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On finite complete rewriting systems and large subsemigroups

K.B. Wong *, P.C. Wong

Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

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Let S be a semigroup and T be a subsemigroup of finite index in S (that is, the set $S \setminus T$ is finite). The subsemigroup T is also called a large subsemigroup of S. It is well known that if T has a finite complete rewriting system, then so does S. In this paper, we will prove the converse, that is, if S has a finite complete rewriting system, then so does T. Our proof is purely combinatorial and also constructive.

1. Introduction

Let S be a semigroup and T be a subsemigroup of finite index in S (that is, the set $S \setminus T$ is finite). Then T is called a *large subsemigroup* of S, and S is called a *small extension* of T. In [4], Ruškuc asked if S is a small extension of T, whether S has a finite complete rewriting system if and only if T has a finite complete rewriting system (see [4, Problem 11.1(iii)] and [6, Remark 4.2]). This problem was partially solved by Wang in [5, Theorem 1], who proved that if T has a finite complete rewriting system, then so does S. However it is still not known whether T has a finite complete rewriting system or not, when S has a finite complete rewriting system. In this paper we shall prove that this is true, i.e., we shall prove the following:

Theorem 1.1. *Suppose S is a small extension of T. If S has a finite complete rewriting system, then so does T.*

* Corresponding author.
E-mail addresses: kbwong@um.edu.my (K.B. Wong), wongpc@um.edu.my (P.C. Wong).

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By Theorem 1.1 and the result of Wang [5, Theorem 1], we have completely answered the problem posed by Ruškuc (see [4, Problem 11.1(iii)]).

Corollary 1.2. *Suppose S is a small extension of T. Then S has a finite complete rewriting system if and only if T has a finite complete rewriting system.*

Let A be a non-empty set. This set A is called the alphabet and the elements of A are called letters. We shall denote the free semigroup and free monoid on A by A^+ and A^*, respectively. The elements of A^+ and A^* are called words. Note that $A^* = A^+ \cup \{\epsilon\}$, where ϵ is the empty word. Given a word $W \in A^*$, we shall denote its length by $\|W\|$, defined as the numbers of letters in W.

A *rewriting system* R over A is a set of rules $U \to V$, which are elements of $A^+ \times A^+$. A word $W_1 \in A^+$ is said to be rewritten to another word $W_2 \in A^+$ by a *one-step reduction* induced by R, if $W_1 = Z_1XZ_2$ and $W_2 = Z_1YZ_2$ for some rule $X \to Y$ in R. In this situation we write $W_1 \to_R W_2$. The reflexive transitive closure and the reflexive symmetric transitive closure of $\to_R$ are denoted by $\to_R^*$ and $\leftrightarrow_R^*$, respectively. The relation $\leftrightarrow_R^*$ is defined to be the congruence on A^+ generated by R and it defines the quotient semigroup $S = A^+/\leftrightarrow_R^*$. S is said to be presented by the *semigroup presentation* $[A\ ;\ R]$. If both A and R are finite, we say the semigroup presentation is finitely presented. For $U \in A^+$, $[U]_R$ shall denote the class of U modulo $\leftrightarrow_R^*$.

Let $\mathrm{Left}(R) = \{X \in A^+\colon\ X \to Y \in R\}$ and $\mathrm{Irr}(R) = A^+ \setminus A^* \mathrm{Left}(R) A^*$. Obviously, $\mathrm{Irr}(R)$ is the set of all words in A^+ that cannot be reduced by any rule in R. A word $W \in A^+$ is called an *irreducible word* if $W \in \mathrm{Irr}(R)$.

We say R is *Noetherian* if there is no infinite reduction sequence,

$$W_1 \to_R W_2 \to_R W_3 \to_R \cdots.$$

R is said to be *confluent* if whenever $U \to_R^* V$ and $U \to_R^* W$, then there is an $X \in A^+$ such that $V \to_R^* X$ and $W \to_R^* X$. If R is both Noetherian and confluent, we say that R is a *complete rewriting system*.

The following fact is well known.

Theorem 1.3. *Suppose R is a complete rewriting system. Then for each $W \in A^+$, there is a unique $W' \in \mathrm{Irr}(R)$ such that $W \to_R^* W'$.*

Theorem 1.3 will be used implicitly in many parts of the paper. Let R be a complete rewriting system on A^+. Then given any word $W \in A^+$, by Theorem 1.3, there is a $U \in \mathrm{Irr}(R)$ such that

$$W \to_R W_1 \to_R W_2 \to_R \cdots \to_R W_n = U.$$

The length of the above reduction sequence starting with W and ends with U is n. The *disorder* of W, denoted by $d_R(W)$, is the maximum of the lengths of all of the reduction sequences starting with W and ends with U. Note that $d_R(W)$ is finite. Suppose it is not. Then there is a $V_1 \in A^+$ such that $W \to_R V_1$ and $d_R(V_1)$ is infinite, for the number of subwords of W that are contained in $\mathrm{Left}(R)$ is finite. Then again there is a $V_2 \in A^+$ such that $V_1 \to_R V_2$ and $d_R(V_2)$ is infinite, and this process can go on indefinitely. So $W \to_R V_1 \to_R V_2 \to_R \cdots$ is an infinite reduction sequence, a contradiction. Note also that $W \in \mathrm{Irr}(R)$ if and only if $d_R(W) = 0$ (see [2] and [3]).

The following useful lemma is obvious.

Lemma 1.4. *If $U \to_R V$, then $d_R(U) > d_R(V)$. Furthermore if W is a subword of U, then $d_R(U) \geqslant d_R(W)$.*

A semigroup is said to have a *finite complete rewriting system* if it has a finitely presented semigroup presentation for which the rewriting system is complete.

2. A criterion

Let $[A\ ;\ R]$ be a finitely presented semigroup presentation for S for which R is complete. Let T be a subsemigroup of S. In this section we first prove a criterion for $[B\ ;\ R_T]$ to be a semigroup presentation for T where B is any non-empty set and R_T is a complete rewriting system over B. This will be done in Theorem 2.2. Then by replacing T with S we can get $[B\ ;\ R_S]$ to be a semigroup presentation for S and R_S is a complete rewriting system over B. This will be done in Corollary 2.3.

Let $A(T)$ be a subset of A^+ such that

$$\{W \in \mathrm{Irr}(R):\ [W]_R \in T\} \subseteq A(T) \subseteq \{W \in A^+:\ [W]_R \in T\}.$$

Let $(B, R_T, A(T), \phi, \rho)$ be a 5-tuple where B is a non-empty set, R_T is a rewriting system over B, $\phi : B^+ \to A^+$ is a homomorphism with $[\phi(U')]_R \in T$ for all $U' \in B^+$, and $\rho : A(T) \to B^+$ is a function. We say the 5-tuple $(B, R_T, A(T), \phi, \rho)$ has *Property* $\mathcal{R}$ relative to $[A\ ;\ R]$, if it satisfies the following:

(P1) for any $U \in A(T)$ and $V_1 \in A^+$ with $U \to_R V_1$, there is a $U' \in B^+$ such that $U \to_R V_1 \to_R^* \phi(U')$ and $\rho(U) \to_{R_T} U'$,

(P2) for any $U', V' \in B^+$ with $U' \to_{R_T}^* V'$, we have $\phi(U') \to_R^* \phi(V')$,

(P3) there does not exist an infinite reduction sequence

$$U_1' \to_{R_T} U_2' \to_{R_T} U_3' \to_{R_T} \cdots,$$

of words from B^+ such that $\phi(U_1') = \phi(U_2') = \phi(U_3') = \cdots$,

(P4) for each $U' \in B^+$ there is a $U'' \in B^+$ such that $\phi(U'') \in A(T)$ and $U' \to_{R_T}^* U''$,

(P5) $\phi(\rho(U)) = U$ for all $U \in A(T)$,

(P6) $U' \to_{R_T}^* \rho(\phi(U'))$ for all $U' \in B^+$ with $\phi(U') \in A(T)$.

Lemma 2.1. *Suppose* (P1), (P2), (P4), *and* (P6) *hold. Then for any* $U \in A(T)$ *and* $V \in \mathrm{Irr}(R)$ *with* $U \to_R^* V$, *we have* $V \in A(T)$ *and* $\rho(U) \to_{R_T}^* \rho(V)$.

Proof. By the definition of $A(T)$, clearly $V \in A(T)$. We shall prove by induction on $d_R(U)$ that $\rho(U) \to_{R_T}^* \rho(V)$.

Suppose $d_R(U) = 0$ then $U = V$. Thus $\rho(U) = \rho(V)$ and $\rho(U) \to_{R_T}^* \rho(V)$. Suppose $d_R(U) > 0$. Assume that it is true for all U_1 with $d_R(U_1) < d_R(U)$.

Let $U \to_R V_1 \to_R^* V$. By (P1), there is a $U' \in B^+$ such that $U \to_R V_1 \to_R^* \phi(U')$ and $\rho(U) \to_{R_T} U'$. By (P4), there is a $U'' \in B^+$ such that $\phi(U'') \in A(T)$ and $U' \to_{R_T}^* U''$. By (P2), $\phi(U') \to_R^* \phi(U'')$. Therefore $U \to_R^* \phi(U'')$ and $\rho(U) \to_{R_T}^* U''$. Since $V \in \mathrm{Irr}(R)$, we have $\phi(U'') \to_R^* V$. Furthermore $d_R(\phi(U'')) < d_R(U)$ (by Lemma 1.4). Therefore by induction $\rho(\phi(U'')) \to_{R_T}^* \rho(V)$. Now by (P6), $U'' \to_{R_T}^* \rho(\phi(U''))$. Hence $\rho(U) \to_{R_T}^* \rho(V)$.

