Phase transition of holographic entanglement entropy in massive gravity

Xiao-Xiong Zeng [a,b], Hongbao Zhang [c,d], Li-Fang Li [e,*]

[a] *School of Material Science and Engineering, Chongqing Jiaotong University, Chongqing 400074, China*
[b] *Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*
[c] *Department of Physics, Beijing Normal University, Beijing 100875, China*
[d] *Theoretische Natuurkunde, Vrije Universiteit Brussel, and The International Solvay Institutes, Pleinlaan 2, B-1050 Brussels, Belgium*
[e] *State Key Laboratory of Space Weather, National Space Science Center, Chinese Academy of Sciences, Beijing 100190, China*

A R T I C L E   I N F O

A B S T R A C T

The phase structure of holographic entanglement entropy is studied in massive gravity for the quantum systems with finite and infinite volumes, which in the bulk is dual to calculating the minimal surface area for a black hole and black brane respectively. In the entanglement entropy–temperature plane, we find for both the black hole and black brane there is a Van der Waals-like phase transition as the case in thermal entropy–temperature plane. That is, there is a first order phase transition for the small charge and a second order phase transition at the critical charge. For the first order phase transition, the equal area law is checked and for the second order phase transition, the critical exponent of the heat capacity is obtained. All the results show that the phase structure of holographic entanglement entropy is the same as that of thermal entropy regardless of the volume of the spacetime on the boundary.

1. Introduction

As any ordinary thermodynamic system, black holes have entropy as well as temperature, and obey the thermodynamical laws [1–3]. But differently from the thermal entropy for the ordinary system, which is proportional to the volume, the entropy of a black hole is proportional to its area of horizon. Among others, AdS/CFT correspondence offers us a natural explanation for such a seemingly bizarre behavior because it maps a gravitational system to an ordinary system on the boundary with one less dimension [4–6]. Moreover, in the context of AdS/CFT, not only can the area of black hole horizon be understood as the thermal entropy for its dual boundary system, but also the area of the bulk co-dimension 2 minimal surface anchored onto the entangling surface on the boundary can be used to calculate the corresponding entanglement entropy for the boundary system [7].

Since the entanglement entropy acquires a similar geometric description to the thermal entropy in the bulk by holography, it is of interest to see whether the entanglement entropy demonstrates the similar behavior as the thermal entropy. Especially in the re-

cent work [8], the authors showed that like the black hole entropy, the entanglement entropy also undergoes a Van der Waals-like phase transition at the same critical temperature in both the fixed charge ensemble and chemical potential ensemble. They also found the second order phase transition occurring for the entanglement entropy at the same critical point as the thermal entropy with nearly the same value as that from thermal phase transition.

In light of these interesting results, this work was then generalized to the extended phase space where the cosmological constant is considered as a thermodynamical variable, and it was found that the entanglement entropy has the similar phase structure as that of the black hole entropy too [9]. Very recently, Nguyen has investigated exclusively the equal area law of holographic entanglement entropy and his result showed that, as black hole entropy, the equal area law also holds for the entanglement entropy regardless of the size of the entangling surface [10]. All of these works reinforce the viewpoint that thermal phase transition can also been captured by the entanglement entropy somehow.

In this paper, we intend to study the phase transition of holographic entanglement entropy in massive gravity. The main motivation of this paper is to explore whether the thermal phase transition in an infinite volume can also be described by holographic entanglement entropy, which has not been reported in the literature until now. Massive gravity, where the graviton is endowed

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* Corresponding author.
   E-mail addresses: xxzeng@itp.ac.cn (X.-X. Zeng), hzhang@vub.ac.be (H. Zhang), lilf@itp.ac.cn (L.-F. Li).

with mass, is one of the gravity theory that attends to modify general relativity for it is not UV complete. An important black hole solution that is ghost-free in massive gravity has been presented recently in [11]. It was found that the graviton plays the same role as the lattice in the holographic conductor model. In light of this interesting property, Ref. [12] generalized this solution to more general cases and studied the corresponding thermal phase transition in the grand canonical ensemble and canonical ensemble. The $P - V$ criticality was also investigated in the extended phase space [13]. The result shows that the phase transition depends not only on the curvature parameter k but also the graviton mass m. In particular, even for the case $k = 0$, the Van der Waals-like phase transition may also occur provided a proper parameter related to the graviton mass is given. This interesting character is the main motivation for us to study phase transition of holographic entanglement entropy in this spacetime because it provides a background with infinite volume on the boundary in which the Van der Waals-like phase transition can take place.

