Existence, Uniqueness, and Asymptotic Behavior of Mild Solutions to Stochastic Functional Differential Equations in Hilbert Spaces

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In this paper we shall consider the existence, uniqueness, and asymptotic behavior of mild solutions to stochastic partial functional differential equations with finite delay $r > 0$: $dX(t) = \left[-AX(t) + f(t, X_t)\right] dt + g(t, X_t)\, dW(t)$, where we assume that $-A$ is a closed, densely defined linear operator and the generator of a certain analytic semigroup. $f: (-\infty, +\infty) \times C_{\alpha} \to H$, $g: (-\infty, +\infty) \times C_{\alpha} \to L^2(K, H)$ are two locally Lipschitz continuous functions, where $C_{\alpha} = C([-r, 0], D(A^\alpha))$, $L^2(K, H)$ are two proper infinite dimensional spaces, $0 < \alpha < 1$. Here, $W(t)$ is a given $K$-valued Wiener process and both $H$ and $K$ are separable Hilbert spaces.

Key Words: stochastic partial functional differential equations; fractional powers of closed operators.

1. INTRODUCTION

In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. It is well known that these topics have been developed mainly...
by using two different methods, that is, the semigroup approach (for example, Chojnowska-Michalik [4], Da Prato et al. [7], Dawson [11] and Kotelenez [14]) and the variational one (for example, Krylov and Rozovskii [15] and Pardoux [19]). On the other hand, although stochastic partial functional differential equations with finite delays also seem very important as stochastic models of biological, chemical, physical and economical systems, the corresponding properties of these systems have not been studied in great detail (cf. [1] and [25]). As a matter of fact, there exist extensive literature on the related topics for deterministic partial functional differential equations with finite delays (for example, see [27] and references of [27] there). We would also like to mention that some similar topics to the above for stochastic ordinary functional differential equations with finite delays have already been investigated by various authors (cf. [18] and [24] and references in [18] among others).

In this paper, by using semigroup methods we shall discuss existence, uniqueness, $p$th moment and almost sure Lyapunov exponents of mild solutions to a class of stochastic partial functional differential equations with finite delays,

$$
\begin{align*}
    dX(t) &= [-AX(t) + f(t, X_t)] \, dt + g(t, X_t) \, dW(t), \quad t \geq t_0, \\
    X_{t_0} &= \phi \in L^p(\Omega, C_a), \quad t_0 \geq 0,
\end{align*}
$$

(1.1)

where $\phi$ is $\mathcal{F}_{t_0}$-measurable and $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)$, $t \geq 0$, on a separable Hilbert space $H$ with the inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Throughout this paper we shall assume $0 < \alpha < 1$, $p > 2$ and define the Banach space $\mathcal{D}(A^\alpha)$ with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in \mathcal{D}(A^\alpha)$, where $\mathcal{D}(A^\alpha)$ denotes the domain of the fractional power operator $A^\alpha : H \to H$ (we refer the reader to [20] for a detailed presentation of the definition and relevant properties of $A^\alpha$). Let $H_\alpha := \mathcal{D}(A^\alpha)$ and $C_\alpha = C([-r, 0], H_\alpha)$ be the space of all continuous functions from $[-r, 0]$ into $H_\alpha$, where $0 < r < \infty$. Let $K$ be another separable Hilbert space with the inner product $(\cdot, \cdot)_K$ and norm $\| \cdot \|_K$. Suppose $W(t)$ is a given $K$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Assume $f : (-\infty, +\infty) \times C_\alpha \to H$ and $g : (-\infty, +\infty) \times C_\alpha \to L_2^Q(K, H)$ are two measurable mappings, satisfying that $f(t, 0)$, $g(t, 0)$ are locally bounded in $H$-norm and $L_2^Q(K, H)$-norm, respectively. Here $L_2^Q(K, H)$ denotes the space of all $Q$-Hilbert–Schmidt operators from $K$ into $H$ (see Definition 2.1 below). We also employ the same notation $\| \cdot \|$ for the norm of $L(K, H)$, where $L(K, H)$ denotes the space of all bounded linear operators from $K$ into $H$.

The contents of this paper are organized as follows. Beginning with some preliminary results which are fundamental for the subsequent developments, we shall investigate in Section 2 the existence and uniqueness of local mild solutions to a class of stochastic evolution equations. Section 3 is
devoted to the study of $p$th moment and almost sure Lyapunov exponential stability properties of mild solutions by using an estimate for stochastic convolution (see Lemma 3.1 below). Finally, we shall present in Section 4 an example which illustrates our main theorems.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t \in [0, +\infty)}$ of complete sub-$\sigma$-algebras of $\mathcal{F}$ is defined. Suppose $X(t): \Omega \to H_a, t \geq t_0 - r$ is a continuous $\mathcal{F}_t$-adapted, $H_a$-valued stochastic process, we can associate with another process $X(t)$, which is generated by the process $X(t)$. Let $b_n(t)$ $(n=1, 2, ...)$ be a sequence of real-valued one dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, P)$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n \gamma_n, \quad t \geq t_0,$$

where $\lambda_n \geq 0$ $(n=1, 2, ...)$ are nonnegative real numbers and $\{e_n\}$ $(n=1, 2, ...)$ is a complete orthonormal basis in $K$. Let $Q \in \mathcal{L}(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then the above $K$-valued stochastic process $W(t)$ is called a $Q$-Wiener process.

**Definition 2.1.** Let $\sigma \in \mathcal{L}(K, H)$ and define

$$\|\sigma\|_{\mathcal{L}^2(K, H)}^2 := \text{tr}(\sigma Q \sigma^*) = \left\{ \sum_{n=1}^{\infty} \|\lambda_n e_n\|^2 \right\}.$$

If $\|\sigma\|_{\mathcal{L}^2(K, H)} < \infty$, then $\sigma$ is called a $Q$-Hilbert–Schmidt operator and let $\mathcal{L}^2(K, H)$ denote the space of all $Q$-Hilbert–Schmidt operators $\sigma: K \to H$.

Let $MC_p(t, p)$, $p > 2$, denote the space of all $\mathcal{F}_t$-measurable functions which belong to $L^p(\Omega, C_a)$, that is, $MC_p(t, p)$, $p > 2$, is the space of all $\mathcal{F}_t$-measurable $C_a$-valued functions $\psi: \Omega \to C_a$ with the norm $E \|\psi\|^p_{C_a} = E\{\sup_{-r \leq \tau \leq 0} \|A^\alpha \psi(\tau)\|^p\} < \infty$. Throughout this paper, we shall impose the following assumptions on the closed, densely defined linear operator $-A: H \to H$:

**Assumption A.** (a) $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$, on the separable Hilbert space $H$.

(b) There exist a constant $M \geq 1$ and a real number $a > 0$ such that $\|S(t) h\| \leq Me^{-at} \|h\|, t \geq 0$, for any $h \in H$.