The proof of this lemma is complete. □

Theorem 2.2. *If* $(B, R_T, A(T), \phi, \rho)$ *has* Property $\mathcal{R}$ *relative to* $[A\ ;\ R]$, *then* $[B\ ;\ R_T]$ *is a semigroup presentation for* T *and* R_T *is complete.*

Proof. We will first prove that $[B\ ;\ R_T]$ is a semigroup presentation for T. Let $\psi : [B\ ;\ R_T] \to T$ be defined by $\psi([U']_{R_T}) = [\phi(U')]_R$ for all $U' \in B^+$. Now we show that ψ is well defined. It is sufficient to prove $U' \to_{R_T} V'$ for $V' \in B^+$ implies that $\phi(U') \to_R^* \phi(V')$. This fact follows from (P2), so ψ is well defined.

Now we show that ψ is a homomorphism. Let $U', V' \in B^+$. Then $\psi([U'V']_{R_T}) = [\phi(U'V')]_R = [\phi(U')\phi(V')]_R = [\phi(U')]_R[\phi(V')]_R = \psi([U']_{R_T})\psi([V']_{R_T})$, where the second equality follows from the fact that ϕ is a homomorphism.

Now we show that ψ is surjective. Let $[W]_R \in T$ for some $W \in A^+$. Since R is complete, we may assume $W \in \mathrm{Irr}(R)$. Note that $W \in A(T)$, so $\psi([\rho(W)]_{R_T}) = [\phi(\rho(W))]_R = [W]_R$, where the last equality follows from (P5). Hence ψ is surjective.

Now we show that ψ is injective. Let $U', V' \in B^+$ with $\psi([U']_{R_T}) = \psi([V']_{R_T})$. Then $[\phi(U')]_R = [\phi(V')]_R$. By (P4), there are $U'', V'' \in B^+$ such that $\phi(U''), \phi(V'') \in A(T)$, $U' \to^*_{R_T} U''$, $V' \to^*_{R_T} V''$. By (P2), $\phi(U') \to^*_R \phi(U'')$ and $\phi(V') \to^*_R \phi(V'')$. So $[\phi(U'')]_R = [\phi(V'')]_R$. Since R is complete, there is a $V_1 \in \mathrm{Irr}(R)$ such that $\phi(U'') \to^*_R V_1$ and $\phi(V'') \to^*_R V_1$. By Lemma 2.1, $\rho(\phi(U'')) \to^*_{R_T} \rho(V_1)$, and then by (P6), $U'' \to^*_{R_T} \rho(V_1)$. Therefore $U' \to^*_{R_T} \rho(V_1)$. Similarly, we have $V' \to^*_{R_T} \rho(V_1)$. Hence $[U']_{R_T} = [\rho(V_1)]_{R_T} = [V']_{R_T}$ and ψ is injective.

Now we have shown that $[B\ ;\ R_T]$ is a semigroup presentation for T, via ψ. We will now proceed to prove that R_T is complete.

Suppose R_T is not Noetherian. Then there exists an infinite reduction sequence

$$U'_1 \to_{R_T} U'_2 \to_{R_T} U'_3 \to_{R_T} \cdots,$$

of words from B^+. By (P2), $\phi(U'_i) \to^*_R \phi(U'_{i+1})$ for all i. Since R is Noetherian, there is an integer i_0 such that for all $i \geqslant i_0$, $\phi(U'_i) = \phi(U'_{i+1})$, but this contradicts (P3). Hence R_T is Noetherian.

Now we prove that R_T is confluent. Suppose $U' \to^*_{R_T} V'_1$ and $U' \to^*_{R_T} V'_2$ with $U', V'_1, V'_2 \in B^+$. By (P4), we may assume $\phi(V'_1), \phi(V'_2) \in A(T)$. Since R is complete, there is a $V_3 \in \mathrm{Irr}(R)$ with $\phi(V'_1) \to^*_R V_3$ and $\phi(V'_2) \to^*_R V_3$. By Lemma 2.1, $\rho(\phi(V'_1)) \to^*_{R_T} \rho(V_3)$, and then by (P6) $V'_1 \to^*_{R_T} \rho(V_3)$. Similarly $V'_2 \to^*_{R_T} \rho(V_3)$. Hence R_T is confluent and is complete. □

In the case when $T = S$ and there is a 5-tuple $(B, R_S, A(S), \phi, \rho)$ that has *Property* $\mathcal{R}$ relative to $[A\ ;\ R]$, we have the following corollary:

Corollary 2.3. *$[B\ ;\ R_S]$ is a semigroup presentation for S and R_S is complete.*

3. Changing the semigroup presentation for *S*

Let $[A\ ;\ R]$ be a finitely presented semigroup presentation for S for which R is complete. Let $W_0 \in A^+$ be such that $\|W_0\| > 1$ and $W_0 \in \mathrm{Irr}(R)$. Now let s be a letter that does not appear in A and set $B = A \cup \{s\}$. We wish to find a complete rewriting system R_S such that $[B\ ;\ R_S]$ is a finitely presented semigroup presentation for S and $W_0 \to^*_{R_S} s$.

By Corollary 2.3, we need to find a 5-tuple $(B, R_S, A(S), \phi, \rho)$ that has Property $\mathcal{R}$ relative to $[A\ ;\ R]$. Note that B has been defined and is finite.

Let $A(S) = A^+$. Let $\phi_1 : B \to A^+$ be defined by $\phi_1(a) = a$ for all $a \in A$ and $\phi_1(s) = W_0$. Clearly ϕ_1 can be extended to a homomorphism $\phi : B^+ \to A^+$ by defining $\phi(U') = \phi_1(b_1)\ldots\phi_1(b_l)$ for all $U' = b_1 \ldots b_l \in B^+$. For convenience, we may define $\phi(\epsilon_B) = \epsilon_A$ where ϵ_B and ϵ_A are empty words in B^* and A^*, respectively.

Recall that we have set $A(S) = A^+$. We define $\rho : A(S) \to B^+$ as follows:

Let $W \in A(S)$.

(a) If W ends with the subword W_0, say $W = X_1 W_0$ for some $X_1 \in A^*$ (we use A^* instead of A^+ because we allow X_1 to be the empty word), then $\rho(W) = \rho(X_1)s$ (in the event $X_1 = \epsilon_A$, set $\rho(W) = s$).

(b) Suppose W does not end with the subword W_0. Let $W = X_2 a$ for some $X_2 \in A^*$ and $a \in A$. Set $\rho(W) = \rho(X_2)a$ (in the event $X_2 = \epsilon_A$, set $\rho(W) = a$).

As for the homomorphism ϕ, we may define $\rho(\epsilon_A) = \epsilon_B$.

Lemma 3.1. *Let $X_1, X_2, X_3 \in A(S)$. If $\rho(X_1X_2X_3) = \rho(X_1X_2)\rho(X_3)$, then $\rho(X_2X_3) = \rho(X_2)\rho(X_3)$.*

Proof. We prove by induction on $\|X_3\|$. Clearly it holds if $\|X_3\| = 0$, i.e., X_3 is the empty word. Suppose $\|X_3\| > 0$. Assume that it holds for all X_4 with $\|X_4\| < \|X_3\|$.

Case 1. Suppose X_3 ends with the subword W_0, say $X_3 = X_4 W_0$ for some $X_4 \in A^*$. Then $\rho(X_1X_2X_3) = \rho(X_1X_2X_4)s$, $\rho(X_2X_3) = \rho(X_2X_4)s$ and $\rho(X_3) = \rho(X_4)s$. Since $\rho(X_1X_2X_3) = \rho(X_1X_2)\rho(X_3)$, we have $\rho(X_1X_2X_4) = \rho(X_1X_2)\rho(X_4)$. By induction, $\rho(X_2X_4) = \rho(X_2)\rho(X_4)$. Therefore $\rho(X_2X_3) = \rho(X_2X_4)s = \rho(X_2)\rho(X_4)s = \rho(X_2)\rho(X_3)$.

Case 2. Suppose X_3 does not end with the subword W_0. Let $X_3 = X_4a$ for some $a \in A$ and $X_4 \in A^*$. Then $\rho(X_3) = \rho(X_4)a$. Now $\rho(X_1X_2X_3) = \rho(X_1X_2)\rho(X_3) = \rho(X_1X_2)\rho(X_4)a$. So $\rho(X_1X_2X_3)$ is a word in B^+ that ends with the letter a.

We claim that $X_1X_2X_3$ does not end with the subword W_0. Suppose the contrary. Then $X_1X_2X_3 = Z_1W_0$ for some $Z_1 \in A^*$ and $\rho(X_1X_2X_3) = \rho(Z_1)s$. So $\rho(X_1X_2X_3)$ is a word in B^+ that ends with the letter s. But this contradicts the last sentence of the previous paragraph. Thus our claim has been established. Therefore $X_1X_2X_3 = X_1X_2X_4a$ and $\rho(X_1X_2X_3) = \rho(X_1X_2X_4)a$. This implies that $\rho(X_1X_2X_4) = \rho(X_1X_2)\rho(X_4)$, and by induction $\rho(X_2X_4) = \rho(X_2)\rho(X_4)$.

Note also that X_2X_3 does not end with the subword W_0, for otherwise $X_1X_2X_3$ would end with the subword W_0. Therefore $X_2X_3 = X_2X_4a$ and $\rho(X_2X_3) = \rho(X_2X_4)a$. Since $\rho(X_2X_4) = \rho(X_2)\rho(X_4)$ and $\rho(X_3) = \rho(X_4)a$, we conclude that $\rho(X_2X_3) = \rho(X_2)\rho(X_3)$. □

Lemma 3.2. *Let $X_1, X_2 \in A(S)$. Then either*

(a) $\rho(X_1X_2) = \rho(X_1)\rho(X_2)$, *or*
(b) $\rho(X_1X_2) = \rho(Z_1)s\rho(Z_4)$ *where* $X_1 = Z_1Z_2$, $X_2 = Z_3Z_4$ *and* $Z_2Z_3 = W_0$ ($Z_1, Z_4 \in A^*$ *and* $Z_2, Z_3 \in A^+$).

Proof. We prove by induction on $\|X_2\|$. Clearly it holds if $\|X_2\| = 0$, i.e., X_2 is the empty word. Suppose $\|X_2\| > 0$. Assume that it holds for all X_3 with $\|X_3\| < \|X_2\|$.