The organization of this paper is as follows. In the next section, after a brief review of the black hole solution in massive gravity, we will study the thermal phase transitions and critical phenomena for a black hole and black brane in the fixed charge ensemble, respectively. In both spacetimes, the equal area laws are checked and the critical exponents are calculated. In Section 3, we will concentrate mainly on studying the phase transition of holographic entanglement entropy in the entanglement entropy–temperature plane. For both the black hole and black brane, we find the equal area law is valid and the critical exponent of the heat capacity in the neighborhood of the critical point is the same as that obtained from the thermal phase transition. The last section is devoted to our conclusions and discussions.

2. Black hole solutions in massive gravity and thermal phase transition

The action for an $(n + 2)$-dimensional massive gravity is given by [11]

$$S = \frac{1}{16\pi G_{n+2}} \int d^{n+2}x \sqrt{-g}[R + \frac{n(n+1)}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 \sum_i^4 c_i \mathcal{U}_i(g, f)]. \quad (2.1)$$

Here G_{n+2} is the Newton's gravitational constant, which will be set to 1 later on. In addition, m is the graviton mass, l is the AdS radius, f is the reference metric. Here the reference metric is given by $f_{\mu\nu}dx^\mu dx^\nu = h_{ij}dx^i dx^j$, where $h_{ij}dx^i dx^j$ is the line element for a 2-dimensional Einstein space with constant curvature $2k$, $k = -1, 0, 1$ corresponds to a sphere, flat, or hyperbolic horizon for the spacetime, individually. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, c_i are constants and $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the $(n + 2) \times (n + 2)$ matrix $\mathcal{K}^\mu_\nu \equiv \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$, i.e.,

$\mathcal{U}_1 = [\mathcal{K}]$,

$\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2]$,

$\mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]$,

$\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]$, $\quad (2.2)$

in which $[\mathcal{K}] = \mathcal{K}^\mu_\mu$. Starting from the action in (2.1), Ref. [12] has recently got the solution of black holes in such a massive gravity as

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 h_{ij}dx^i dx^j, \quad i, j = 1, 2, 3, \cdots, n \quad (2.3)$$

with a chemical potential

$$\mu = \frac{Q}{(n-1)r_+^{n-1}}, \quad (2.4)$$

in which

$$f(r) = k + \frac{r^2}{l^2} - \frac{M}{r^{n-1}} + \frac{Q^2}{2n(n-1)r^{2(n-1)}} + \frac{c_0 c_1 m^2}{n}r + c_0^2 c_2 m^2$$
$$+ \frac{(n-1)c_0^3 c_3 m^2}{r} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r^2}, \quad (2.5)$$

r_+ is the event horizon which satisfies $f(r_+) = 0$, M is the black hole mass, and Q is the black hole charge. In this paper, we only consider the four-dimensional spacetime, namely $c_3 = c_4 = 0$ as shown in [12]. For convenience, we will relabel the remanent graviton mass dependent parameters, $c_0 c_1 m^2/2$, $c_0^2 c_2 m^2$, as a, b respectively later in this paper.

The thermal phase structure of the spacetime in (2.3) has been investigated extensively [12,13]. In this paper, we are interested in the Van der Waals-like phase transition.[1] To compare the thermal entropy phase transition with entanglement entropy phase transition directly, we focus only on the phase transition taking place at the T–S plane. In particular we will discuss the phase structure of the thermal entropy and entanglement entropy for both the black hole and black brane, namely $k = 1$ and $k = 0$ in (2.5).[2]

2.1. The case of $k = 1$

In this case, the metric in (2.5) can be simplified as

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{4r^2} + \frac{r^2}{l^2} + ar + b. \quad (2.6)$$

The temperature and entropy of the black hole are

$$T_{bh} = \frac{f(r)'}{4\pi}\Big|_{r_+} = \frac{12r_+^4 + l^2(-Q^2 + 4r_+^2(1 + b + 2ar_+))}{16l^2\pi r_+^3}, \quad (2.7)$$

$$S = \pi r_+^2. \quad (2.8)$$

Substituting (2.8) into (2.7) and eliminating the parameter r_+, we can get the relation between the temperature T_{bh} and entropy S, namely

$$T_{bh} = \frac{12S^2 + l^2(-\pi^2 Q^2 + 4(1 + b)\pi S + 8a\sqrt{\pi}S^{3/2})}{16l^2\pi^{3/2}S^{3/2}}. \quad (2.9)$$

Based on this relation, one can plot the isocharge curves on the T_{bh}–S plane for a given charge. The plot is presented in Fig. 1, which is obviously similar to that of the Van der Waals phase transition. As can be seen from this plot, for different charges, the curves behave differently. For the small charge in (a), we can see that the curve is not monotonic. According to the definition of the specific heat capacity

$$C_Q = T_{bh}\frac{\partial S}{\partial T_{bh}}\Big|_Q, \quad (2.10)$$

we know that the specific heat capacity changes from positive to negative and then to positive again. That is, for an intermediate

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[1] This behavior can be observed in both the T–S plane in general phase space and P–V plane in extended phase space, where the cosmological constant is viewed as a dynamical pressure of the black hole system. It has been pointed out that these two perspectives are related to each other by a duality similar to the T-duality of string theory [14].

