$ is a closed subset in X).*

What is a pity in this definition of upper-semicontinuity is that the product $\Gamma_1 \times \Gamma_2: x \rightrightarrows \Gamma_1(x) \times \Gamma_2(x)$ of two upper-semicontinuous multifunctions $\Gamma_1: X \rightrightarrows Y_1$ and $\Gamma_2: X \rightrightarrows Y_2$ is not upper-semicontinuous. The next example is illustrative of that.

EXAMPLE 1. Let $\Gamma: \mathbb{R} \rightrightarrows \mathbb{R}$ be defined as:

$$\Gamma(x) = \{1/|x|\} \quad \text{if} \quad x \neq 0, \; \Gamma(0) = \mathbb{R}.$$

The multifunction $\Gamma_g: x \rightrightarrows \{x\} \times \Gamma(x)$ is not upper-semicontinuous at 0.

If, however, in addition to the upper-semicontinuity, both multifunctions Γ_1 and Γ_2 are compact-valued, then so is $\Gamma_1 \times \Gamma_2$ [4, p. 120].

The multifunction Γ_g built up from the multifunction Γ in the example above will be considered on several occasions in the sequel, since the image of X by Γ_g is precisely the graph of Γ. A multifunction Γ such that the corresponding Γ_g is upper-semicontinuous (at a given point $x_0 \in X$) is called *graphically upper-semicontinuous* (at x_0) by Penot [26, Definition 1.1]. As a general rule, Γ is upper-semicontinuous at x_0 whenever it is graphically upper-semicontinuous at x_0. However, in the absence of a compactness property on $\Gamma(x_0)$, the latter definition turns out to be a more restrictive one (cf. the example above). For more on the relationship between the graphical upper-semicontinuity and other continuity concepts for multifunctions, we refer the reader to Penot's paper [26].

Another very useful topological property of multifunctions is what traditionally is called lower-semicontinuity. $\Gamma: X \rightrightarrows Y$ is said to be *lower-semicontinuous* at $x_0 \in X$ if, for every open subset $\mathcal{O}$ meeting $\Gamma(x_0)$, there exists a neighbourhood N of x_0 such that $\mathcal{O}$ meets $\Gamma(x)$ for all $x \in N$. The counterpart of (1.1) for lower-semicontinuous Γ is as follows:

(1.2) *$\Gamma: X \rightrightarrows Y$ is lower-semicontinuous if and only if for each open subset $\mathcal{O}$ in Y, $\Gamma^{-1}(\mathcal{O})$ is an open subset in X (resp. for each closed subset F in Y, $\Gamma^{-1}(F)$ is a closed subset in X).*

Although the lower-semicontinuity and the upper-semicontinuity cannot be compared, the practice of both concepts shows that it is "easier" for a multifunction to be upper-semicontinuous than lower-semicontinuous. As often as not, the definition itself of the multifunction Γ under consideration, combined with other properties like the closedness of the graph, suffices to secure that Γ is upper-semicontinuous.

Unlike for the previous case, the product of two lower-semicontinuous multifunctions is lower-semicontinuous. In particular, the lower-semicontinuity of $\Gamma: X \rightrightarrows Y$ implies that of $\Gamma_g: X \rightrightarrows X \times Y$.

II. COMPACTNESS OF THE IMAGE, OF THE GRAPH

Consider a continuous function $\gamma: X \to Y$ and a subset K of X. The next statements are familiar ones in general topology: the image $\gamma(K)$ is compact whenever K is compact; K is compact if and only if the graph of γ_K is compact. Extending these results has been of first concern for those authors who introduced continuity concepts for multifunctions. These extensions are as follows:

THEOREM 2.1. *Let $\Gamma: X \rightrightarrows Y$ be upper-semicontinuous. Suppose that $\Gamma(x)$ is compact for all x in a compact subset $K \subset X$. Then*

(2.1) *the image $\Gamma(K)$ is compact;*

(2.2) *the graph of Γ_K is compact.*

It is clear that, as it is when Γ is single-valued on K, K is compact whenever the graph of Γ_K is compact. Note that, under the assumptions of the theorem above, the $\cap$-inverse of a compact subset T of Y by Γ_K is indeed compact but that nothing can be claimed concerning $\Gamma_K^{-1}(T)$. Also note that neither (2.1) nor (2.2) holds if one substitutes the upper-semicontinuity for the lower-semicontinuity of Γ. The next example illustrates this difference.