Case 1. Suppose X_2 ends with the subword W_0, say $X_2 = X_3W_0$ for some $X_3 \in A^*$. Then $\rho(X_1X_2) = \rho(X_1X_3)s$. If X_3 is the empty word, then $\rho(X_1X_2) = \rho(X_1)s = \rho(X_1)\rho(X_2)$, we are done. If X_3 is not the empty word, then $\rho(X_1X_2) = \rho(X_1X_3)s$, and by induction ($\|X_3\| < \|X_2\|$), either $\rho(X_1X_3) = \rho(X_1)\rho(X_3)$ or $\rho(X_1X_3) = \rho(Z_1)s\rho(Z_4)$, where $X_1 = Z_1Z_2$, $X_3 = Z_3Z_4$ and $Z_2Z_3 = W_0$ ($Z_1, Z_4 \in A^*$ and $Z_2, Z_3 \in A^+$). Suppose the former holds. Then $\rho(X_2) = \rho(X_3W_0) = \rho(X_3)s$ and $\rho(X_1X_2) = \rho(X_1X_3)s = \rho(X_1)\rho(X_3)s = \rho(X_1)\rho(X_2)$.

Suppose the latter holds. Then $X_2 = Z_3Z_4W_0 = Z_3Z_5$ ($Z_5 = Z_4W_0$) and $\rho(X_1X_2) = \rho(X_1X_3)s = \rho(Z_1)s\rho(Z_4)s = \rho(Z_1)s\rho(Z_4W_0) = \rho(Z_1)s\rho(Z_5)$. Thus the lemma holds.

Case 2. Suppose X_2 does not end with the subword W_0 but X_1X_2 ends with the subword W_0, say $X_1X_2 = X_3W_0$ for some $X_3 \in A^*$. Then $\|W_0\| > \|X_2\|$ and $X_1 = X_3X_4$ where $X_4X_2 = W_0$ ($X_4 \in A^+$). Note that $\rho(X_1X_2) = \rho(X_3)s$ and the lemma holds.

Case 3. Suppose X_1X_2 does not end with the subword W_0. Let $X_2 = X_3a$ where $a \in A$ and $X_3 \in A^*$. Then $\rho(X_1X_2) = \rho(X_1X_3)a$. Since $\|X_3\| < \|X_2\|$, by induction and using an argument similar to Case 1, we conclude that the lemma holds. □

Lemma 3.3. $\phi(\rho(U)) = U$ *for all* $U \in A(S)$. (Property (P5).)

Proof. Let $U \in A(S)$. We shall prove by induction on $\|U\|$ that $\phi(\rho(U)) = U$. If $\|U\| = 1$, then $U = a$ for some $a \in A$ and clearly $\phi(\rho(U)) = a = U$. Suppose $\|U\| > 1$. Assume the lemma holds for all $U_1 \in A(S)$ with $\|U_1\| < \|U\|$.

Suppose U ends with the subword W_0, say $U = X_1 W_0$ for some $X_1 \in A^*$. Then $\rho(U) = \rho(X_1)s$ and $\phi(\rho(U)) = \phi(\rho(X_1))\phi(s) = \phi(\rho(X_1))W_0 = X_1 W_0 = U$, where the first equality follows from the fact that ϕ is a homomorphism, and the second last equality follows from induction (clearly $\|X_1\| < \|U\|$).

Suppose U does not end with the subword W_0. Let $U = X_2 a$ for some $X_2 \in A^*$ and $a \in A$. Now $\rho(U) = \rho(X_2)a$ and similarly by induction $\phi(\rho(U)) = \phi(\rho(X_2))\phi(a) = \phi(\rho(X_2))a = X_2 a = U$. Hence the lemma holds. □

Now we define the rules in R_S. Recall that $W_0 \in \mathrm{Irr}(R)$ and $\|W_0\| > 1$.

($\mathcal{C}1$) for each $X \to Y \in R$ put $\rho(X) \to \rho(Y)$ in R_S;
($\mathcal{C}2$) put $W_0 \to s$ in R_S;
($\mathcal{C}3$) if there is a rule $X_1 X_2 \to Y_1 \in R$ such that $W_0 = Z_1 X_1$ ($X_1, Y_1 \in A^+$ and $X_2, Z_1 \in A^*$), put $\rho(Z_1 X_1 X_2) \to \rho(Y')$ in R_S where $Z_1 X_1 X_2 \to_R^* Y'$ and $Y' \in \mathrm{Irr}(R)$;
($\mathcal{C}4$) if there is a rule $X_2 X_1 \to Y_1 \in R$ such that $W_0 = X_1 Z_1$ ($X_1, Y_1 \in A^+$ and $X_2, Z_1 \in A^*$), put $\rho(X_2 X_1 Z_1) \to \rho(Y')$ in R_S where $X_2 X_1 Z_1 \to_R^* Y'$ and $Y' \in \mathrm{Irr}(R)$;
($\mathcal{C}5$) if there is a rule $X_2 X_3 X_4 \to Y_1 \in R$ such that $W_0 = X_4 X_5 = X_1 X_2$ ($X_2, X_4, Y_1 \in A^+$ and $X_3, X_5, X_1 \in A^*$), put $\rho(X_1(X_2 X_3 X_4)X_5) \to \rho(Y')$ in R_S where $X_1(X_2 X_3 X_4)X_5 \to_R^* Y'$ and $Y' \in \mathrm{Irr}(R)$;
($\mathcal{C}6$) if there are $X_1, X_2, X_3 \in A^+$ such that $W_0 = X_1 X_2 = X_2 X_3$, put $sX_3 \to X_1 s$ in R_S (in the event of this we must have $\|X_1\| = \|X_3\|$).

Note that the number of rules of the form $\mathcal{C}1$ and $\mathcal{C}2$ that we put in R_S is finite. The number of rules of the form $\mathcal{C}3$ that we put in R_S is also finite because R is finite and W_0 is a fixed word. Similarly for the number of rules of the form $\mathcal{C}4$ up to $\mathcal{C}6$. Therefore R_S is a finite rewriting system.

Remark. Note that one can subsume the rules ($\mathcal{C}1$), ($\mathcal{C}3$), and ($\mathcal{C}4$) all within ($\mathcal{C}5$) by just allowing $X_1 X_2$ and $X_4 X_5$ be empty, as well as equal to W_0.

Since $A(S) = A^+$, the condition $\phi(U') \in A(S)$ for $U' \in B^+$ is vacuously always true. So Property (P6) takes the following form.

Lemma 3.4. *$U' \to_{R_S}^* \rho(\phi(U'))$ for all $U' \in B^+$.* (Property (P6).)

Proof. Let $U' \in B^+$. We shall prove by induction on $\|U'\|$ that $U' \to_{R_S}^* \rho(\phi(U'))$. Suppose $\|U'\| = 1$. Then $U' = a$ for some $a \in A$ or $U' = s$ (recall that $B = A \cup \{s\}$). In either cases, we have $\rho(\phi(U')) = U'$. So $U' \to_{R_S}^* \rho(\phi(U'))$.

Suppose $\|U'\| > 1$. Assume the lemma holds for all $U'_1 \in B^+$ with $\|U'_1\| < \|U'\|$.

Case 1. Suppose $U' \in A^+$. Then $\phi(U') = U'$. If U' ends with the subword W_0, say $U' = X_1 W_0$ for some $X_1 \in A^*$, then $\rho(U') = \rho(X_1)s = \rho(\phi(X_1))s$. Since $W_0 \to s \in R_S$ (the rule of the form ($\mathcal{C}2$)), we see that $U' \to_{R_S} X_1 s$. Clearly $\|X_1\| < \|U'\|$. So by induction, $X_1 \to_{R_S}^* \rho(\phi(X_1))$. Thus $U' = X_1 W_0 \to_{R_S} X_1 s \to_{R_S}^* \rho(\phi(X_1))s = \rho(\phi(U'))$.

If U' does not end with the subword W_0, then $U' = X_2 a$ for some $X_2 \in A^+$ and $a \in A$. Note that $\rho(U') = \rho(X_2)a = \rho(\phi(X_2))a$. By induction, $X_2 \to_{R_S}^* \rho(\phi(X_2))$. Thus $U' \to_{R_S}^* \rho(\phi(U'))$.

Case 2. Suppose $U' = U'_1 s U'_2$ for some $U'_2 \in A^+$ and $U'_1 \in B^*$. Note that $\phi(U') = \phi(U'_1) W_0 U'_2$. If U'_2 ends with the subword W_0, say $U'_2 = X_1 W_0$ for some $X_1 \in A^*$, then $\rho(\phi(U')) = \rho(\phi(U'_1) W_0 X_1 W_0) = \rho(\phi(U'_1) W_0 X_1)s = \rho(\phi(U'_1 s X_1))s$. By induction, $U'_1 s X_1 \to_{R_S}^* \rho(\phi(U'_1 s X_1))$. Also $U' \to_{R_S} U'_1 s X_1 s$ by the rule $W_0 \to s \in R_S$ (rule ($\mathcal{C}2$)). Thus $U' \to_{R_S}^* \rho(\phi(U'))$.

Suppose U'_2 does not end with the subword W_0, but $W_0 U'_2$ ends with the subword W_0, say $W_0 U'_2 = X_2 W_0$ for some $X_2 \in A^+$. Then there is an $X_3 \in A^+$ such that $W_0 = X_2 X_3 = X_3 U'_2$. So $sU'_2 \to X_2 s \in R_S$ (a rule of the form ($\mathcal{C}6$)) and $U' \to_{R_S} U'_1 X_2 s$. On the other hand, $\rho(\phi(U')) = \rho(\phi(U'_1) X_2 W_0) = \rho(\phi(U'_1) X_2)s = \rho(\phi(U'_1 X_2))s$, and also $\|U'_1 X_2\| = \|U'_1\| + \|X_2\| = \|U'_1\| + \|U'_2\| <$

$\|U'\|$. Therefore by induction, $U'_1X_2 \to^*_{R_S} \rho(\phi(U'_1X_2))$. Thus $U' \to_{R_S} U'_1X_2s \to^*_{R_S} \rho(\phi(U'_1X_2))s = \rho(\phi(U'))$.