(c) The fractional power $A^\alpha$ satisfies that $\|A^\alpha S(t) h\| \leq M_\alpha e^{-a\alpha t} \|h\|, t > 0$, for any $h \in H$, where $M_\alpha \geq 1$.

(d) $\|S(t) h - h\| \leq N_\alpha t^\alpha \|A^\alpha h\|, h \in \mathcal{D}(A^\alpha), N_\alpha \geq 1.$
Under Assumption A, we shall consider the following stochastic integral equation instead of (1.1) by carrying out a semigroup type argument mentioned above:

\begin{equation}
X(t) = S(t-t_0) \phi(0) + \int_{t_0}^{t} S(t-u) f(u, X_u) \, du \\
+ \int_{t_0}^{t} S(t-u) g(u, X_u) \, dW(u), \quad t \geq t_0,
\end{equation}

(2.1)

\[ X_{t_0} = \phi \in MC_x(t_0, p), \quad t_0 \geq 0. \]

We also need the following lemma (see p. 104, Proposition 4.15 [9]).

**Lemma 2.1.** Given a predictable, \( \mathcal{F}_t \)-adapted process \( \Phi(t): \Omega \rightarrow \mathcal{L}^2(K, H) \), \( t \geq 0 \). If for arbitrary \( k \in K \) and \( \int_0^t E \| \Phi(u) \|_{L^2}^2 \, du < \infty \), \( \int_0^t E \| A^t \Phi(u) \|_{L^2}^2 \, du < \infty \), then

\[ A^t \int_0^t \Phi(s) \, dW(s) = \int_0^t A^s \Phi(s) \, dW(s). \]

**Assumption B.** For arbitrary \( \gamma, \xi \in C_a \) and \( t_0 \leq t \leq T \), suppose that there exist positive real constants \( N_1 = N_1(T) \) and \( N_2 = N_2(T) > 0 \) such that

\[
\| f(t, \gamma) - f(t, \xi) \|^p \leq N_1 \| \gamma - \xi \|^p_{C_a} \\
\| g(t, \gamma) - g(t, \xi) \|_{L^2}^p \leq N_2 \| \gamma - \xi \|^p_{C_a}.
\]

Under Assumption B, we may suppose that there exists a real number \( N_3 = N_3(T) > 0 \) such that \( \| f(t, \xi) \|^p + \| g(t, \xi) \|_{L^2}^p \leq N_3 (1 + \| \xi \|^p_{C_a}) \) for \( t_0 \leq t \leq T \), where \( T \) is any fixed time.

**Theorem 2.2.** Let \( 0 < \alpha < \frac{1}{2p} \). Suppose that the assumptions A and B hold. Then there exists a unique local continuous solution to (2.1) for any initial value \((t_0, \phi)\) with \( t_0 \geq 0 \) and \( \phi \in MC_x(t_0, p) \).

To prove this theorem, assume \( T > t_0 \) is a fixed time to be determined later and \( D_T \) is the subspace of all continuous processes \( Z \) which belong to \( C([t_0-r, T], L^p(\Omega, H)) \) with \( \| Z \|_{D_T} < \infty \), where

\[
\| Z \|_{D_T} := \sup_{t_0 \leq t \leq T} (E \| Z_t \|^p_{C_a})^{1/p}
\]

(2.2)

and

\[
E \| Z_t \|^p_{C_a} := E \left\{ \sup_{-r \leq s \leq 0} \| Z_s(s) \|^p \right\}
\]
Introduce the following mapping $\Phi$ on $D_T$:

$$(\Phi Z)(t) = S(t) A^*\phi(0) + \int_{t_0}^t A^*S(t-s) f(s, A^{-a}Z_s) \, ds$$

$$+ \int_{t_0}^t A^*S(t-s) g(s, A^{-a}Z_s) \, dW(s), \quad t \geq t_0,$$

$$\Phi Z(t) = A^*\phi(t), \quad t_0 \leq t \leq t_0 - r.$$

**Lemma 2.3.** For arbitrary $Z \in D_T$, $(\Phi Z)(t)$ is continuous on the interval $[t_0, T]$ in the $L^p$-sense.

**Proof.** Let $t_0 < t_1 < t_2 < T$. Then for any fixed $Z \in D_T$,

$$E \left\| (\Phi Z)(t_2) - (\Phi Z)(t_1) \right\| ^p \leq 3^{p-1} E \left\| (S(t_2) - S(t_1)) A^*\phi(0) \right\| ^p$$

$$+ 3^{p-1} E \left\| A^*S(t_2-s) f(s, A^{-a}Z_s) \, ds \right\| ^p$$

$$- \int_{t_0}^{t_1} A^*S(t_1-s) f(s, A^{-a}Z_s) \, ds \right\| ^p$$

$$+ 3^{p-1} E \left\| A^*S(t_2-s) g(s, A^{-a}Z_s) \, dW(s) \right\| ^p$$

$$- \int_{t_0}^{t_1} A^*S(t_1-s) g(s, A^{-a}Z_s) \, dW(s) \right\| ^p \leq I_1 + I_2 + I_3.$$

Thus, we obtain by the condition (d) of Assumption A and Assumption B that

$$I_1 \leq 3^{p-1} E \left\| S(t_2-t_1) S(t_1) A^*\phi(0) - S(t_1) A^*\phi(0) \right\| ^p$$

$$\leq 3^{p-1} N^p_a (t_2-t_1)^p \ E \left\| A^*S(t_1) A^*\phi(0) \right\| ^p$$

and

$$I_2 \leq 6^{p-1} E \left\| A^*S(t_2-s) f(s, A^{-a}Z_s) \, ds \right\| ^p$$

$$+ 6^{p-1} E \left\| A^*(S(t_2-s) - S(t_1-s)) f(s, A^{-a}Z_s) \, ds \right\| ^p$$

$$\leq 6^{p-1} E \left( \int_{t_1}^{t_2} \left\| A^*S(t_2-s) f(s, A^{-a}Z_s) \right\| \, ds \right) ^p$$

$$+ 6^{p-1} E \left( \int_{t_1}^{t_2} \left\| A^*S(t_1-s)(S(t_2-t_1) - I) f(s, A^{-a}Z_s) \right\| \, ds \right) ^p$$