EXAMPLE 2. Let $\mathbb{R} \rightrightarrows \mathbb{R}$ be defined as follows: $\Gamma(x)$ is the segment with end points $-1/x$ and $1/x$ if $x \neq 0$ and $\Gamma(0) = [-1, +1]$. Γ is lower-semicontinuous and compact-valued but $\Gamma(K)$ is unbounded for any compact subset K containing the origin.

The results of Theorem 2.1, especially (2.1), are well known in the context of multifunctions (see [4, p. 116], for example); they are also widely used in applications, more particularly in nondifferentiable optimization where multifunctions arise naturally. Let us illustrate the use of Theorem 2.1 in convex optimization. Given a convex function $f: \mathbb{R}^p \to \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$, the ε-subdifferential of f at $x \in \mathbb{R}^p$ is defined as the set $\partial_\varepsilon f(x)$ of x^* satisfying

$$f(x') \geqslant f(x) + \langle x^*, x' - x \rangle - \varepsilon \qquad \text{for all} \quad x' \in \mathbb{R}^p.$$

An interesting property of the multifunction $\partial_\cdot f(\cdot): \mathbb{R}_+ \times \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ which assigns $\partial_\varepsilon f(x)$ to (ε, x) is that it is upper-semicontinuous and compact-valued. A lot of minimization methods using the ε-subdifferential have been developed in recent years (the so-called ε-subgradient methods). In all these procedures, the definition of the next iterate x_{n+1} from x_n requires to know one or several elements of $\partial_\varepsilon f(x_n)$. We therefore are usually faced with three sequences: $(\varepsilon_n) \subset \mathbb{R}_+$, $(x_n) \subset \mathbb{R}^p$, (x_n^*) such that $x_n^* \in \partial_{\varepsilon_n} f(x_n)$ for all n.

A common situation is when (ε_n) is assumed to be bounded (possibly converging to 0) and (x_n) is secured to be bounded. Now, what about the sequence (x_n^*) ? By a direct application of (2.1), we get that (x_n^*) is a *boun-*

ded sequence. Moreover, the graph of $\partial_{.} f(\cdot)$ is *closed* in $\mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^p$. As a result, subsequencing if necessary, we may suppose that $(\varepsilon_n, x_n, x_n^*)$ converges to some $(\varepsilon_\infty, x_\infty, x_\infty^*)$ and therefore $x_\infty^* \in \partial_{\varepsilon_\infty} f(x_\infty)$. This way of doing it is classical and represents a key step in proving convergence theorems.

III. Connectedness of the Image, of the Graph, of the Inverse Images

Let $\gamma: X \to Y$ be a continuous function, and let C be a subset of X. It is a well-known fact that C is connected if and only if the graph of γ_C is connected; if so, the image of C under γ is also connected. The extension of these results to multifunctions can be carried out for both lower- and upper-semicontinuities. The reason is that, while compactness of K relies on a property of coverings of K by *open* subsets, connectedness of C can be defined by means of coverings of C by two *open* or two *closed* subsets.

THEOREM 3.1. *Let $\Gamma: X \rightrightarrows Y$ be lower-semicontinuous (or upper-semicontinuous). Suppose that $C \subset X$ is connected and that $\Gamma(x)$ is nonempty and connected for all $x \in C$. Then the image of C under Γ is connected.*

One may wonder whether, under the assumptions above on Γ, the graph of Γ_C is connected. The answer is positive if Γ is lower-semicontinuous, since Γ_g is lower-semicontinuous in such a case (recall that $\Gamma_g(C)$ is precisely gr Γ_C). As for upper-semicontinuous Γ, we know that the answer is negative (cf. Example 1). Γ should be a graphically upper-semicontinuous multifunction, and a way of securing it is to suppose that Γ is compact-valued (see Section I). We summarize these situations in the following statements.