Suppose $W_0U'_2$ does not end with the subword W_0. Let $U'_2 = U'_3a$ for some $a \in A$ and $U'_3 \in A^*$. Note that $\rho(\phi(U')) = \rho(\phi(U'_1)W_0U'_3a) = \rho(\phi(U'_1)W_0U'_3)a = \rho(\phi(U'_1sU'_3))a$. By induction $U'_1sU'_3 \to^*_{R_S} \rho(\phi(U'_1sU'_3))$. Thus $U' \to^*_{R_S} \rho(\phi(U'))$.

Case 3. Suppose $U' = U'_1s$ for some $U'_1 \in B^+$. Note that $\phi(U') = \phi(U'_1)W_0$ and $\rho(\phi(U')) = \rho(\phi(U'_1))s$. By induction, $U'_1 \to^*_{R_S} \rho(\phi(U'_1))$, and thus $U' \to^*_{R_S} \rho(\phi(U'))$.

The proof of this lemma is complete. □

Since $A(S) = A^+$, we have $\phi(U') \in A(S)$ for all $U' \in B^+$. Therefore the following lemma holds by choosing $U'' = U'$.

Lemma 3.5. *For each $U' \in B^+$ there is a $U'' \in B^+$ such that $\phi(U'') \in A(S)$ and $U' \to^*_{R_S} U''$.* (Property (P4).)

Lemma 3.6. *Suppose $U' \to_{R_S} V'$ by one of the rules of the form $(\mathcal{C}1)$, $(\mathcal{C}3)$, $(\mathcal{C}4)$ or $(\mathcal{C}5)$. Then $\phi(U') \neq \phi(V')$.*

Proof. Note that all the rules $(\mathcal{C}1)$, $(\mathcal{C}3)$, $(\mathcal{C}4)$ or $(\mathcal{C}5)$ have the form $\rho(X) \to \rho(Y)$ where $X \neq Y$ and $X \to^*_R Y$.

Let $U' = Z'_1\rho(X)Z'_2$ with $Z'_1, Z'_2 \in B^*$. Then $V' = Z'_1\rho(Y)Z'_2$. Note that $\phi(U') = \phi(Z'_1)X\phi(Z'_2)$ and $\phi(V') = \phi(Z'_1)Y\phi(Z'_2)$ (by Lemma 3.3 and the fact that ϕ is a homomorphism). If $\phi(U') = \phi(V')$, then $X = Y$ and

$$X \to_R Y \to_R X \to_R Y \to_R \cdots,$$

would be an infinite reduction sequence, contrary to the fact that R is complete. Hence $\phi(U') \neq \phi(V')$. □

Lemma 3.7. *There does not exist an infinite reduction sequence*

$$U'_1 \to_{R_S} U'_2 \to_{R_S} U'_3 \to_{R_S} \cdots,$$

of words from B^+ such that $\phi(U'_1) = \phi(U'_2) = \phi(U'_3) = \cdots$. (Property (P3).)

Proof. Suppose that such a sequence exists. Since $\phi(U'_i) = \phi(U'_{i+1})$, by Lemma 3.6, we conclude that $U'_i \to_{R_S} U'_{i+1}$ by one of the rules of the form $(\mathcal{C}2)$ or $(\mathcal{C}6)$. Note that if a rule of the form $(\mathcal{C}2)$ is applied to $U'_i \to_{R_S} U'_{i+1}$, then $\|U'_{i+1}\| < \|U'_i\|$. If a rule of the form $(\mathcal{C}6)$ is applied to $U'_i \to_{R_S} U'_{i+1}$, then $\|U'_{i+1}\| = \|U'_i\|$ and one of the letter s in U'_{i+1} will be further to the right than it is in U'_i. Thus $\|U'_i\| \geqslant \|U'_{i+1}\|$ for all i.

There is an integer i_0 such that for all $i \geqslant i_0$, $\|U'_i\| = \|U'_{i+1}\|$. So the only rule that can be applied on $U'_i \to_{R_S} U'_{i+1}$ is a rule of the form $(\mathcal{C}6)$. Since one of the letter s in U'_{i+1} will be further to the right than it is in U'_i, this process cannot go on indefinitely. We have obtained a contradiction. Hence the lemma holds. □

Lemma 3.8. *For any $U', V' \in B^+$ with $U' \to^*_{R_S} V'$, we have $\phi(U') \to^*_R \phi(V')$.* (Property (P2).)

Proof. It is sufficient to show $U' \to_{R_S} V'$ with $U', V' \in B^+$ implies that $\phi(U') \to^*_R \phi(V')$.

Suppose $U' \to_{R_S} V'$ by a rule of the form $(\mathcal{C}1)$, say $\rho(X) \to \rho(Y) \in R_S$ where $X \to Y \in R$. Let $U' = Z'_1\rho(X)Z'_2$ with $Z'_1, Z'_2 \in B^*$. Then $V' = Z'_1\rho(Y)Z'_2$. By Lemma 3.3, $\phi(U') = \phi(Z'_1)X\phi(Z'_2)$ and $\phi(V') = \phi(Z'_1)Y\phi(Z'_2)$. Clearly $\phi(U') \to_R \phi(V')$ by the rule $X \to Y$.

Suppose $U' \to_{R_S} V'$ by a rule of the form ($\mathcal{C}2$). Let $U' = Z'_1 W_0 Z'_2$ with $Z'_1, Z'_2 \in B^*$. Then $V' = Z'_1 s Z'_2$. By Lemma 3.3, $\phi(U') = \phi(Z'_1) W_0 \phi(Z'_2) = \phi(V')$. Clearly $\phi(U') \to^*_R \phi(V')$.

Suppose $U' \to_{R_S} V'$ by a rule of the form ($\mathcal{C}3$), say $\rho(Z_1X_1X_2) \to \rho(Y')$, where $X_1X_2 \to Y_1 \in R$, $W_0 = Z_1X_1$ and $Z_1X_1X_2 \to^*_R Y'$ ($X_1, Y_1 \in A^+$, $X_2, Z_1 \in A^*$ and $Y' \in \mathrm{Irr}(R)$). Let $U' = Z'_3\rho(Z_1X_1X_2)Z'_4$ with $Z'_3, Z'_4 \in B^*$. Then $V' = Z'_3\rho(Y')Z'_4$. By Lemma 3.3, $\phi(U') = \phi(Z'_3)Z_1X_1X_2\phi(Z'_4)$ and $\phi(V') = \phi(Z'_3)Y'\phi(Z'_4)$. So $\phi(U') \to^*_R \phi(V')$, for $Z_1X_1X_2 \to^*_R Y'$.

Similarly we can show that if $U' \to_{R_S} V'$ by a rule of the form ($\mathcal{C}4$), ($\mathcal{C}5$) or ($\mathcal{C}6$), then $\phi(U') \to^*_R \phi(V')$. The proof of this lemma is complete. □

Lemma 3.9. *For any $U \in A(S)$ and $V_1 \in A^+$ with $U \to_R V_1$, there is a $U' \in B^+$ such that $U \to_R V_1 \to^*_R \phi(U')$ and $\rho(U) \to_{R_S} U'$.* (Property (P1).)

Proof. Let $U \to_R V_1$ by a rule $X_2 \to Y_2 \in R$. Let $U = X_1X_2X_3$ where $X_1, X_3 \in A^*$. Then $V_1 = X_1Y_2X_3$.

Case 1. Suppose $\rho(X_1X_2X_3) = \rho(X_1X_2)\rho(X_3)$.

SubCase 1.1. Suppose $\rho(X_1X_2) = \rho(X_1)\rho(X_2)$. Then $\rho(U) = \rho(X_1)\rho(X_2)\rho(X_3)$ and also $\rho(U) \to_{R_S} \rho(X_1)\rho(Y_2)\rho(X_3)$ by the rule $\rho(X_2) \to \rho(Y_2) \in R_S$ (a rule of the form ($\mathcal{C}1$)). Let $U' = \rho(X_1) \times \rho(Y_2)\rho(X_3)$. By Lemma 3.3, $\phi(U') = X_1Y_2X_3 = V_1$ and thus the lemma holds.

SubCase 1.2. Suppose $\rho(X_1X_2) \neq \rho(X_1)\rho(X_2)$. By Lemma 3.2, there are $Z_1, Z_4 \in A^*$ and $Z_2, Z_3 \in A^+$ with $X_1 = Z_1Z_2$, $X_2 = Z_3Z_4$ and $Z_2Z_3 = W_0$ such that $\rho(X_1X_2) = \rho(Z_1)s\rho(Z_4)$. Note that $\rho(Z_2Z_3Z_4) \to \rho(Y') \in R_S$ where $Z_2Z_3Z_4 \to^*_R Y'$ and $Y' \in \mathrm{Irr}(R)$ (a rule of the form ($\mathcal{C}3$)). Furthermore $\rho(Z_1Z_2Z_3Z_4) = \rho(X_1X_2) = \rho(Z_1)s\rho(Z_4) = \rho(Z_1Z_2Z_3)\rho(Z_4)$. So by Lemma 3.1, $\rho(Z_2Z_3Z_4) = \rho(Z_2Z_3)\rho(Z_4) = s\rho(Z_4)$. Therefore $\rho(X_1X_2) = \rho(Z_1)s\rho(Z_4) \to_{R_S} \rho(Z_1)\rho(Y')$ and

$$\rho(U) = \rho(X_1X_2)\rho(X_3) \to_{R_S} \rho(Z_1)\rho\big(Y'\big)\rho(X_3).$$

Let $U' = \rho(Z_1)\rho(Y')\rho(X_3)$. Then by Lemma 3.3, $\phi(U') = Z_1Y'X_3$. Note that $Z_2Z_3Z_4 \to_R Z_2Y_2 \to^*_R Y'$ (for $Y' \in \mathrm{Irr}(R)$). Therefore

$$U = (Z_1Z_2)(Z_3Z_4)X_3 \to_R V_1 = (Z_1Z_2)Y_2X_3 \to^*_R \phi\big(U'\big),$$

and thus the lemma holds.