$$= I_{21} + I_{22}.$$
Therefore, there exist positive constants, $Q_{21}, Q_{22} > 0$ and $\epsilon_1 = 1 - \alpha + \frac{p}{2} > 0$ such that

$$I_{21} \leq 6^{p-1} M^*_e N_2 E \left( \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} e^{-a(t_2 - s)(1 + \|Z\|_C)} \frac{1}{\rho} \, ds \right)^p$$

$$\leq Q_{21} (t_2 - t_1)^{1 + \epsilon_1} (1 + \|Z\|_C)$$

and

$$I_{22} = 6^{p-1} E \left( \int_{t_0}^{t_1} A^rS \left( \frac{t_1 - s}{2} \right) (S(t_2 - t_1) - 1) S \left( \frac{t_1 - s}{2} \right) f(s, A^{-s}Z_s) \, ds \right)^p$$

$$\leq 6^{p-1} N_2 N_4 E \left( \int_{t_0}^{t_1} \left( \frac{t_1 - s}{2} \right)^{-\alpha} e^{-a(t_2 - s)} (S(t_2 - t_1) - 1) S \left( \frac{t_1 - s}{2} \right) f(s, A^{-s}Z_s) \, ds \right)^p$$

$$\leq 6^{p-1} N_2 N_4 E \left( \int_{t_0}^{t_1} \left( \frac{t_1 - s}{2} \right)^{-\alpha} e^{-2a(t_2 - s)} (t_2 - t_1)^\alpha \|f(s, A^{-s}Z_s)\| \, ds \right)^p$$

$$\leq Q_{22} (t_2 - t_1)^{1 + \epsilon_1} (1 + \|Z\|_C).$$

In a similar way, we have by using Lemma 7.2 in [9] that

$$I_3 \leq 6^{p-1} E \left( \int_{t_1}^{t_2} A^rS(t_2 - s) g(s, A^{-s}Z_s) \, dW(s) \right)^p$$

$$+ 6^{p-1} E \left( \int_{t_0}^{t_1} A^r(S(t_2 - s) - S(t_1 - s)) g(s, A^{-s}Z_s) \, dW(s) \right)^p$$

$$\leq 6^{p-1} E \left( \int_{t_0}^{t_1} \|A^rS(t_2 - s) g(s, A^{-s}Z_s)\|_2 \, ds \right)^{p/2}$$

$$+ 6^{p-1} E \left( \int_{t_0}^{t_1} \|A^r(S(t_2 - s) - S(t_1 - s)) g(s, A^{-s}Z_s)\|_2 \, ds \right)^{p/2}$$

$$= I_{31} + I_{32}.$$

Then, it follows that there exist positive constants $Q_{31} > 0$ and $\epsilon_2 = (p - 1 - 2p\alpha)/2p > 0$ such that

$$I_{31} \leq 6^{p-1} M^*_e N_3 E \left( \int_{t_1}^{t_2} (t_2 - s)^{-2\alpha} e^{-2a(t_2 - s)} (1 + \|Z\|_C) \, ds \right)^{p/2}$$

$$\leq Q_{31} (t_2 - t_1)^{1 + \epsilon_2} (1 + \|Z\|_C).$$
Let \( \{e_n\} \), \( n \geq 1 \), be a complete orthonormal basis of the separable Hilbert space \( K \) such that \( Q^{1/2}e_n = \sqrt{\lambda_n} e_n \), where \( Q \) is the covariance operator of Wiener process \( W \). Then, we obtain that there exists a positive constant \( Q_{32} > 0 \) such that

\[
I_{32} = 6^{p-1} E \left( \int_{t_0}^{t_1} \left| A^* S \left( \frac{t_1 - s}{2} \right) (S(t_2 - t_1) - I) S \left( \frac{t_1 - s}{2} \right) g(s, A^{-z}Z_s) \right|^2 \frac{ds}{\sigma^2} \right)^{p/2} \\
\leq 6^{p-1} N_n C_0 E \left( \int_{t_0}^{t_1} \left| S \left( \frac{t_1 - s}{2} \right) \sqrt{\lambda_n} g(s, A^{-z}Z_s) e_n \right|^2 ds \right)^{p/2} \\
\leq 6^{p-1} N_n C_0 E \left( \int_{t_0}^{t_1} \left| e_n \right|^2 \frac{ds}{\sigma^2} \right)^{p/2} \\
\leq Q_{32} (t_2 - t_1)^p \left( 1 + \|Z\|_{L^p} \right).
\]

Since \( Z \in D_T \), it follows that \( I_1, I_2, I_3 \) tend to zero, as \( t_2 \to t_1 \). Therefore, the proof of the lemma is complete.

**Lemma 2.4.** Suppose the operator mapping \( \Phi \) and the corresponding domain \( D_T \) are defined as above; then \( \Phi(D_T) \subset D_T \).

**Proof.** Let \( Z \in D_T \). Then we have that

\[
E \| (\Phi Z)_t \|_C \leq 3^{p-1} E \sup_{-r \leq t \leq 0} \| S(t + \theta) A^* \phi(0) \|^p \\
= I_4 + I_5 + I_6.
\]

Let \( q = \frac{p}{p-1} \); then we obtain that

\[
I_4 \leq 3^{p-1} M_T E \| \phi \|_{L^q}, \\
I_5 \leq 3^{p-1} M_T^p E \sup_{-r \leq t \leq 0} \left( \int_{t_0}^{t+\theta} e^{-a(t+\theta-s)} (t+\theta-s)^{-\alpha} \| f(s, A^{-z}Z_s) \| ds \right)^p \\
\leq 3^{p-1} N_3 M_T^p (T-t_0) (T(1-\alpha q)(aq)^{\alpha q})^{p/q} (1 + \|Z\|_{L^p}^p).
\]
Let \( R(s) = \int_{t_o}^{s} (s-\theta)^{-\rho} S(s-\rho) g(\sigma, A^{-\rho}Z_\sigma) dW(\sigma) \) with \( \frac{1}{2}+\alpha < \rho < \frac{1}{2} \),

\[
I_8 \leq 3^{p-1} E \sup_{-r < \theta < 0} \left| A^s \int_{t_0}^{t+\theta} (t+\theta-s)^{p-1} S(t+\theta-s) R(s) \, ds \right|^p
\]

\[
\leq 3^{p-1} M_\rho^\alpha E \sup_{-r < \theta < 0} \left( \int_{t_0}^{t+\theta} (t+\theta-s)^{p-1-\alpha} e^{-a(t+\theta-s)} \|R(s)\| \, ds \right)^p
\]

\[
\leq 3^{p-1} N_1 M_\rho^\alpha (T-t_0) (T(t_0))^{(1-\alpha)(aq)}^{(1-\alpha)(aq)-1} \cdot \frac{T^{-p+\alpha}}{(1-2\rho)^{p/2}} \cdot \frac{1}{(2(1-\alpha)(aq))^{p/2}}
\]

Therefore, we obtain that \( \|\Phi Z\|_{L_T} < \infty \). Thus this completes the proof. 