THEOREM 3.2. *Let $C \subset X$ be a connected subset and let $\Gamma: X \rightrightarrows Y$ be such that $\Gamma(x)$ is nonempty and connected for all $x \in C$. Either of the next assumptions ensure that the graph of Γ_C is connected:*

(3.1) *Γ is lower-semicontinuous;*

(3.2) *Γ is upper-semicontinuous and compact-valued.*

A set which is both compact and connected is called a *continuum*. We thus have:

COROLLARY 3.3. *Let $\Gamma: X \rightrightarrows Y$ be upper-semicontinuous. Assume that $K \subset X$ is a continuum and that $\Gamma(x)$ is a nonempty continuum for all $x \in K$. Then the image of K under Γ as well as the graph of Γ_K are continua.*

Before proving Theorem 3.1, some remarks are in order especially in regard to the historical outline. To our knowledge, the first mention of a result on the connectedness of the image $\Gamma(C)$ of a connected set C by a semicontinuous multifunction Γ goes back to Hahn's book [14].[1] Hahn's arguing was, however, in metric spaces and $\Gamma(x) \neq \varnothing$ was implicitly part of his definition of a multifunction. Later on, several authors have noticed this result or part of it: Smithson [28, Proposition 2.2], Davy [10, Theorem 2.3], Valadier [30, Lemme 5], Penot [25, p. 34], and Naccache [22, Lemma 2.1]. It has not always been observed that the theorem holds true for both semicontinuities and, more important, some authors have overlooked the fact that $\Gamma(x)$ has to be nonempty for all $x \in C$. The necessity of this assumption is conspicuous in simple examples as well as in the proof of the theorem.

Proof of Theorem 3.1. By definition, S is a connected subset of a topological space Z if and only if, for all coverings of S by two open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ in Z (or by two closed subsets in Z) such that $S \cap \mathcal{O}_1 \neq \varnothing$ and $S \cap \mathcal{O}_2 \neq \varnothing$, one has $S \cap \mathcal{O}_1 \cap \mathcal{O}_2 \neq \varnothing$.

Suppose first that the multifunction Γ is lower-semicontinuous. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two open subsets of Y such that

$$\begin{aligned} &\text{(i)} \quad \Gamma(C) \subset \mathcal{O}_1 \cup \mathcal{O}_2; \\ &\text{(ii)} \quad \Gamma(C) \cap \mathcal{O}_1 \neq \varnothing, \ \Gamma(C) \cap \mathcal{O}_2 \neq \varnothing. \end{aligned} \tag{3.3}$$

We have to prove that $\Gamma(C) \cap \mathcal{O}_1 \cap \mathcal{O}_2$ is nonempty. For that purpose, let us show first that

$$C \cap \Gamma^{-1}(\mathcal{O}_1) \cap \Gamma^{-1}(\mathcal{O}_2) = \varnothing \tag{3.4}$$

leads to a contradiction.

According to the characterization (1.2) of lower-semicontinuous multifunctions in terms of $\cap$-inverse of open sets, both $\Gamma^{-1}(\mathcal{O}_1)$ and $\Gamma^{-1}(\mathcal{O}_2)$ are open in X. Moreover, according to (3.3), we have that

(i′) $C \subset \Gamma^{-1}(\mathcal{O}_1) \cup \Gamma^{-1}(\mathcal{O}_2)$ (because $\Gamma(x) \neq \varnothing$ whenever $x \in C$!),

(ii′) $C \cap \Gamma^{-1}(\mathcal{O}_1) \neq \varnothing$, $C \cap \Gamma^{-1}(\mathcal{O}_2) \neq \varnothing$.

These relations plus relation (3.4) contradict the fact that C is connected. Let now x be in $C \cap \Gamma^{-1}(\mathcal{O}_1) \cap \Gamma^{-1}(\mathcal{O}_2)$. By definition,

$$\Gamma(x) \cap \mathcal{O}_1 \neq \varnothing, \qquad \Gamma(x) \cap \mathcal{O}_2 \neq \varnothing.$$

According to (i) of (3.3), we have that $\Gamma(x) \subset \mathcal{O}_1 \cup \mathcal{O}_2$. Since $\Gamma(x)$ is supposed to be connected, we therefore have that $\Gamma(x) \cap \mathcal{O}_1 \cap \mathcal{O}_2$ is nonempty. Whence $\Gamma(C) \cap \mathcal{O}_1 \cap \mathcal{O}_2$ is nonempty.