[2] In principle the case of $k = -1$ can also be analysed, but in practice it is hard to choose a proper entangling surface to implement the calculation.
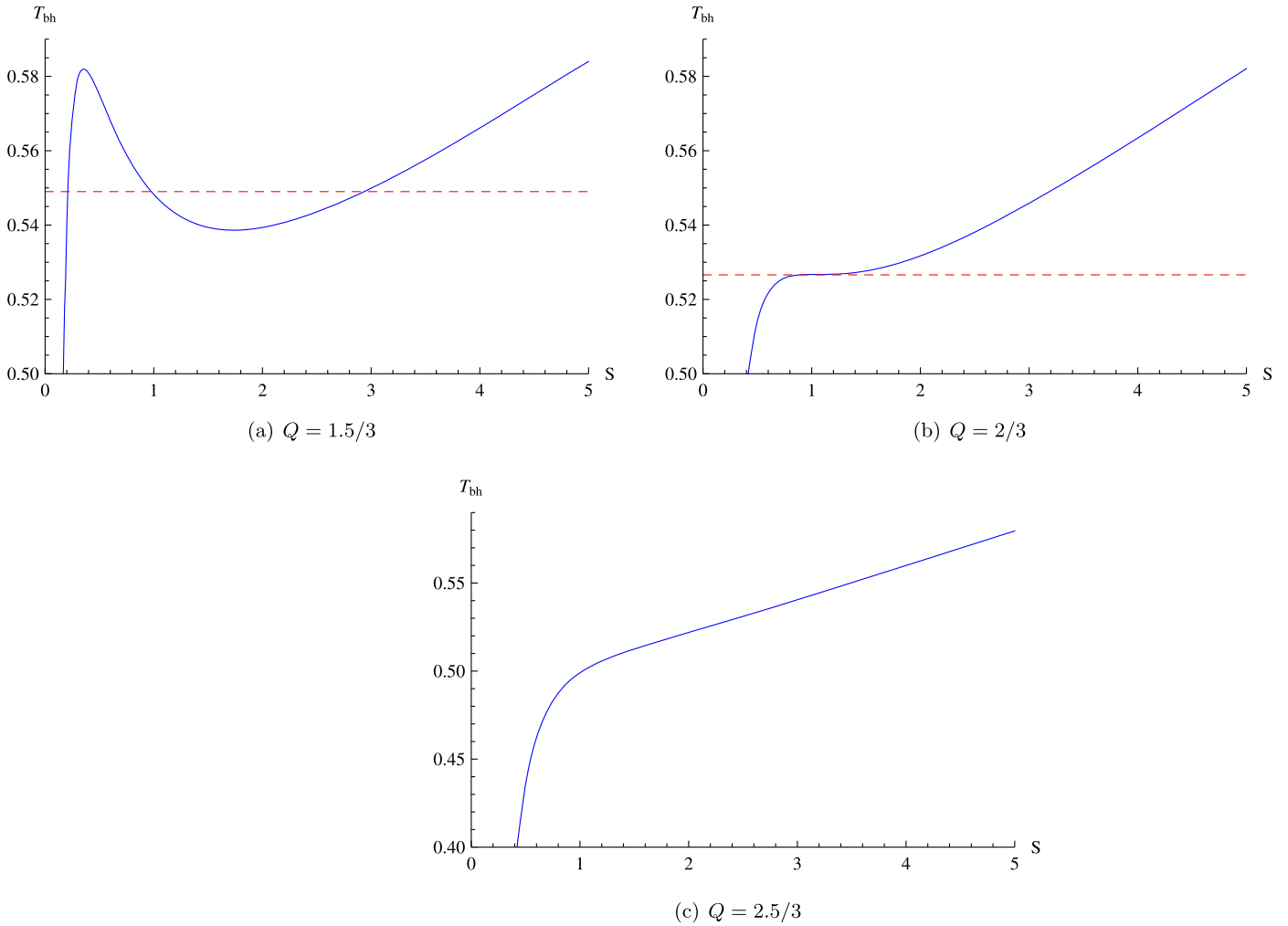
(a) $Q = 1.5/3$

(b) $Q = 2/3$

(c) $Q = 2.5/3$

Fig. 1. Relation between the entropy and temperature for a black hole in a fixed charge ensemble. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.