**Proof of Theorem 2.2.** Let \( X, Y \in D_T \), then for any fixed \( t \in [t_0, T] \),

\[
E \| (\Phi X)_t - (\Phi Y)_t \|_{L_T}^p
\]

\[
\leq E \sup_{-r < \theta < 0} \left\| (\Phi X)(t+\theta) - (\Phi Y)(t+\theta) \right\|_{L_T}^p
\]

\[
\leq 2^{p-1} E \sup_{-r < \theta < 0} \left\| A^s S(t+\theta-s)(f(s, A^{-\rho}X_s) - f(s, A^{-\rho}Y_s)) \, ds \right\|_{L_T}^p
\]

\[
+ 2^{p-1} E \sup_{-r < \theta < 0} \left\| A^s S(t+\theta-s)(g(s, A^{-\rho}X_s) - g(s, A^{-\rho}Y_s)) \, dW(s) \right\|_{L_T}^p
\]

\[
= I_7 + I_8
\]

and

\[
I_7 \leq 2^{p-1} N_1 M_\rho^\alpha E \sup_{-r < \theta < 0} \left( \int_{t_0}^{t+\theta} e^{-a(t+\theta-s)} \left\| X_s - Y_s \right\| \, ds \right)^p
\]

\[
\leq 2^{p-1} N_1 M_\rho^\alpha (T-t_0) (T(t_0))^{(1-aq)(aq)-1} \cdot \frac{T^{-p+\alpha}}{(1-2\rho)^{p/2}} \cdot \frac{1}{(2(1-\alpha)(aq))^{p/2}} \|X - Y\|_{L_T}^p.
\]

Next, let \( 1/p + \alpha < \rho < 1/2 \) and

\[
A(s) = \int_{t_0}^{s} (s-\rho)^{-\rho} S(s-\rho) (g(\sigma, A^{-\rho}X_\sigma) - g(\sigma, A^{-\rho}Y_\sigma)) \, dW(\sigma);
\]

then we have

\[
I_8 \leq 2^{p-1} E \sup_{-r < \theta < 0} \left\| A^s S(t+\theta-s) A(s) \, ds \right\|_{L_T}^p
\]

\[
\leq 2^{p-1} N_1 M_\rho^\alpha E \sup_{-r < \theta < 0} \left( \int_{t_0}^{t+\theta} e^{-a(t+\theta-s)} \left\| A(s) \right\| \, ds \right)^p
\]
\[ \leq 2^{p-1} N_p M_p^\varepsilon (T-t_0) (T-1-(1+\alpha-\rho)(aq)^{\delta(1+\alpha-\rho)-1})^{p/q} \]

\[ \cdot c_p M_p^\varepsilon \frac{T^{-p+\frac{p}{2}}}{(1-2p)^{p/2}} \|X-Y\|_{D_T}. \]

Hence, by taking a suitable \( T > t_0 \) such that \( T-t_0 > 0 \) is sufficiently small, we obtain a positive real number \( B(T) \in (0, 1) \) such that
\[ \|\Phi X - \Phi Y\|_{D_T} \leq B(T) \|X - Y\|_{D_T} \]
for any \( X, Y \in D_T \). Thus, by the well known Banach fixed point theorem we have a unique fixed point \( U \in D_T \) which, setting \( X(t) = A^{-1} U(t) \), immediately yields
\[
X(t) = S(t) \phi(t-t_0) + \int_{t_0}^t S(t-s) f(s, X(s)) \, ds \\
+ \int_{t_0}^t S(t-s) g(s, X(s)) \, dW(s), \quad t \geq t_0,
\]
which proves the existence of a local solution of (2.1). The uniqueness of the solution is proved similarly. Therefore the proof is complete.

**Theorem 2.5.** Assume \( 0 < \alpha < \frac{p-2}{2} \) and let \( f: (-\infty, +\infty) \times C_a \to H \), \( g: (-\infty, +\infty) \times C_a \to \mathcal{L}_0^2(K, H) \) satisfy the assumptions A and B. If there exists a constant \( B_2 > 0 \) such that
\[ \|f(t, \psi)\|_{\mathcal{L}_0^2} + \|g(t, \psi)\|_{\mathcal{L}_0^2} \leq B_2 (1 + \|\psi\|_{L_0^2}), \]
for all \( \psi \in C_a, \ t \geq t_0, \) then there exists a unique, global continuous solution \( X(t): \Omega \to H \) to the equation (2.1) for any initial value \( (t_0, \phi) \) with \( \phi \in MC_a(t_0, p) \).

**Proof.** If \( f \) and \( g \) satisfy the global Lipschitz condition, then the proof of the theorem can be given similarly as a corollary of Theorem 2.2. If \( f \) and \( g \) satisfy the local Lipschitz condition, then the proof is given by the truncation method [6, p. 17]. Hence, we omit the proof.

**3. ALMOST SURE EXPONENTIAL STABILITY**

Suppose \( \Phi(t), \ t \geq 0 \) is a \( \mathcal{L}_0^2(K, H) \)-valued predictable process with \( E \int_0^T \|\Phi(s)\|_{\mathcal{L}_0^2}^p \, ds < \infty , \ t \geq t_0. \) In [6], Da Prato and Zabczyk proved
\[
E \left( \sup_{0 \leq t \leq T} \left\| \int_0^T S(t-u) \Phi(u) \, dW(u) \right\|^p \right) \\
\leq c_p \sup_{0 \leq t \leq T} \|S(t)\|^p T^{(p/2) - 1} \int_0^T \|\Phi(s)\|_{\mathcal{L}_0^2}^p \, ds
\]
for any $T \geq 0$, where $c_p = \left(\frac{p(p-1)}{2}\right)^{\frac{1}{2}}$, by using the equation
\[
\int_0^t S(t-u) \Phi(u) \, dW(u) = \left(\frac{\sin \pi \beta}{\pi}\right) \int_0^t (t-u)^{\beta-1} S(t-u) \, Y(u) \, du, \quad t \geq 0,
\]
with $Y(u) = \mathbf{1}_{[0, u]}(u-s) - b S(u-s) F(s) \, dW(s)$, $0 \leq u \leq T$, $1/p < \beta < 1/2$. The following lemma is obtained as a consequence of the above mentioned inequality. However, we would like to give the proof here for the reader’s convenience. For this end, assume $q = \frac{p}{p-1}$ for any $p > 2$.