The same arguments are valid for an upper-semicontinuous Γ by considering a covering of $\Gamma(C)$ by two closed subsets and the second characterization of upper-semicontinuous multifunctions given in (1.1). ∎

[1] We are indebted to Professor W. Oettli for pointing out this reference.

In the realm of continuous functions, we have the following result: if $\gamma: X \to Y$ is continuous on S and if T is a connected component of $\gamma(S)$, then $\gamma^{-1}(T)$ is a union of connected components of S. It turns out that we have the same type of properties for the inverse images by a semi-continuous multifunction.

THEOREM 3.4. *Let $\{S_i\}_{i \in I}$ be a partition of $S \subset X$ into connected components, and let $\Gamma: S \rightrightarrows Y$ be nonempty valued on S and such that $\Gamma(S_i)$ is connected for each i. Then, the inverse images of a connected component of $\Gamma(S)$ are unions of connected components of S.*

Sufficient conditions for $\Gamma(S_i)$ to be connected are precisely given in Theorem 3.1. Even if the connectedness of $\Gamma(x)$ for $x \in S_i$ is not quite necessary, it is easy to see that the announced result does not hold if $\Gamma(x)$ is allowed to be empty on S.

Proof of Theorem 3.4. Let $\{T_j\}_{j \in J}$ be a partition of $\Gamma(S)$ into connected components and let us consider the $\cap$-inverse image of a component T_j. For each x in $\Gamma^{-1}(T_j)$, let $S_{i(x)}$ denote the connected component of x in S; we thus have

$$\Gamma^{-1}(T_j) \subset \bigcup S_{i(x)}.$$

Clearly, for each x in $\Gamma^{-1}(T_j)$, $\Gamma(S_{i(x)}) \cap T_j$ is nonempty. Since, by hypothesis, $\Gamma(S_{i(x)})$ is connected, it must lie entirely in T_j. Therefore,

$$\Gamma^{-1}(\Gamma(S_{i(x)})) \subset \Gamma^{-1}(T_j).$$

Now, because Γ is nonempty valued on $S_{i(x)}$, we also have

$$S_{i(x)} \subset \Gamma^{-1}(\Gamma(S_{i(x)})).$$

Hence, $\Gamma^{-1}(T_j)$ is nothing other than $\bigcup S_{i(x)}$ and the announced result is proved for the $\cap$-inverse image of T_j.

As for the $\subset$-inverse image of T_j, the nonemptiness of Γ on S is used in an earlier stage in the proof. For each x in $\Gamma^{-1}(T_j)$, $\Gamma(S_{i(x)}) \cap T_j$ contains $\Gamma(x)$ which is assumed nonempty. Whence $\Gamma(S_{i(x)}) \cap T_j$ is nonempty and the rest of the proof follows unchanged. ∎

IV. APPLICATIONS

We have collected in this section several examples from various areas where a result like the connectedness of a connected set by a semicontinuous multifunction is meaningful. To begin with, we consider what has actually motivated our interest in the matter developed in the present study, namely, the search for sharp mean value theorems for nondifferentiable functions.

IV.1. *Mean Value Theorems for Locally Lipschitz Functions*

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$, and let F be a locally Lipschitz function from $\mathcal{O}$ into $\mathbb{R}^n$. The *generalized Jacobian matrix* in Clarke's sense [8] of F at $x_0 \in \mathcal{O}$, denoted by $\mathscr{J}F(x_0)$, is the set of matrices defined by

$$\mathscr{J}F(x_0) = \operatorname{co}\{\lim_{i \to +\infty} JF(x_i)\}; \tag{4.1}$$

in this definition, x_i converges to x_0, F is differentiable at x_i for each i, and $JF(x_i)$ is the Jacobian matrix of F at x_i.

When $f: \mathcal{O} \to \mathbb{R}$, we denote by $\partial f(x_0)$ the subset of $\mathbb{R}^m$ (instead of $(\mathbb{R}^m)^*$) constructed in (4.1) through the gradients of f; $\partial f(x_0)$ is then called the *generalized gradient* of f at x_0. If $\mathcal{O}$ is convex and $f: \mathcal{O} \to \mathbb{R}$ is convex on $\mathcal{O}$, $\partial f(x_0)$ reduces to the *subdifferential* of f at x_0.