Case 2. Suppose $\rho(X_1X_2X_3) \neq \rho(X_1X_2)\rho(X_3)$. By Lemma 3.2, there are $Z_1, Z_4 \in A^*$ and $Z_2, Z_3 \in A^+$ with $X_1X_2 = Z_1Z_2$, $X_3 = Z_3Z_4$ and $Z_2Z_3 = W_0$ such that $\rho(X_1X_2X_3) = \rho(Z_1)s\rho(Z_4)$. Since $W_0 \in \mathrm{Irr}(R)$, we must have $\|Z_2\| < \|X_2\|$ (if not, then X_2 would be a subword of W_0 and $W_0 \notin \mathrm{Irr}(R)$ because $X_2 \to Y_2 \in R$). Let $X_2 = X_4Z_2$ for some $X_4 \in A^+$. Then $Z_1 = X_1X_4$.

SubCase 2.1. Suppose that $\rho(X_1X_4) = \rho(X_1)\rho(X_4)$. Note that $\rho(X_4Z_2Z_3) \to \rho(Y') \in R_S$ where $X_4Z_2Z_3 \to^*_R Y'$ and $Y' \in \mathrm{Irr}(R)$ (a rule of the form ($\mathcal{C}4$)). Furthermore $\rho(X_4Z_2Z_3) = \rho(X_4)s$ and $\rho(U) = \rho(X_1X_2X_3) = \rho(Z_1)s\rho(Z_4) = \rho(X_1X_4)s\rho(Z_4) = \rho(X_1)\rho(X_4)s\rho(Z_4) \to_{R_S} \rho(X_1)\rho(Y')\rho(Z_4)$. Let $U' = \rho(X_1)\rho(Y')\rho(Z_4)$. Then by Lemma 3.3, $\phi(U') = X_1Y'Z_4$. As before $X_4Z_2Z_3 \to_R Y_2Z_3 \to^*_R Y'$ (recall that $X_2 = X_4Z_2$) and

$$U = (Z_1Z_2)(Z_3Z_4) = (X_1X_4Z_2)(Z_3Z_4) \to_R X_1Y_2Z_3Z_4 = V_1 \to^*_R \phi\big(U'\big).$$

So the lemma holds.

SubCase 2.2. Suppose $\rho(X_1X_4) \neq \rho(X_1)\rho(X_4)$. By Lemma 3.2, there are $Z_5, Z_8 \in A^*$ and $Z_6, Z_7 \in A^+$ with $X_1 = Z_5Z_6$, $X_4 = Z_7Z_8$ and $Z_6Z_7 = W_0$ such that $\rho(X_1X_4) = \rho(Z_5)s\rho(Z_8)$. Note that

$$U = X_1X_2X_3 = Z_5Z_6(Z_7Z_8Z_2)Z_3Z_4,$$

and $X_2 = Z_7Z_8Z_2$. Also $\rho(Z_6(Z_7Z_8Z_2)Z_3) \to \rho(Y') \in R_S$ where $Z_6(Z_7Z_8Z_2)Z_3 \to_R^* Y'$ and $Y' \in \operatorname{Irr}(R)$ (a rule of the form $(\mathcal{C}5)$). Since $\rho(Z_5Z_6Z_7Z_8) = \rho(X_1X_4) = \rho(Z_5)s\rho(Z_8) = \rho(Z_5Z_6Z_7)\rho(Z_8)$, by Lemma 3.1, $\rho(Z_6Z_7Z_8) = \rho(Z_6Z_7)\rho(Z_8) = s\rho(Z_8)$. So $\rho(Z_6(Z_7Z_8Z_2)Z_3) = \rho(Z_6Z_7Z_8)s = s\rho(Z_8)s$ and $s\rho(Z_8)s \to \rho(Y') \in R_S$.

Recall that

$$\begin{aligned}\rho\big(Z_5Z_6(Z_7Z_8Z_2)Z_3Z_4\big) = \rho(U) &= \rho(X_1X_2X_3)\\ &= \rho(Z_1)s\rho(Z_4)\\ &= \rho(X_1X_4)s\rho(Z_4)\\ &= \rho(Z_5)s\rho(Z_8)s\rho(Z_4).\end{aligned}$$

Therefore $\rho(U) = \rho(Z_5)s\rho(Z_8)s\rho(Z_4) \to_{R_S} \rho(Z_5)\rho(Y')\rho(Z_4)$. Let $U' = \rho(Z_5)\rho(Y')\rho(Z_4)$. Then by Lemma 3.3, $\phi(U') = Z_5Y'Z_4$. As before $Z_6(Z_7Z_8Z_2)Z_3 \to_R Z_6Y_2Z_3 \to_R^* Y'$ (recall that $X_2 = X_4Z_2 = Z_7Z_8Z_2$) and

$$U = Z_5Z_6(Z_7Z_8Z_2)Z_3Z_4 \to_R Z_5Z_6Y_2Z_3Z_4 = V_1 \to_R^* \phi\big(U'\big).$$

The proof of this lemma is complete. □

By Corollary 2.3, Lemmas 3.9, 3.8, 3.7, 3.5, 3.3 and 3.4, we have shown that $[B\ ;\ R_S]$ is a semigroup presentation for S, R_S is a finite complete rewriting system and $W_0 \to_{R_S}^* s$. Now note that if $U' \to V' \in R_S$ is a rule of the form $(\mathcal{C}2)$, $(\mathcal{C}3)$, $(\mathcal{C}4)$, $(\mathcal{C}5)$ or $(\mathcal{C}6)$, then $\|U'\| > 1$. From this we conclude that $s \in \operatorname{Irr}(R_S)$. Note also that if $X \in A^+$, $X \neq W_0$ and $\|X\| > 1$, then $\|\rho(X)\| > 1$. Therefore if $X \to Y \in R$ with $\|X\| > 1$, then $\rho(X) \to \rho(Y) \in R_S$ and $\|\rho(X)\| > 1$ (a rule of the form $(\mathcal{C}1)$). This implies that if $a \in A \cap \operatorname{Irr}(R)$, then $a \in \operatorname{Irr}(R_S)$.

Thus we have proved the following theorem.

Theorem 3.10. *Let $[A\ ;\ R]$ be a finitely presented semigroup presentation for S for which R is complete. Let $W_0 \in A^+$ be such that $\|W_0\| > 1$ and $W_0 \in \operatorname{Irr}(R)$. Now let s be a symbol that does not appear in A and set $B = A \cup \{s\}$. Then there is complete rewriting system R_S such that $[B\ ;\ R_S]$ is a finitely presented semigroup presentation for S and $W_0 \to_{R_S}^* s$. Furthermore $s \in \operatorname{Irr}(R_S)$, and $a \in \operatorname{Irr}(R_S)$ for all $a \in A \cap \operatorname{Irr}(R)$.*

4. Reduction process

In this section we will make further refinements and improvements (we call them reductions) to Theorem 3.10. The reason for such reductions is that we need a finitely presented semigroup presentation for S, which can be handled easily.

Let S be a semigroup and T be a large subsemigroup of S. Let $[A\ ;\ R]$ be a finitely presented semigroup presentation for S for which R is complete. Let $S \setminus T = \{[W_1]_R, [W_2]_R, \dots, [W_n]_R\}$ with $W_i \in \operatorname{Irr}(R)$ and $\|W_1\| \leqslant \|W_2\| \leqslant \cdots \leqslant \|W_n\|$. Suppose that $\|W_1\| = \|W_2\| = \cdots = \|W_{i_0-1}\| = 1$ and $\|W_{i_0}\| > 1$. By Theorem 3.10, there is a finitely presented semigroup presentation $[B_{i_0}\ ;\ R_{i_0}]$ for S such that $B = A \cup \{s_{i_0}\}$ for some symbol s_{i_0} that does not appear in A, R_{i_0} is complete, $W_{i_0} \to_{R_{i_0}}^* s_{i_0}$ and $W_1, W_2, \dots, W_{i_0-1}, s_{i_0} \in \operatorname{Irr}(R_{i_0})$.

Now in this new semigroup presentation $[B_{i_0}\ ;\ R_{i_0}]$, we see that

$$S \setminus T = \left\{[W_1]_{R_{i_0}}, [W_2]_{R_{i_0}}, \ldots, [W_{i_0-1}]_{R_{i_0}}, [s_{i_0}]_{R_{i_0}}, \left[W'_{i_0+1}\right]_{R_{i_0}}, \ldots, \left[W'_n\right]_{R_{i_0}}\right\},$$

with $W_1, \ldots, W_{i_0-1}, s_{i_0}, W'_{i_0+1}, \ldots, W'_n \in \mathrm{Irr}(R_{i_0})$.

Note that this process can be continued (in at most n steps) until we obtain a finitely presented semigroup presentation $[B_n\ ;\ R_n]$ for S such that R_n is complete and $S \setminus T = \{[s_1]_{R_n}, [s_2]_{R_n}, \ldots, [s_n]_{R_n}\}$ with $s_1, \ldots, s_n \in \mathrm{Irr}(R_n) \cap B_n$.

In fact by a standard procedure described in [1, Section 2.2], we may further assume that for each $X \to Y \in R_n$, we have $Y \in \mathrm{Irr}(R)$, and for each $X \to Y \in R_n$, there is no $X' \in B_n^+$ for which $X \to_{R_n} X'$ by any rule in $R_n \setminus \{X \to Y\}$. This is the form of the presentation that we will use.

5. The main result

Let S be a semigroup and T be a large subsemigroup of S. As stated in Section 4, we may assume that $[A\ ;\ R]$ is a finitely presented semigroup presentation for S for which R is complete and

(Q1) $S \setminus T = \{[s_1]_R, [s_2]_R, \ldots, [s_n]_R\}$ with $s_1, \ldots, s_n \in \mathrm{Irr}(R) \cap A$,
(Q2) for each $X \to Y \in R$, we have $Y \in \mathrm{Irr}(R)$,
(Q3) for each $X \to Y \in R$, there is no $X' \in A^+$ for which $X \to_R X'$ by any rule in $R \setminus \{X \to Y\}$.