**Lemma 3.1.** Let $0 < \theta < \frac{p^2}{2p}$ and assume $\beta$ is any fixed real number such that $1/p + \theta < \beta < 1/2$. Then for all $t \in (t_0 + r, T)$
\[
E \left( \sup_{-r \leq s \leq 0} \left\| \int_{t_0}^{t+s} S(t+s-u) \Phi(u) \, dW(u) \right\|^p \right) \\
\leq c_p e^\omega M^p C(p, \theta, \beta) \left( 2a - \frac{2a}{p} \right)^{2\beta-1} \\
\cdot \int_{t_0}^t e^{-a(t-s)} \mathbb{E} \left( \mathbb{E}(\sigma) \right) \|Y(u)\|^p \, du,
\]
where $C(p, \theta, \beta) := \left\{ \Gamma(1-q(1+\theta - \beta)) a^{q(1+\theta - \beta)-1} \right\}^{p/q}$ and $\Gamma(\cdot)$ is the usual Gamma function.

**Proof.** Using (3.1) and the Hölder inequality we obtain that
\[
E \left( \sup_{-r \leq s \leq 0} \left\| \int_{t_0}^{t+s} S(t+s-u) \Phi(u) \, dW(u) \right\|^p \right) \\
\leq M^p e^\omega E \left( \sup_{-r \leq s \leq 0} \left\| \int_{t_0}^{t+s} (t+s-u)^{\beta-1} A^S(t+s-u) \, Y(u) \, du \right\|^p \right) \\
\leq M^p e^\omega \left\{ \sup_{-r \leq s \leq 0} \left[ \int_{t_0}^{t+s} (t+s-u)^{q(\beta-1-\theta-1)} e^{-a(t+s-u)} \, du \right]^{p/q} \right\}^{p/q} \\
\cdot \int_{t_0}^t e^{-a(t-s)} \mathbb{E} \left( \mathbb{E}(\sigma) \right) \|Y(u)\|^p \, du.
\]
On the other hand, we have that
\[
E \left\{ \sup_{-r \leq s \leq 0} \left( \int_{t_0}^{t+s} e^{-a(t+s-u)} \|Y(u)\|^p \, du \right) \right\} \\
\leq e^{\omega T} \int_{t_0}^t e^{-a(t-u)} \mathbb{E} \left( \mathbb{E}(\sigma) \right) \|Y(u)\|^p \, du
\[ e^{-\epsilon t} c_p M^p \int_0^t e^{-a(t-s)} \| \Phi(s) \|_{2p}^2 \, ds \leq e^{-\epsilon t} c_p M^{p}\int_0^t (u(t-s)^{-\beta})^{(2a-2ap)} e^{-a(t-s)} \| \Phi(s) \|_{2p}^2 \, ds \]

which, by the Young inequality, immediately yields

\[ E \left( \sup_{-r \leq s \leq 0} \left( \int_0^s e^{-a(t-s)} \| Y(u) \|_p \, du \right) \right) \]
\[ \leq c_p e^{-\epsilon t} M^{p} \left( \int_0^t v^{-2\beta} e^{-a(t-s)} v^{2\beta-1} \, dv \right) \left( \int_0^t e^{-a(t-s)} \| \Phi(s) \|_{2p}^2 \, ds \right) \]
\[ \leq c_p e^{-\epsilon t} M^{p} \Gamma(1-2\beta) \left( 2a - \frac{2a}{p} \right)^{2\beta-1} \left( \int_0^t e^{-a(t-s)} \| \Phi(s) \|_{2p}^2 \, ds \right). \]

Therefore, the proof of the lemma is complete.

Now, we are in a position to present the stability results of the solution to (2.1).

**Theorem 3.2.** Assume \( 0 < \theta < \frac{p-1}{p} \). Let \( f: (-\infty, \infty) \times C_0 \rightarrow H, g: (-\infty, \infty) \times C_0 \rightarrow \mathcal{L}^p(K, H) \) satisfy the local Lipschitz condition B. Furthermore assume that Assumption A is satisfied and there exist nonnegative real numbers \( Q_1, Q_2 > 0 \) and continuous functions \( \xi_1, \xi_2: [0, \infty) \rightarrow \mathbb{R}^+ \) such that

\[ E \| f(t, X_t) \|_p^p \leq Q_1 E \| X_t \|_{E_{Q_1}}^p + \xi_1(t), \quad t \geq t_0, \]
\[ E \| g(t, X_t) \|_{L^p_0}^p \leq Q_2 E \| X_t \|_{E_{Q_2}}^p + \xi_2(t), \quad t \geq t_0, \]

for any solution \( X(t) \) to (2.1). Let \( L_0 = L_1 Q_1 + L_2 Q_2 \) with \( L_1 = 3^{p-1} M_0^p (\Gamma(1-\theta) a^{p+1})^{\theta/p} \) and \( L_2 = 3^{p-1} M_0^p c_p C(p, \theta, \beta) \Gamma(1-2\beta)(2a - \frac{2a}{p})^{2\beta-1} \), where \( C(p, \theta, \beta) \) is the positive real number given in Lemma 3.1. Suppose \( a > L_0 \), where \( \| S(t) \| \leq M e^{-at}, t \geq 0 \), and there exist nonnegative real numbers \( P_1 \) and \( P_2 \geq 0 \) such that \( \| \xi_j(t) \| \leq P_j e^{-(a-L_0)t} (j = 1, 2) \), then there exist positive constants \( \epsilon > 0 \) and \( K(p, \epsilon, \Phi) > 0 \) such that for each \( t \geq t_0 + 2r \)

\[ E \| X_t \|_{E_{Q_1}}^p \leq K(p, \epsilon, \Phi) \cdot e^{-a(t-t_0)}. \]

In other words, the solution is the \( p \)th moment exponentially stable.