As usual, the space $\mathscr{M}(n, m)$ of (n, m)-matrices is equipped with a matricial norm. As consequences of the definition itself of $\mathscr{J}F$, we have that:

(i) for all $x \in \mathcal{O}$, $\mathscr{J}F(x)$ is a nonempty compact convex subset of $\mathscr{M}(n, m)$;

(ii) the multifunction $\mathscr{J}F: \mathcal{O} \rightrightarrows \mathscr{M}(n, m)$ is upper-semicontinuous.

Thus, according to Theorems 3.1 and 3.2, *for all compact connected sets S in $\mathcal{O}$, $\bigcup_{x \in S} \mathscr{J}F(x)$ and $\{(x, M) \mid x \in S,\ M \in \mathscr{J}F(x)\}$ are compact and connected.*

To prove mean value theorems, we begin by the simplest case, that is, the one dealing with real-valued functions.

THEOREM 4.1. *Let $f: I \to \mathbb{R}$ be locally Lipschitz on an open interval I of $\mathbb{R}$. Then*

$$\frac{f(b) - f(a)}{b - a} \in \partial f(]a, b[) \tag{4.2}$$

for all different a and b in I.

This theorem may be proved in several ways, the usual one relying on the following optimality condition: if $c \in]a, b[$ is a maximum or a minimum of g on $[a, b]$, then $0 \in \partial g(c)$. We simply derive the result (4.2) from the connectedness of the image of an interval by the generalized gradient.

Proof of Theorem 4.1. Let $g: I \to \mathbb{R}$ be defined by $g(x) = f(x) - \bar{m}(x-a)$, where $\bar{m}$ stands for the mean value $(f(b) - f(a))/(b-a)$. It is clear that "$g'(c) \leqslant 0$ for all c in the subset of full measure of $]a, b[$ where g' exists" is equivalent to "$\partial g(c) \subset \mathbb{R}_-$ for all $c \in]a, b[$"; in such a case, due to the integral representation $g(b) - g(a) = \int_a^b g'(t)\, dt$ (g is indeed absolutely continuous on $[a, b]$), g is decreasing on $[a, b]$, hence constant on $[a, b]$. Therefore, if g is not constant on $[a, b]$ (i.e., if f is not affine on $[a, b]$), $\partial g(]a, b[)$ meets $\mathbb{R}_-^*$ and $\mathbb{R}_+^*$; consequently, $\partial g(]a, b[)$, which is known to be an interval of $\mathbb{R}$, contains 0. Now, according to the definition of the generalized gradient, $\partial g(]a, b[) = \partial f(]a, b[) - \bar{m}$; hence the result (4.2) is proved. ▮

As seen in the proof above, $\partial f(]a, b[)$ is either reduced to one point (when f is affine on $[a, b]$) or is an interval with the mean value $(f(b) - f(a))/(b-a)$ lying in its interior. This can be viewed as an analogue to the Darboux property for differentiable functions. In the same vein, the following Dini-like property on the difference quotients of f is easy to derive:

$$\Delta = \bigcup_{\substack{a \leqslant c, d \leqslant b \\ c \neq d}} \frac{f(d) - f(c)}{d - c}$$

is an interval and the closure of Δ and that of $\partial f(]a, b[)$ are the same.

Generalizing Theorem 4.1 to the case where a, b lie in $\mathcal{O} \subset \mathbb{R}^m$ or a more general space X is a matter of applying chain rules for generalized gradients. If, for example, $f: \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz on an open set $\mathcal{O}$ of $\mathbb{R}^m$ and if the segment $[a, b]$ lies in $\mathcal{O}$, we have that

$$f(b) - f(a) \in \bigcup_{c \in]a,b[} \langle \partial f(c), b - a \rangle. \tag{4.3}$$

Passing to the case whare *vector-valued* functions are involved is more subtle. The key idea consists in "scalarizing" the vector-valued function $F: \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ by considering

$$F_{x^*}: t \to \langle F(a + t(b-a)), x^* \rangle \qquad \text{for all} \quad x^* \in \mathbb{R}^n,$$

and then applying aforementioned results to F_{x^*}. For more details and references, we refer the reader to [16]. In particular, the following mean value theorem is derived.