In order to show that T has a finite complete rewriting system, we shall find a 5-tuple $(B, R_T, A(T), \phi, \rho)$ that has Property $\mathcal{R}$ relative to $[A\ ;\ R]$ and apply Theorem 2.2.

Let $A_1 = \{a \in A\colon\ [a]_R \in T\}$ and $A_S = \{s_1, s_2, \ldots, s_n\}$. Note that in general the union of A_S and A_1 is not necessary equal to A. This is because there might exist an element $b \in A$ such that $[b]_R \in S \setminus T$. If this happens, we would have $b \to_R^* s_i$ for some i.

Lemma 5.1. *Let $X \to Y \in R$ with $[X]_R \in T$. Then*

(a) *if $W \in A^+$ is a subword of X and $[W]_R \in S \setminus T$, then $W = s_i$ for some i,*
(b) *if $W \in A^+$ is a subword of Y and $[W]_R \in S \setminus T$, then $W = s_i$ for some i.*

Proof. (a) Suppose $W \notin A_S$. Then by (Q1) $W \to_R^* s_i$ for some i. To be precise there is a $W_1 \in A^+$ such that $W \to_R W_1 \to_R^* s_i$. Let $W \to_R W_1$ by the rule $X_1 \to Y_1$. Since $[X]_R \in T$, we cannot have $W = X$. Therefore $X_1 \neq X$ and $X_1 \to Y_1 \in R \setminus \{X \to Y\}$. Let $X = Z_1 W Z_2$ where $Z_1, Z_2 \in A^*$. Then $X \to_R Z_1 W_1 Z_2$ by the rule $X_1 \to Y_1$, contrary to (Q3). Hence $W = s_i$ for some i.

(b) can be proved similarly using the fact that $Y \in \mathrm{Irr}(R)$ (see (Q2)). □

We now begin to define the 5-tuple $(B, R_T, A(T), \phi, \rho)$. Let $A(T)(0)$ be the set of all $W \in A^+$, such that $[W]_R \in T$, and if X_1 is a subword W with $[X_1]_R \in S \setminus T$, then $\|X_1\| = 1$ and $X_1 \in A_S$. In other word,

$$A(T)(0) = \bigl\{W \in (A_1 \cup A_s)^+\colon\ [W]_R \in T,\ \text{and } W \text{ does not contain any subword}$$
$$X_1 \text{ with } [X_1]_R \in S \setminus T \text{ and } \|X_1\| > 1\bigr\}.$$

The following lemma is clear from the definition of $A(T)(0)$.

Lemma 5.2. *Let $W \in A(T)(0)$ and W' be a subword of W. If $[W']_R \in T$, then $W' \in A(T)(0)$.*

Next let

$$F_1 = A_1,$$
$$F_2 = \{sb\colon\ s \in A_S,\ b \in A_1 \cup A_S \text{ and } [sb]_R \in T\},$$
$$F_3 = \{as\colon\ a \in A_1,\ s \in A_S \text{ and } [as]_R \in T\},$$
$$F_4 = \{sbs'\colon\ s, s' \in A_S,\ b \in A_1 \cup A_S \text{ and } [sb]_R, [bs']_R, [sbs']_R \in T\}.$$

It is not hard to see that if $W \in F_1 \cup F_2 \cup F_3 \cup F_4$, then $[W]_R \in T$. Furthermore $F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq A(T)(0)$. For convenience, for each $G \subseteq A^+$ and $X \in A^+$, we set $XG = \{XW\colon\ W \in G\}$.

Now we shall define $A(T)$. Let $A(T)(1) = F_1 \cup F_2 \cup F_3 \cup F_4$ and for each $i \geqslant 1$, let

$$A(T)(i+1) = \left(\bigcup_{a \in A_1} \big(\big(aA(T)(i)\big) \cap A(T)(0)\big)\right) \cup \left(\bigcup_{X \in F_2} \big(\big(XA(T)(i)\big) \cap A(T)(0)\big)\right).$$

Set $A(T) = \bigcup_{i \geqslant 1} A(T)(i)$. In the following lemma we shall prove some properties of $A(T)$.

Lemma 5.3.

(a) $A(T) = A(T)(0)$.
(b) $A(T)$ *contains the set* $\{W \in \mathrm{Irr}(R)\colon\ [W]_R \in T\}$.
(c) *Let* $X \to Y \in R$ *with* $[X]_R \in T$. *Then* $X, Y \in A(T)$.

Proof. (a) Clearly $A(T) \subseteq A(T)(0)$. Let $W \in A(T)(0)$. We shall prove by induction on $\|W\|$ that $W \in A(T)$.

Suppose $\|W\| = 1$. Since $[W]_R \in T$, we must have $W \in A_1$. So $W \in A(T)(1) \subseteq A(T)$.

Suppose $\|W\| = 2$. Then $W = a'a$, or $W = as$, or $W = sa$, or $W = ss'$ ($a, a' \in A_1$, $s, s' \in A_S$). If $W = a'a$, then $W \in (a'A(T)(1)) \cap A(T)(0) \subseteq A(T)(2) \subseteq A(T)$. If $W = as$, then $W \in F_3 \subseteq A(T)(1) \subseteq A(T)$. If $W = sa$ or $W = ss'$, then $W \in F_2 \subseteq A(T)(1) \subseteq A(T)$.

Suppose $\|W\| \geqslant 3$. Assume that it is true for all $W' \in A(T)(0)$ with $\|W'\| < \|W\|$.

If W begins with a letter $a \in A_1$, say $W = aW'$ where $W' \in A^+$, then $\|W'\| \geqslant 2$. Note that $[W']_R \in T$, for if $[W']_R \in S \setminus T$, then by the definition of $A(T)(0)$, $W' \in A_S$ and $\|W'\| = 1$, contrary to the fact that $\|W'\| \geqslant 2$. Therefore by Lemma 5.2, $W' \in A(T)(0)$. By induction, $W' \in A(T)$. Let $W' \in A(T)(i)$ for some $i \geqslant 1$. Then $W \in (aA(T)(i)) \cap A(T)(0) \subseteq A(T)(i+1) \subseteq A(T)$.

If W begins with a letter $s \in A_S$, say $W = sbW'$ where $b \in A_1 \cup A_S$ and $W' \in A^+$, then $\|W'\| \geqslant 1$. If $[W']_R \in S \setminus T$, then by the definition of $A(T)(0)$, $W' = s'$ for some $s' \in A_S$, and $W = sbs'$. Since $W \in A(T)(0)$, we have $[sb]_R, [bs']_R, [sbs']_R \in T$ (definition of $A(T)(0)$). This means $W \in F_4 \subseteq A(T)(1) \subseteq A(T)$.

If $[W']_R \in T$, then by Lemma 5.2, $W' \in A(T)(0)$. By induction, $W' \in A(T)$. Let $W' \in A(T)(i)$ for some $i \geqslant 1$. Then $W \in (sbA(T)(i)) \cap A(T)(0) \subseteq A(T)(i+1) \subseteq A(T)$.

The proof of part (a) of the lemma is complete.

Part (b) follows from part (a) and the fact that $A(T)(0)$ contains the set $\{W \in \mathrm{Irr}(R)\colon\ [W]_R \in T\}$.

(c) By part (a) of Lemma 5.1, we conclude that X does not contain any subword X_1 with $[X_1]_R \in S \setminus T$ and $X_1 \notin A_S$. So $X \in A(T)(0) = A(T)$. Similarly by part (b) of Lemma 5.1, $Y \in A(T)$. □

Now we shall define the set B and the homomorphism ϕ. Let

$$C_R = \{c_{as}\colon\ [as]_R \in T \text{ with } a \in A_1 \text{ and } s \in A_S\},$$
$$C_{L_1} = \{c_{sa}\colon\ [sa]_R \in T \text{ with } a \in A_1 \text{ and } s \in A_S\},$$

$$C_{L_2} = \{c_{ss'}\colon [ss']_R \in T \text{ with } s, s' \in A_S\},$$
$$C_{M_1} = \{c_{s'as}\colon [s'as]_R, [s'a]_R, [as]_R \in T \text{ with } a \in A_1 \text{ and } s, s' \in A_S\},$$
$$C_{M_2} = \{c_{ss's''}\colon [ss's'']_R, [ss']_R, [s's'']_R \in T \text{ with } s, s', s'' \in A_S\}.$$

Set $C = C_R \cup C_{L_1} \cup C_{L_2} \cup C_{M_1} \cup C_{M_2}$ and $B = A_1 \cup C$. Since A_1 and A_S are finite, it is not hard to see that B is finite. Let $\phi_1 : B \to A^+$ be defined by $\phi_1(a) = a$ for all $a \in A_1$ and $\phi_1(c_u) = u$ for all $c_u \in C$ (for example $\phi_1(c_{as}) = as$ for $c_{as} \in C_R$). Clearly ϕ_1 can be extended to a homomorphism $\phi : B^+ \to A^+$ by defining $\phi(U') = \phi_1(b_1)\dots\phi_1(b_l)$ for all $U' = b_1 \dots b_l \in B^+$. Furthermore $[\phi(U')]_R \in T$ for all $U' \in B^+$. For convenience, we may define $\phi(\epsilon_B) = \epsilon_A$ where ϵ_B and ϵ_A are empty words in B^* and A^*, respectively. The following lemma is obvious.

Lemma 5.4. *For all $U' \in B^+$, $\|\phi(U')\| \geqslant \|U'\|$.*

We define $\rho : A(T) \to B^+$ as follows:
Let $W \in A(T)$.