**Proof.** Without loss of generality, we suppose \( t_0 = 0 \). Let \( -r \leq s \leq 0 \); then for each \( t > 2r \)
$E \|X(t+s)\|_p^p \leq 3^{p-1} E \|S(t+s-t_0) \phi(0)\|_p^p$

$+ 3^{p-1} E \left\| \int_{t_0}^{t+s} S(t+s-u) f(u, X_u) \, du \right\|_p^p$

$+ 3^{p-1} E \left\| \int_{t_0}^{t+s} S(t+s-u) g(u, X_u) \, dW(u) \right\|_p^p$

$= I_1 + I_2 + I_3,$

$I_1 \leq 3^{p-1} M_p^p e^{-p(t-r)} (t-r)^{-p} E \|\phi(0)\|_p^p.$

On the other hand, by the Hölder inequality we can obtain that

$I_2 \leq 3^{p-1} M_p^p E \left( \int_{t_0}^{t+s} (t+s-u)^{-\theta} e^{-a(t+s-u)} \|f(u, X_u)\| \, du \right)^p$

$\leq 3^{p-1} M_p^p e^{-a(t+s)} \left( \int_{t_0}^{t+s} u^{-\theta} e^{-au} \, du \right)^{p/q} \left( \int_{t_0}^{t+s} e^{-a(t-s)} \|f(u, X_u)\|_p^p \, du \right)^q$

$\leq 3^{p-1} M_p^p e^{-a(t+s)(1-q\theta)\alpha^{q-1}} \left( \int_{t_0}^{t+s} e^{-a(t-s)} [Q_2 E \|X_u\|_{C^q} + \zeta_2(u)] \, du \right)^{p/q}.$

By virtue of Lemma 3.1, we can deduce

$I_3 \leq 3^{p-1} M_p^p M' e^{-at} C(p, \theta, \beta) \Gamma(1-2\beta) \left(2a - \frac{2a}{p}\right)^{2\beta-1} \int_{t_0}^{t+s} e^{-a(t-s)} [Q_2 E \|X_u\|_{C^q} + \zeta_2(u)] \, du.$

Suppose that $M_0 = 3^{p-1} M_p^p r^{-\theta} e^{a\theta} E \|\phi(0)\|_p^p,$ $L_1 = 3^{p-1} M_p^p \Gamma(1-q\theta)\alpha^{q-1} r^{p/q},$ and $L_2 = 3^{p-1} M_p^p M' e^{C(p, \theta, \beta) \Gamma(1-2\beta)(2a - \frac{2a}{p})^{2\beta-1}};$ then we have

$E \|X(t+s)\|_p^p \leq M_0 \cdot e^{-at} + L_0 e^{-a(t+s)} \int_{t_0}^{t+s} e^{aw} \|X_u\|_{C^q} \, du$

$+ e^{-at} \int_{t_0}^{t+s} e^{aw} (L_1 \zeta_1(u) + L_2 \zeta_2(u)) \, du \quad (3.3)$
for each $t \geq 2r$, where $L_0 = L_1Q_1 + L_2Q_2$. Therefore, we have for arbitrary $\varepsilon \in \mathbb{R}_+$ with $0 < \varepsilon < a$ and $T > 0$ large enough

$$
\int_0^T e^{\varepsilon t} E \|X(t+s)\|_\psi^p \, dt \leq M_0 \int_0^T e^{-\alpha t} \varepsilon + L_0 \int_0^T e^{\alpha(-t)} \varepsilon^\mu E \|X_s\|_{C_0}^p \, du \, dt
$$

$$+
\int_0^T e^{-\alpha t} \varepsilon^\mu E \|X_s\|_{C_0}^p \, du \, dt
$$

$$= M_0 \int_0^T e^{-\varepsilon t} \varepsilon^\mu E \|X_t\|_{C_0}^p \, dt
$$

which, by virtue of the continuity of $X(t), t \geq 0$, immediately implies

$$
\int_0^T e^{\varepsilon t} E \|X_s\|_\psi^p \, dt
$$

$$\leq M_0 \int_0^T e^{-\alpha t} \varepsilon + L_0 \int_0^T e^{\alpha(-t)} \varepsilon^\mu E \|X_s\|_{C_0}^p \, du \, dt
$$

$$+ \int_0^T e^{-\alpha t} \varepsilon^\mu E \|X_s\|_{C_0}^p \, du \, dt
$$

$$= M_0 \int_0^T e^{-\varepsilon t} \varepsilon^\mu E \|X_t\|_{C_0}^p \, dt
$$

(3.4)
On the other hand, since $a > L_0$ by assumption, it is possible to choose a suitable $\varepsilon \in \mathbb{R}$, with $0 < \varepsilon < a - L_0$ small enough so that
\[ L_0 e^{\alpha'} / (a - \varepsilon) < 1, \]
which, letting $T > 0$ tend to infinity and using (3.6), immediately yields
\[
\int_0^\infty e^{\alpha} E \| X_t \|_{\ell^p} \, dt \leq \frac{1}{1 - L_0 e^{\alpha'}} \left( M_0 \cdot \int_0^\infty e^{-(a - \varepsilon') t} \, dt + \frac{L_0 e^{\alpha'}}{a - \varepsilon} \int_0^\infty e^{\alpha} E \| X_t \|_{\ell^p} \, du \right)
\]
\[
\quad + \int_0^T e^{-\alpha t} e^{\alpha(t+1)} \int_0^{t+1} e^{\alpha} (L_1 \xi_1(u) + L_2 \xi_2(u)) \, du \, dt
\]
\[ := K(p, \varepsilon, \phi) < \infty. \quad (3.7) \]
By virtue of (3.2), (3.3) and (3.7), we can easily deduce (note $0 < \varepsilon < a - L_0$)
\[
E \| X_t \|_{\ell^p}^p \leq M_0 \cdot e^{-\alpha t} + M_1(\varepsilon) \cdot e^{-\alpha t} + L_0 \cdot e^{\alpha} \int_0^\infty e^{\alpha} E \| X_t \|_{\ell^p} \, du
\]
\[ \leq (M_0 + M_1(\varepsilon) + L_0 \cdot e^{\alpha} K'(p, \varepsilon, \phi)) \cdot e^{-\alpha t} := K(p, \varepsilon, \phi) \cdot e^{-\alpha t}. \]
where $M_1(\varepsilon) = \int_0^\varepsilon e^{\alpha(L_1 \xi_1(u) + L_2 \xi_2(u))} \, du < \infty$, i.e.,
\[ E \| X_t \|_{\ell^p}^p \leq K(p, \varepsilon, \phi) \cdot e^{-\alpha t}. \]