THEOREM 4.2. *Let $F: \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function on an open subset $\mathcal{O}$ of $\mathbb{R}^m$, and let $[a, b]$ be a segment lying in $\mathcal{O}$. Then*

$$F(b) - F(a) \in \text{co} \left\{ \bigcup_{c \in]a,b[} \mathcal{J}F(c) \cdot (b-a) \right\}. \tag{4.4}$$

At this stage, something in the results seems to be inaccurate. We know that for $n = 1$, $\bigcup_{c \in]a,b[} \langle \partial f(c), b-a \rangle$ is an interval so that the convex hull operation is unnecessary in (4.4) (see the formula (4.3)). At the same time, expressing $F(b) - F(a)$ by means of Carathéodory's theorem in (4.4) requires us to use $n+1$ points of $\bigcup_{c \in]a,b[} \mathscr{J}F(c) \cdot (b-a)$. Thus, for $n = 1$, two points are necessary a priori and one thereby does not retrieve the previous result. Here again the connectedness of $\bigcup_{c \in]a,b[} \mathscr{J}F(c) \cdot (b-a)$ will be helpful.

Carathéodory's theorem states that any x in the convex hull of $S \subset \mathbb{R}^n$ can be expressed as a convex combination of $n+1$ points of S. In general, the number $n+1$ cannot be replaced by any smaller number. There is, however, a theorem initiated by Fenchel (1929) and pursued by Bunt (1934) which says that, in some cases, the convex combinations of n terms of S yield all of co S (cf. [15]). The exact statement is as follows.

THEOREM (Fenchel–Bunt). *If S has at most n connected components in $\mathbb{R}^n$, then the convex combinations having n terms of S yield all of* co S.

This result, which is very expressive from the geometrical viewpoint (draw some sketches in $\mathbb{R}^2$ or $\mathbb{R}^3$), does not seem to be well known even from specialists of convex analysis; it is mentioned with a proof in [11, Theorem 18(ii)] or [18, Lemma B.2.2].

In regard to the mean value theorems, everything becomes coherent now and we can state:

THEOREM 4.3. *Let $F: \mathcal{O} \subset \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function on an open subset $\mathcal{O}$ of $\mathbb{R}^m$, and let $[a, b]$ be a segment in $\mathcal{O}$. There then exist real numbers λ_k, vectors c_k, matrices M_k, $k = 1, \ldots, n$, such that*

$$\lambda_k \geqslant 0,\ c_k \in]a, b[,\ M_k \in \mathscr{J}F(c_k) \qquad \text{for all } k,$$

$$\sum_{k=1}^{n} \lambda_k = 1$$

and

$$F(b) - F(a) = \sum_{k=1}^{n} \lambda_k M_k (b-a).$$

IV.2. *Existence Theorems in Problems of Calculus of Variations*

Consider the following minimization problem:

$$\inf_{v \in H_0^1(0,1)} \int_0^L f(x, v'(x))\, dx, \qquad (\mathscr{P})$$

where f: $[0, L] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping satisfying the usual growth condition, namely: there exist $c_2 \geqslant c_1 > 0$, α and β in $L^1(0, L)$, such that

$$c_1 |\xi|^2 + \alpha(x) \leqslant f(x, \xi) \leqslant c_2 |\xi|^2 + \beta(x)$$

for all $\xi \in \mathbb{R}$ and almost all x in $[0, L]$.

To prove the existence of a solution to the problem $(\mathscr{P})$, Aubert and Tahraoui [1] have used the fact that the image of an interval by an upper-semicontinuous multifunction Γ: $\mathbb{R} \rightrightarrows \mathbb{R}$ is an interval. Their device consisted in introducing the following perturbed problem:

$$\inf_{u \in L^2(0,L)} \left\{ \int_0^L f(x, u(x))\, dx - r \int_0^L u(x)\, dx \right\}. \tag{$\mathscr{P}_r$}$$

The proof is then divided in to three steps:

1st step. For all $r \in \mathbb{R}$, define a multifunction Γ: $\mathbb{R} \rightrightarrows \mathbb{R}$ by

$$\forall r \in \mathbb{R} \qquad \Gamma(r) = \left\{ \int_0^L u_r(x)\, dx \mid u_r \text{ solution of } (\mathscr{P}_r) \right\}.$$

For all $r \in \mathbb{R}$, $\Gamma(r)$ is a nonempty compact interval which can also be written as $[\int_0^L \underline{u}_r(x)\, dx, \int_0^L \bar{u}_r(x)\, dx]$, where $\underline{u}_r$ (resp. $\bar{u}_r$) stands for the smallest (resp. the largest) solution of $(\mathscr{P}_r)$. Γ is moreover an upper-semicontinuous multifunction.