(a) Suppose $W \in A(T)(1)$. If $W \in F_1$, then set $\rho(W) = W$. If $W \in F_2 \cup F_3 \cup F_4$, set $\rho(W) = c_W$ (for example if $W = as \in F_3$, then $\rho(W) = c_{as}$).
(b) Suppose $W \in A(T)(i+1)$ for some $i \geqslant 1$. Then $W = aW_1$ or $W = sbW_1$ ($a \in A_1$, $s \in A_S$, $b \in A_1 \cup A_S$ and $W_1 \in A(T)(i)$). If the former holds, set $\rho(W) = a\rho(W_1)$. If the latter holds, set $\rho(W) = c_{sb}\rho(W_1)$.

The function ρ is well-defined can be easily proved by observing that a word from $A(T)(i+1)$ is obtained in a unique way from a unique word from $A(T)(i)$. As for the homomorphism ϕ, we may define $\rho(\epsilon_A) = \epsilon_B$.

Lemma 5.5. *Let $U \in A(T)(l)$ for some $l \geqslant 1$. Then $\rho(U) = b'_1 \dots b'_l$ where $b'_i \in B$. Furthermore if $l > 1$, then $b'_i \in A_1 \cup C_{L_1} \cup C_{L_2}$ for all $1 \leqslant i \leqslant l-1$.*

Proof. We prove by induction on l. Suppose $l = 1$. Then $\rho(U) = b'_1$ by the definition of ρ. Suppose $l > 1$. Assume that it is true for all l' with $l' < l$.

Since $U \in A(T)(l)$, we have either $U = aU_1$ or $U = sbU_1$ ($a \in A_1$, $sb \in F_2$ and $U_1 \in A(T)(l-1)$). Suppose $U = aU_1$. Then $\rho(U) = a\rho(U_1)$. This means $b'_1 = a \in A_1$. By induction $\rho(U_1) = b'_2 \dots b'_l$. Furthermore if $l-1 > 1$ (i.e. $l > 2$), then $b'_2, \dots, b'_{l-1} \in A_1 \cup C_{L_1} \cup C_{L_2}$.

Suppose $U = sbU_1$. Then $\rho(U) = c_{sb}\rho(U_1)$. This means $b'_1 = c_{sb} \in C_{L_1} \cup C_{L_2}$. By induction $\rho(U_1) = b'_2 \dots b'_l$. Furthermore if $l-1 > 1$ (i.e. $l > 2$), then $b'_2, \dots, b'_{l-1} \in A_1 \cup C_{L_1} \cup C_{L_2}$.

Hence in either cases the lemma holds. □

Lemma 5.6. *$\phi(\rho(U)) = U$ for all $U \in A(T)$.* (Property (P5).)

Proof. We just need to show that for all $i \geqslant 1$, if $U \in A(T)(i)$, then $\phi(\rho(U)) = U$.

Suppose $U \in A(T)(1)$. If $U \in F_1$, then $\rho(U) = U$ and $\phi(\rho(U)) = U$. If $U \in F_2 \cup F_3 \cup F_4$, then $\rho(U) = c_U$ and $\phi(\rho(U)) = \phi(c_U) = U$. Assume that it is true for all $U' \in A(T)(i)$.

Let $U \in A(T)(i+1)$. Then $U = aU_1$ or $U = sbU_1$ where $a \in A_1$, $sb \in F_2$ and $U_1 \in A(T)(i)$. If the former holds, then $\rho(U) = a\rho(U_1)$ and by induction $\phi(\rho(U)) = a\phi(\rho(U_1)) = aU_1 = U$. If the latter holds, then $\rho(U) = c_{sb}\rho(U_1)$, and by induction $\phi(\rho(U)) = \phi(c_{sb})\phi(\rho(U_1)) = sbU_1 = U$. Hence the lemma holds. □

Lemma 5.7. *Let $U' = b'_1 \dots b'_l \in B^+$ where $b'_i \in A_1 \cup C_{L_1} \cup C_{L_2}$ for all $1 \leqslant i \leqslant l-1$ and $b'_l \in B$. If $\phi(U') \in A(T)$, then $\phi(U') \in A(T)(l)$ and $\rho(\phi(U')) = U'$.*

Proof. We prove by induction on l. Suppose $l = 1$. If $b'_1 = a \in A_1$, then $\phi(b'_1) = a$, and $\rho(\phi(b'_1)) = b'_1$. If $b'_1 = c_z \in C$, then $\phi(b'_1) = z \in A(T)(1)$, and $\rho(\phi(b'_1)) = b'_1$.

Suppose $l > 1$. Assume that it is true for all l' with $l' < l$. Let $U' = b'_1 U'_1$ where $U'_1 = b'_2 \dots b'_l$. By induction, $\phi(U'_1) \in A(T)(l-1)$ and $\rho(\phi(U'_1)) = U'_1$. Since $b'_1 \in A_1 \cup C_{L_1} \cup C_{L_2}$, we have $\phi(b'_1) \in A_1 \cup F_2$. Therefore $\phi(U') = \phi(b'_1)\phi(U'_1) \in A(T)(l)$, and $\rho(\phi(U')) = b'_1 \rho(\phi(U'_1)) = b'_1 U'_1 = U'$. Hence the lemma holds. □

We are now ready to define the rules in R_T. Let us begin by recalling some of the results of Lemma 5.3. For each $X \to Y \in R$ with $[X]_R \in T$, we have $X, Y \in A(T)$ (part (c) of Lemma 5.3). Furthermore if $Y \in \mathrm{Irr}(R)$ and $[Y]_R \in T$, then $Y \in A(T)$ (part (b) of Lemma 5.3). Recall that $C = C_R \cup C_{L_1} \cup C_{L_2} \cup C_{M_1} \cup C_{M_2}$, ϵ_A is the empty word in A^*, ϕ is a homomorphism of B^+ into A^+ (furthermore $[\phi(U')]_R \in T$ for all $U' \in B^+$), and ρ is a function of $A(T)$ into B^+. As R is a finite complete rewriting system, $\mathrm{Left}(R) = \{X \in A^+ \colon\ X \to Y \in R\}$ is finite. Let $N = (\max_{X \in \mathrm{Left}(R)} \|X\|) + 4$. The rules are grouped into two forms, $(\mathcal{D}1)$ and $(\mathcal{D}2)$:

($\mathcal{D}$1) for each $U' \in B^+$ with $\|\phi(U')\| \leqslant N$ and $\phi(U') \notin \mathrm{Irr}(R)$, put $U' \to \rho(\overline{\phi(U')})$ in R_T where $\phi(U') \to_R^* \overline{\phi(U')}$ and $\overline{\phi(U')} \in \mathrm{Irr}(R)$;

($\mathcal{D}$2) for each $U' \in B^+$ with $\|U'\| = 2$, $\phi(U') \in A(T)$ and $U' \neq \rho(\phi(U'))$, put $U' \to \rho(\phi(U'))$ in R_T.

Note that the number of rules of the form $(\mathcal{D}1)$ that we put in R_T is finite, for by Lemma 5.4 the length of U' is bounded and B is finite. Similarly the number of rules of the form $(\mathcal{D}2)$ that we put in R_T is also finite. Therefore R_T is finite and $[B\ ;\ R_T]$ is finitely presented. Note that by the main result in [4, Theorem 6.1], one can get a finite presentation for T by taking N sufficiently large.

Lemma 5.8. *Let $U', V' \in B^+$. If $U' \to_{R_T} V'$ by a rule of the form $(\mathcal{D}2)$, then $\phi(U') = \phi(V')$. Furthermore either*

(i) *the number of elements in $C_R \cup C_{M_1} \cup C_{M_2}$ which appear as letters in the word V' is less than that in the word U', or*

(ii) *the number of elements in $C_R \cup C_{M_1} \cup C_{M_2}$ which appear as letters in the word V' is the same as that in the word U', $\|U'\| = \|V'\|$, and there is an element in $C_R \cup C_{M_1} \cup C_{M_2}$ in which it "moves" further right in the resulting word V' than it is in the word U' (the element may have changed).*

Proof. Let $U' \to_{R_T} V'$ by the rule $X' \to \rho(\phi(X'))$ where $X' \in B^+$, $\|X'\| = 2$, $\phi(X') \in A(T)$ and $X' \neq \rho(\phi(X'))$. By Lemma 5.6, $\phi(\rho(\phi(X'))) = \phi(X')$. Since ϕ is a homomorphism, we have $\phi(U') = \phi(V')$. Now we will show that either (i) or (ii) holds.

If the first letter that appears in X' is not from $C_R \cup C_{M_1} \cup C_{M_2}$, then by Lemma 5.7, $\rho(\phi(X')) = X'$, a contradiction. So we may assume that the first letter that appears in X' is from $C_R \cup C_{M_1} \cup C_{M_2}$.

By Lemma 5.5, $\rho(\phi(X'))$ has at most one letter from $C_R \cup C_{M_1} \cup C_{M_2}$, which is then the last letter. If $\rho(\phi(X'))$ has no letter from $C_R \cup C_{M_1} \cup C_{M_2}$, then (i) holds.

Suppose $\rho(\phi(X'))$ has a letter from $C_R \cup C_{M_1} \cup C_{M_2}$. Then $\phi(X') = \phi(\rho(\phi(X')))$ ends with a letter from A_S. Let $X' = cy$ where $c \in C_R \cup C_{M_1} \cup C_{M_2}$ and $y \in B$. Then $y \notin A_1 \cup C_{L_1}$. If $y \in C_R \cup C_{M_1} \cup C_{M_2}$, then (i) holds. So we may assume that $y \in C_{L_2}$. Let $y = c_{s'''s''''}$. If $c = c_{as}$, then $\rho(\phi(X')) = ac_{ss'''s''''}$, if $c = c_{sas'}$, then $\rho(\phi(X')) = c_{sa}c_{s's'''s''''}$, and if $c = c_{ss's''}$, then $\rho(\phi(X')) = c_{ss'}c_{s''s'''s''''}$. Therefore $\|\rho(\phi(X'))\| = \|X'\|$ and (ii) holds. □

Lemma 5.9. *$U' \to_{R_T}^* \rho(\phi(U'))$ for all $U' \in B^+$ with $\phi(U') \in A(T)$.* (Property (P6).)