**Theorem 3.3.** Suppose that all the conditions of Theorem 3.2 are satisfied. Then the solution is almost surely exponentially stable. Moreover, there exists a positive constant $\varepsilon > 0$ such that
\[ \limsup_{t \to \infty} \frac{\log \| X(t) \|_{\ell^p}}{t} \leq -\frac{\varepsilon}{2p} \quad a.s. \]

**Proof.** For each fixed positive real number $\varepsilon_N > 0$ ($N = 2, 3, \ldots$),
\[
P \left( \sup_{N \leq t \leq N+1} \| X(t) \|_{\ell^p} > \varepsilon_N \right)
\]
\[ \leq \left( \frac{3}{\varepsilon_N} \right)^p E \left( \sup_{N \leq t \leq N+1} \| S(t-N+1) X(N-1) \|_{\ell^p} \right)
\]
\[ + \left( \frac{3}{\varepsilon_N} \right)^p E \left( \sup_{N \leq t \leq N+1} \left\| \int_{t-N+1}^t S(t-u) f(u, X_u) \, du \right\|_{\ell^p} \right)
\]
\[ + \left( \frac{3}{\varepsilon_N} \right)^p E \left( \sup_{N \leq t \leq N+1} \left\| \int_{t-N+1}^t S(t-u) g(u, X_u) \, dW(u) \right\|_{\ell^p} \right)
\]
\[ = I_1 + I_2 + I_3. \]
Hence, in view of Theorem 3.2, we have

\[ I_1 = \left( \frac{3}{\varepsilon_N} \right)^p E \left( \sup_{N \leq t \leq N+1} \|S(t-N+1) A^\varepsilon X(N-1)\|^p \right) \]

\[ \leq \left( \frac{3M}{\varepsilon_N} \right)^p E \|A^\varepsilon X(N-1)\|^p \]

\[ \leq \left( \frac{3M}{\varepsilon_N} \right)^p E \|X_{N-1}\|^p \varepsilon_N \]

\[ \leq \left( \frac{3M}{\varepsilon_N} \right)^p \left\{ M_0 e^{-\alpha(N-\theta_0-1)} + \int_{N-1}^N (L_1 \xi_1(u) + L_2 \xi_2(u)) e^{-\alpha(N-u-1)} du \right\} \]

\[ \leq \left( \frac{3M}{\varepsilon_N} \right)^p \left\{ M e^{\alpha} + \frac{2}{\theta} (L_1 P_1 + L_2 P_2) \right\} e^{-\alpha(N-\theta_0)}. \]

Therefore, there exists a positive real number \( R_0 > 0 \) such that

\[ I_1 \leq \left( \frac{3M}{\varepsilon_N} \right)^p R_0 e^{-\alpha(N-\theta_0)}. \]

Next, by a straightforward computation we have

\[ I_2 \leq \left( \frac{3}{\varepsilon_N} \right)^p M_2 E \left( \sup_{N \leq t \leq N+1} \left( \int_{N-1}^t (t-u)^{-\theta} e^{-\alpha(t-u)} \|f(u, X_u)\| du \right)^p \right) \]

\[ \leq \left( \frac{3}{\varepsilon_N} \right)^p M_2 E \left( \sup_{N \leq t \leq N+1} \left( \int_{N-1}^t (t-u)^{-\theta} du \right)^{p/\theta} \int_{N-1}^t \|f(u, X_u)\|^p du \right) \]

\[ \leq \left( \frac{3}{\varepsilon_N} \right)^p M_2 \left( \frac{2^{1-\theta}}{1-q\theta} \right)^{p/\theta} \left( \int_{N-1}^{N+1} \left[ \beta_1 E \|X_u\| \xi_0 + \xi_1(u) \right] du \right). \]

Hence, by using Theorem 3.2 we can get there exists a positive real number \( R_1 > 0 \) such that

\[ I_2 \leq \left( \frac{R_1}{\varepsilon_N} \right) e^{-\alpha(N-\theta_0)}. \]

Finally, by virtue of Lemma 3.1 we have that there exists a positive real number \( R_2 > 0 \) such that

\[ I_3 \leq \left( \frac{3M}{\varepsilon_N} \right)^p c_\varepsilon C(p, \beta, \beta) \Gamma(1-2\beta) a^{2\beta-1} \left( \int_{N-1}^{N+1} \left[ \beta_1 E \|X_u\| \xi_0 + \xi_2(u) \right] du \right) \]

\[ \leq \left( \frac{R_2}{\varepsilon_N} \right) e^{-\alpha(N-\theta_0)}. \]
Thus, there exists a positive real number $p_1 > 0$ such that

$$P \left( \sup_{N \leq t < N+1} \|X(t)\|_\theta > \epsilon_N \right) \leq \left( \frac{p_1}{\epsilon_N^p} \right) e^{-\frac{1}{2} \beta(N-t_0)}. \quad (3.8)$$

Therefore, we have the desired conclusion by a Borel–Cantelli lemma argument and consequently the proof is complete.

4. EXAMPLE

In this section we shall present an example on the almost sure Lyapunov exponential stability of solutions of stochastic partial functional differential equations. Let $H = L^2(0, \pi)$ and $K = \mathbb{R}^1$ with the norms $\| \cdot \|$ and $| \cdot |$, respectively. Let $A : L^2(0, \pi) \to L^2(0, \pi)$ be the linear operator defined by

$$A \zeta = -\left( \frac{d^2}{dx^2} \right) \zeta,$$

where $\mathcal{D}(A) = \{ \zeta \in H : \zeta, \frac{d}{dx} \zeta \text{ are absolutely continuous}, \text{ and } (\frac{d^2}{dx^2}) \zeta \in H, \zeta(0) = \zeta(\pi) = 0 \}$. Let $p > 2$, $0 < \theta < \frac{p-2}{2p}$, and suppose $r > 0$ is a positive real number. Set $H_0 = \mathcal{D}(A^p)$ and $C_0 = C([-r, 0], H_0)$.

It is well known that $A$ is a closed, densely defined linear operator. $B(t)$ denotes a one-dimensional standard Brownian motion.