2nd step. There exist r_1 and r_2 such that $\int_0^L \underline{u}_{r_1}(x)\, dx \leqslant 0$ and $\int_0^L \bar{u}_{r_2}(x)\, dx \geqslant 0$. In other words, $\Gamma(\mathbb{R})$ meets $\mathbb{R}_-$ and $\mathbb{R}_+$. Therefore, $\Gamma(\mathbb{R})$, which is an interval, contains 0.

3rd step. According to what has been shown just above, there exists $r_0 \in \mathbb{R}$ such that u_{r_0} is solution of $(\mathscr{P}_{r_0})$ and $\int_0^L u_{r_0}(x)\, dx = 0$.

It remains to check that v_{r_0}: $x \to \int_0^x u_{r_0}(t)\, dt$ is indeed a solution of $(\mathscr{P})$, which is an easy matter.

IV.3. *Connectedness of the Set of Nondominated outcomes in Multicriteria Optimization*

Let the following be given:

(i) $X \subset \mathbb{R}^n$, called the set of feasible decisions;

(ii) $F = (f_1, \ldots, f_m)$: $\mathbb{R}^n \to \mathbb{R}^m$, called the criterion function; the image $Y = F(X)$ is called the set of feasible outcomes;

(iii) the cone $K \subset \mathbb{R}^m$ used to order outcomes: $y_1, y_2 \in Y$, y_1 dominates y_2 if and only if $y_2 - y_1 \in K$.

The set of maximal elements under this relation is called the set of nondominated outcomes and is denoted by $\mathscr{E}(Y, K)$.

Naccache [22] and Bitran and Magnanti [5] have proved connectedness results pertaining to $\mathscr{E}(Y, K)$; these results were partly generalized by Nieuwenhuis [23]. In the first two references, especially in the first one, the connectedness of $\mathscr{E}(Y, K)$ is attained by showing that there is some connected set S for which $S \subset \mathscr{E}(Y, K) \subset \bar{S}$. More precisely, S is the image of a convex cone (in fact the interior of the dual of K) by the multifunction

$$\Gamma: x^* \rightrightarrows \Gamma(x^*) = \{\bar{y} \in Y \mid \langle x^*, \bar{y}\rangle = \sup_{y \in Y} \langle x^*, y\rangle\}.$$

Γ is known to be the subdifferential of the support function of Y; it is therefore upper-semicontinuous. The connectedness of S then follows.

IV.4. *Structure of the Set of Solutions of Nonlinear Equations*

Consider an ordinary differential equation

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) \qquad \text{for all} \quad t \in [a, b], \\ x(t_0) &= x_0, \end{aligned} \qquad \text{(O.D.E.)}$$

where $(t_0, x_0) \in [a, b] \times \mathbb{R}^n$ and $f: [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded and continuous function. Peano's classical theorem states that there exists a solutions to (O.D.E). It is also well known that the set $S(x_0)$ of solutions is compact and connected in $\mathscr{C}([a, b], \mathbb{R}^n)$; in short, $S(x_0)$ is a nonempty continuum of $\mathscr{C}([a, b], \mathbb{R}^n)$. This kind of theorem is a prototype for answers to questions on the topological properties of solutions of abstract nonlinear equations $T(x) = y$. Peano's result as a substitute for the uniqueness of solutions of O.D.E. gave rise to the appellation *Peano phenomenon*. Its abstract formulation, due to Stampacchia (1949), is as follows: a continuous map $T: X \to Y$, X and Y being Banach spaces, is said to have Peano phenomenon at $y \in Y$ if $T^{-1}(y)$ is a nonempty continuum (cf. [31, 32]). Given a nonlinear equation $T(x) = y$, what are the topological properties of $\{x \in X, T(x) \in C\}$? Clearly, the answer depends on the structure of $T^{-1}(y)$ for all $y \in C$ and on properties of semicontinuity of the multifunction T^{-1}. Such properties of T^{-1} are evidently deduced from those of T. Recall, for example, that $T: X \to Y$ is said to be *open* (resp. *closed*) if $T(A)$ is an open (resp. a closed) subset of Y whenever A is an open (resp. a closed) subset of X. Translated into properties of T^{-1}, we have: T is closed (resp. open) if and only if $T^{-1}: Y \rightrightarrows X$ is upper-semicontinuous (resp. lower-semicontinuous). Further definitions combine the above-mentioned properties of T with the requirement that $T^{-1}(y)$ is connected for all $y \in Y$ [32]. For instance, a mapping T of X onto Y is called *quasi-invertible* (resp. *semi-invertible*) if $T^{-1}(y)$ is connected for each $y \in T$ and T is open (resp. closed). The following, already observed by Vidossich [32, Sect. 2], is now an easy consequence of our Theorem 3.1.