Proof. Let $U' = b'_1 \dots b'_l \in B^+$ where $b'_i \in B$ for all i. If $b'_i \in A_1 \cup C_{L_1} \cup C_{L_2}$ for all $1 \leqslant i \leqslant l-1$, then by Lemma 5.7, $\rho(\phi(U')) = U'$. Hence $U' \to_{R_T}^* \rho(\phi(U'))$.

So we may assume that $b'_i \in C_R \cup C_{M_1} \cup C_{M_2}$ for some $1 \leqslant i \leqslant l-1$. By Lemma 5.2 and part (a) of Lemma 5.3, $\phi(b'_i b'_{i+1}) \in A(T)$. By Lemma 5.5, $b'_i b'_{i+1} \neq \rho(\phi(b'_i b'_{i+1}))$. Therefore $b'_i b'_{i+1} \to \rho(\phi(b'_i b'_{i+1}))$ is a rule of the form $(\mathcal{D}2)$ in R_T.

Let $V' = b'_1 \dots b'_{i-1}\rho(\overline{\phi(b'_i b'_{i+1})})b'_{i+2}\dots b'_l$. Then $U' \to_{R_T} V'$, and by Lemma 5.6, $\phi(U') = \phi(b'_1 \dots b'_l) = \phi(V')$. By Lemma 5.8, we conclude that after applying rules of the form $(\mathcal{D}2)$ a finite number of times, there is a $U'' = d'_1 \dots d'_r \in B^+$ with $d'_i \in A_1 \cup C_{L_1} \cup C_{L_2}$ for all $1 \leqslant i \leqslant r-1$ and $d'_r \in B$, such that $U' \to^*_{R_T} U''$ and $\phi(U') = \phi(U'')$. Again by Lemma 5.7, $\rho(\phi(U'')) = U''$. So $U' \to^*_{R_T} \rho(\phi(U'')) = \rho(\phi(U'))$. □

Lemma 5.10. *Let $U' \in B^+$ and $V \in A^+$. If $\phi(U') \to_R V$, then there is a $V' \in B^+$ such that $U' \to_{R_T} V'$ by a rule of the form $(\mathcal{D}1)$, and $V \to^*_R \phi(V')$.*

Proof. Let $U' = b'_1 \dots b'_l$ where $b'_i \in B$, and $\phi(U') \to_R V$ by a rule $X \to Y$ in R. Then for some non-negative integers j_1, j_2, X is a subword of $\phi(b'_{j_1} \dots b'_{j_1+j_2})$. We may assume that X is not a subword of $\phi(b'_{j_1+1} \dots b'_{j_1+j_2})$ or $\phi(b'_{j_1} \dots b'_{j_1+j_2-1})$. Since $\phi(b'_{j_1})$ and $\phi(b'_{j_1+j_2})$ are at most of length 3, we deduce that $\|\phi(b'_{j_1} \dots b'_{j_1+j_2})\| \leqslant \|X\| + 4 \leqslant N$. So $b'_{j_1} \dots b'_{j_1+j_2} \to \rho(\overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})})$ is a rule of the form $(\mathcal{D}1)$ in R_T, where $\phi(b'_{j_1} \dots b'_{j_1+j_2}) \to^*_R \overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})}$, $\phi(b'_{j_1} \dots b'_{j_1+j_2}) \notin \mathrm{Irr}(R)$ and $\overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})} \in \mathrm{Irr}(R)$. Set $V' = b'_1 \dots b'_{j_1-1}\rho(\overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})})b'_{j_1+j_2+1} \dots b'_l$. Then $U' \to_{R_T} V'$.

By Lemma 5.6, $\phi(V') = \phi(b'_1 \dots b'_{j_1-1})(\overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})})\phi(b'_{j_1+j_2+1} \dots b'_l)$. Let $\phi(b'_{j_1} \dots b'_{j_1+j_2}) = W_1 X W_2$ where $W_1, W_2 \in A^*$ (we allow W_1, W_2 to be empty word). Then $\phi(b'_{j_1} \dots b'_{j_1+j_2}) \to_R W_1 Y W_2 \to^*_R \overline{\phi(b'_{j_1} \dots b'_{j_1+j_2})}$. Hence $V = \phi(b'_1 \dots b'_{j_1-1})(W_1 Y W_2)\phi(b'_{j_1+j_2+1} \dots b'_l) \to^*_R \phi(V')$. □

Lemma 5.11. *For each $U' \in B^+$ there is a $U'' \in B^+$ such that $\phi(U'') \in A(T)$ and $U' \to^*_{R_T} U''$.* (Property (P4).)

Proof. We shall prove by induction on $d_R(\phi(U'))$. Suppose $d_R(\phi(U')) = 0$. Then $\phi(U') \in A(T)$ (part (b) of Lemma 5.3). So we may choose $U'' = U'$. Suppose $d_R(\phi(U')) > 0$. Assume that it is true for all $U'_1 \in B^+$ with $d_R(\phi(U'_1)) < d_R(\phi(U'))$.

Since $d_R(\phi(U')) > 0$, there is a $V \in A^+$ such that $\phi(U') \to_R V$. By Lemma 5.10, there is a $V' \in B^+$ such that $U' \to_{R_T} V'$ and $V \to^*_R \phi(V')$. Therefore $\phi(U') \to^*_R \phi(V')$, and $d_R(\phi(V')) < d_R(\phi(U'))$. By induction, there is a $U'' \in B^+$ such that $\phi(U'') \in A(T)$ and $V' \to^*_{R_T} U''$. Hence $U' \to^*_{R_T} U''$. □

Lemma 5.12. *Suppose $U' \to_{R_T} V'$ by one of the rules of the form $(\mathcal{D}1)$. Then $\phi(U') \neq \phi(V')$ and $\phi(U') \to^*_R \phi(V')$.*

Proof. Suppose $U' \to_{R_T} V'$ by a rule of the form $(\mathcal{D}1)$, say $X' \to Y'$. Then $\|\phi(X')\| \leqslant N$, $\phi(X') \notin \mathrm{Irr}(R)$, and $Y' = \rho(\overline{\phi(X')})$, where $\phi(X') \to^*_R \overline{\phi(X')}$ and $\overline{\phi(X')} \in \mathrm{Irr}(R)$.

Let $U' = W'_1 X' W'_2$ where $W'_1, W'_2 \in B^*$ (we allow W'_1 and W'_2 to be empty word). Note that $V' = W'_1 \rho(\overline{\phi(X')}) W'_2$. By Lemma 5.6 and the fact that ϕ is a homomorphism, we must have $\phi(V') = \phi(W'_1)\overline{\phi(X')}\phi(W'_2) \neq \phi(U')$, for otherwise we would have $\phi(X') = \overline{\phi(X')}$. Furthermore $\phi(U') \to^*_R \phi(V')$. □

Lemma 5.13. *There does not exist an infinite reduction sequence*

$$U'_1 \to_{R_T} U'_2 \to_{R_T} U'_3 \to_{R_T} \cdots,$$

of words from B^+ such that $\phi(U'_1) = \phi(U'_2) = \phi(U'_3) = \cdots$. (Property (P3).)

Proof. Suppose that such a sequence exists.

Since $\phi(U'_i) = \phi(U'_{i+1})$, by Lemma 5.12, we conclude that $U'_i \to_{R_T} U'_{i+1}$ by a rule of the form $(\mathcal{D}2)$. By Lemma 5.8, the number of elements in $C_R \cup C_{M_1} \cup C_{M_2}$ which appear as letters in the word U'_{i+1} is either less than that in the word U'_i, or the number are the same and $\|U'_i\| = \|U'_{i+1}\|$, but it 'moves' to the right. So we deduce that there is an integer i_0 such that for all $i \geqslant i_0$, the number of elements in $C_R \cup C_{M_1} \cup C_{M_2}$ which appear as letters in the word U'_i is the same as in the word U'_{i+1}, and $\|U'_i\| = \|U'_{i+1}\|$. So a letter (an element in $C_R \cup C_{M_1} \cup C_{M_2}$) in the word U'_i will 'move' further right in the word U'_{i+1}. But this process cannot be continued indefinitely as $\|U'_i\| = \|U'_{i+1}\|$. We have obtained a contradiction. □

Lemma 5.14. *For any $U', V' \in B^+$ with $U' \to^*_{R_T} V'$, we have $\phi(U') \to^*_R \phi(V')$.* (Property (P2).)

Proof. It is sufficient to show $U' \to_{R_T} V'$ with $U', V' \in B^+$ implies that $\phi(U') \to^*_R \phi(V')$.

Suppose $U' \to_{R_T} V'$ by a rule of the form ($\mathcal{D}1$). By Lemma 5.12, $\phi(U') \to^*_R \phi(V')$. Suppose $U' \to_{R_T} V'$ by a rule of the form ($\mathcal{D}2$). By Lemma 5.8, $\phi(U') = \phi(V')$, and thus $\phi(U') \to^*_R \phi(V')$. □

Lemma 5.15. *For any $U \in A(T)$ and $V_1 \in A^+$ with $U \to_R V_1$, there is a $U' \in B^+$ such that $U \to_R V_1 \to^*_R \phi(U')$ and $\rho(U) \to_{R_T} U'$.* (Property (P1).)

Proof. By Lemma 5.6, $U = \phi(\rho(U))$. By Lemma 5.10, there is a $U' \in B^+$ such that $\rho(U) \to_{R_T} U'$ by a rule of the form ($\mathcal{D}1$), and $V_1 \to^*_R \phi(U')$. The lemma follows. □

Proof of Theorem 1.1. Let $[A\ ;\ R]$ be a finitely presented semigroup presentation for S for which R is complete. By the reduction process described in Section 4, we may assume that (Q1), (Q2) and (Q3) hold. Now the 5-tuple $(B, R_T, A(T), \phi, \rho)$ has been defined. By Theorem 2.2, it is sufficient to show that $(B, R_T, A(T), \phi, \rho)$ has Property $\mathcal{R}$ relative to $[A\ ;\ R]$. This has been done in Lemmas 5.6, 5.9, 5.11, 5.13, 5.14 and 5.15. □

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