Example 4.1. Consider the stochastic delay reaction-diffusion equation

$$dX(t, x) = \left[ \frac{\partial^2}{\partial x^2} X(t, x) + F(t, X(t-r_1(t), x)) \right] dt + G(t, X(t-r_2(t), x)) \, dB(t)$$

$$X(t, 0) = X(t, \pi) = 0, \quad t \geq 0,$$

$$X(s, x) = \phi(s, x), \quad -r \leq s \leq 0, \quad 0 \leq x \leq \pi,$$

where $r_1, r_2$ are continuous functions with $0 < r_1(t) < r, 0 < r_2(t) < r$ for all $t \geq 0$ and $\phi \in C_0$. Suppose $F : (-\infty, +\infty) \times \mathbb{R}^1 \to \mathbb{R}^1$ and $G : (-\infty, +\infty) \times \mathbb{R}^1 \to \mathbb{R}^1$ are continuous and global Lipschitz continuous in the second variable. Assume

(H) there exist positive real numbers $Q_1, Q_2 > 0$ and continuous functions $\xi_1, \xi_2 : [0, \infty) \to \mathbb{R}_+$ such that

$$|F(t, y)|^p \leq Q_1 |y|^p + \xi_1(t), \quad t \geq 0, \quad y \in \mathbb{R}^1,$$

$$|G(t, y)|^p \leq Q_2 |y|^p + \xi_2(t), \quad t \geq 0, \quad y \in \mathbb{R}^1.$$

Suppose that $Q_1, Q_2 > 0$ are sufficiently small positive numbers such that $a > L_0$, where $L_0$ is given in Theorem 3.2. If there exist nonnegative real numbers $P_1, P_2 \geq 0$ such that $|\xi_i(t)| \leq P_i e^{-(a-L_0)t}, \quad (i = 1, 2)$ for all
\( t \geq 0 \), then there exists a global solution \( X(t) = X(t, \theta, \phi) \in \mathcal{H}_\theta \) and furthermore a positive constant \( \varepsilon > 0 \) such that
\[
\limsup_{t \to \infty} \frac{\log \|X(t)\|_\theta}{t} \leq -\frac{\varepsilon}{p}, \quad \text{a.s.}
\]

**Proof.** First of all, note that there exists a complete orthonormal set \( \{\zeta_n\} \ (n = 1, 2, \ldots) \) of eigenvectors of \( A \) with \( \zeta_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \) and the analytic semigroup \( S(t), t \geq 0 \) which is generated by \( -A \) such that
\[
A \zeta = \sum_{n=1}^\infty n^2(\zeta, \zeta_n) \zeta_n, \quad \zeta \in \mathcal{D}(A),
\]
\[
S(t) \zeta = \sum_{n=1}^\infty \exp(-n^2t)(\zeta, \zeta_n) \zeta_n, \quad \zeta \in H.
\]

We define \( A^\theta \) (actually \( |A|^\theta \)) for self-adjoint operator \( A \) by the classical spectral theorem and it is easy to deduce
\[
|A|^\theta e^{-At} \zeta = \sum_{n=1}^\infty (n^2)^\theta e^{-n^2t}(\zeta, \zeta_n) \zeta_n,
\]
which immediately implies
\[
\|A^\theta e^{-At}\|_2^2 = \sum_{n=1}^\infty n^{2\theta} e^{-2n^2t} |(\zeta, \zeta_n)|^2 = e^{-2at} \sum_{n=1}^\infty (n^2)^{2\theta} e^{-(2n^2-2a)t} |(\zeta, \zeta_n)|^2.
\]
(4.1)

On the other hand, define the function
\[
\delta(x) = x^\theta e^{-(x-a)} = e^{\theta \ln x - (x-a)}, \quad x > 0
\]
which achieves its maximum when \( x = \theta \), i.e., \( x = \theta = n^2t \) in (4.1). Therefore,
\[
x^\theta e^{-(x-a)} \leq \theta^\theta e^{-(\theta-a)} \leq \theta^\theta e^{-\theta(1-a)}
\]
as \( t = \theta/n^2 < \theta \) for \( n \geq 1 \) and \( a > 0 \). However, if \( 1 > a \), then
\[
e^{-(1-a)} \leq \frac{1}{1-a}
\]
which immediately implies that
\[
x^\theta e^{-(x-a)} \leq \theta^\theta (1-a)^{-\theta} = \left(\frac{\theta}{1-a}\right)^\theta.
\]
Hence, let $0 < a < 1$ and $M \geq 1$ be two any fixed real numbers and let $\mathcal{M} = \{[\frac{\bar{p}}{1-a}]\}^\mu$; then we have that
\[
\left\| S(t) \zeta \right\| \leq M e^{-at} \left\| \zeta \right\|, \quad t > 0,
\]
\[
\left\| \mathcal{A} S(t) \zeta \right\| \leq M_\varphi e^{-at} \left\| \zeta \right\|, \quad t > 0,
\]
for all $\zeta \in H$. Now let $f(t, \psi)(x) = F(t, \psi(-r_1(t))(x))$ and $g(t, \psi)(x) = G(t, \psi(-r_2(t))(x))$ for all $\psi \in C_\varphi = C([-r, 0], H_\varphi)$ and any $x \in [0, \pi]$. Since we have that for any fixed $s \in [-r, 0]$,
\[
\left\| \psi(s) - \phi(s) \right\|^2 = \int_0^s |\psi(s) - \phi(s)(x)|^2 \, dx
\]
\[
= \sum_{n=1}^\infty (\psi(s) - \phi(s), \zeta_n)^2
\]
\[
\leq \sum_{n=1}^\infty n^\varphi (\psi(s) - \phi(s), \zeta_n)^2
\]
\[
= \left\| \mathcal{A} \psi(s) - \phi(s) \right\|^2
\]
\[
\leq \left\| \psi - \phi \right\|_{C_\varphi}^2,
\]
we then get that the functions $f(t, \psi)$ and $g(t, \psi)$ are globally Lipschitz continuous in $\psi \in C_\varphi$. Furthermore since $\left\| \psi(-r_1(t)) \right\|^p \leq \left\| \psi \right\|_{C_\varphi}^p$, $t \geq 0$, and by Condition (H), we have
\[
\left\| f(t, \psi) \right\|^p = \int_0^s |F(t, \psi(-r_1(t))(x))|^p \, dx
\]
\[
\leq \int_0^s \{Q_1 |\psi(-r_1(t))(x)|^p + \zeta_1(t) \} \, dx
\]
\[
\leq Q_1 \left\| \psi \right\|_{C_\varphi}^p + \pi \zeta_\xi(t), \quad t \geq 0,
\]
and similarly
\[
\left\| g(t, \psi) \right\|^p = Q_2 \left\| \psi \right\|_{C_\varphi}^p + \pi \zeta_\xi(t), \quad t \geq 0
\]
for all $\psi \in C_\varphi$. Hence, all the conditions in Theorem 3.2 are satisfied. Therefore, we have the desired conclusion by Theorem 3.3.

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