PROPOSITION 4.4. *Quasi-invertible and semi-invertible maps both have the property that inverse images by them of connected sets are connected.*

The set $S(x_0)$ of solutions to (O.D.E.) actually enjoys a stronger property than connectedness, namely, *acyclicity*.[2] This kind of result, initiated by Aronszajn (1942), has been extended to various contexts like that of differential inclusions with delays, integral equations [27], functional differential inclusions [13], etc. Roughly speaking, acyclicity is expected when the problem lies on the "boundary of uniqueness," in the sense that it can be approximated by a sequence of problems for which conditions for the uniqueness of solution are secured.

The multifunction $x_0 \rightrightarrows S(x_0)$ which associates to each x_0 the set of trajectories of (O.D.E.) issued from this point as well as the multifunction $\mathscr{R}_T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ which assigns to x_0 the reachable set at time $T \in [a, b]$, i.e.,

$$\mathscr{R}_T(x_0) = \{x(T) \mid x \in S(x_0)\}$$

are upper-semicontinuous. Several authors have put in perspective this property, among others in the context of differential inclusions (see [10, 19, 30, 27, 20, 3]). To help the reader in this jungle, we go through an example.

Let $F: \mathbb{R} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be an upper-semicontinuous multifunction taking nonempty compact convex values and let $[a, b]$ be an interval of $\mathbb{R}$. We consider the following differential inclusion: find an absolutely continuous mapping $x: [a, b] \to \mathbb{R}^n$ satisfying

$$\begin{aligned} &\dot{x}(t) \in F(t, x(t)) \qquad \text{for almost all} \quad t \in [a, b], \\ &x(a) = x_0. \end{aligned} \tag{D.I.}$$

We assume, for the sake of simplicity, that $F([a, b] \times \mathbb{R}^n)$ is bounded. Under these conditions, it has been proved (cf. the above-referenced papers) that *the multifunction*

$$S: \mathbb{R}^n \rightrightarrows \mathscr{C}([a, b], \mathbb{R}^n)$$

which assigns to x_0 the set of solutions of (D.I.) issued from x_0 as well as the multifunction

$$\begin{aligned} &\mathscr{R}_T: \mathbb{R}^n \rightrightarrows \mathbb{R}^n \\ &\qquad x_0 \rightrightarrows \mathscr{R}_T(x_0) = \{x(T) \mid x \in S(x_0)\} \end{aligned}$$

which associates to x_0 the reachable set at time $T \in [a, b]$ from x_0 are upper-

[2] Without going into details, let us say that acyclicity is a concept stronger than connectedness but weaker than convexity; it is a substitute for convexity in fixed point theorems.

semicontinuous. Both take nonempty compact connected values in their respective image spaces.

We thus have:

PROPOSITION 4.5. *Under the assumptions displayed above, assume that the initial point x_0 lies in a compact connected set X_0. Then the set of reachable points at time T by following a trajectory of (D.I.) issued from $x_0 \in X_0$ is a (compact) connected set.*

An interesting application of the connectedness of $\mathscr{R}_T(x_0)$ is a Hukuhara-type theorem: let x_1 be on the boundary bd $\mathscr{R}_T(x_0)$ of $\mathscr{R}_T(x_0)$; there then exists a solution $\bar{x}$ of (D.I.) such that $\bar{x}(T) = x_1$ and $\bar{x}(t) \in \text{bd}\, \mathscr{R}_t(x_0)$ for all $t \in [a, b]$. For more details and developments, we refer the reader to [3, Chap. 2].

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