range of event horizon r_+ the black hole is thermodynamically unstable. In other words, there are three holes competing thermodynamically. The smallest hole continues to win until a temperature $T_\star$, above which the system jumps to the large hole. As stressed in [15], this kind of transition is a first order phase transition. When the charge increases to the critical charge, the smallest hole and the largest hole merge into one and squeeze out the unstable black hole. So there is an inflection point in the T_{bh}–S plane, which is shown in (b) of Fig. 1. One can see that the specific heat capacity is divergent at the inflection point and it has been shown that this phase transition is a second order phase transition. At the inflection point, the critical charge and critical entropy are determined by the following equations

$$\left(\frac{\partial T_{bh}}{\partial S}\right)_Q = \left(\frac{\partial^2 T_{bh}}{\partial S^2}\right)_Q = 0. \tag{2.11}$$

Substituting (2.9) into (2.11), we find the critical charge, critical entropy and critical temperature

$$Q_c = \frac{l(1+b)}{3}, \tag{2.12}$$

$$S_c = \frac{l^2(1+b)\pi}{6}, \tag{2.13}$$

$$T_c = \frac{4 + 4b + \sqrt{6}a\sqrt{(1+b)l^2}}{2\sqrt{6}\sqrt{(1+b)l^2}\pi}. \tag{2.14}$$

Obviously these critical parameters depend on the parameters associated with the graviton mass.[3] As a and b vanish, these values are the same as that of the AdS-RN black hole. As $b > -1$, we can observe that there is always a Van der Waals-like phase transition. In this paper, we choose $a = b = 1$, and for convenience, we also set the AdS radius $l = 1$.

For the first order phase transition, we will check whether Maxwell's equal area law holds in this background. The first order transition temperature $T_\star$ plays an important role on the equal area law. To find $T_\star$, we first plot the curve about the free energy $F = M - TS$ [12] with respect to the temperature T. The plot is shown in Fig. 2. We see that there is a swallowtail structure, which corresponds to the unstable phase in (a) of Fig. 1. The non-smoothness of the junction indicates that the phase transition is a first order. The critical temperature $T_\star$ is apparently given by the horizontal coordinate of the junction. From Fig. 2, we find $T_\star = 0.549$. Substituting this temperature into (2.9), we can get three values of the entropy $S_1 = 0.210442$, $S_2 = 0.977147$, $S_3 = 2.93091$. With these values, we can now check Maxwell's equal area law, which states

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[3] In this paper, we are not interested in the effect of the graviton mass on the critical behavior, for more information about this topic please see [12,13].
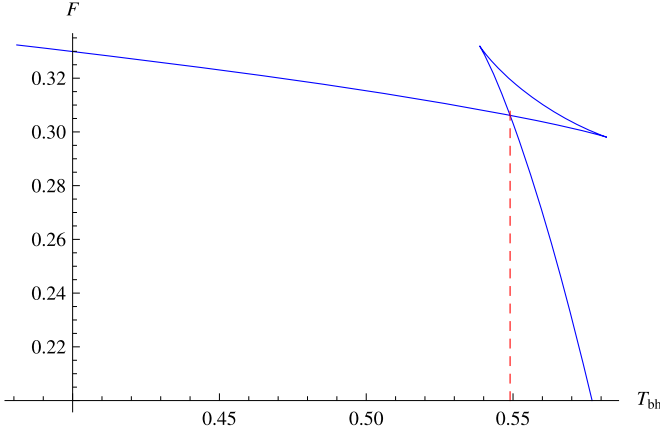
Fig. 2. Relation between the free energy and temperature for a black hole at $Q = 1.5/3$. The intersection point between the red dotted line and horizontal coordinate is the temperature of first order phase transition.

$$T_\star(S_3 - S_1) = \int_{S_1}^{S_3} T_{bh}(S, Q)dS. \tag{2.15}$$

After simplying the whole calculation, we find both the left and right of (2.15) equal to 1.49354 exactly. Thus, Maxwell's equal area law is verified in this background.

For the second order phase transition, we are interested in the critical exponent associated with the heat capacity defined in (2.10). Near the critical point, we write the entropy as $S = S_c + \delta$, where S_c is defined in (2.13). Expanding the temperature in small δ, we find

$$
\begin{aligned}
T_{bh} = & \frac{12S_c^2 + l^2\left(-\pi^2 Q^2 + 4(1+b)\pi S_c + 8a\sqrt{\pi}S_c^{3/2}\right)}{16l^2\pi^{3/2}S_c^{3/2}} \\
& + \frac{\left(12S_c^2 + l^2\pi\left(3\pi Q^2 - 4(1+b)S_c\right)\right)\delta}{32l^2\pi^{3/2}S_c^{5/2}} \\
& + \frac{3\left(-4S_c^2 + l^2\pi\left(-5\pi Q^2 + 4(1+b)S_c\right)\right)\delta^2}{128l^2\pi^{3/2}S_c^{7/2}} \\
& + \frac{\left(12S_c^2 + 5l^2\pi\left(7\pi Q^2 - 4(1+b)S_c\right)\right)\delta^3}{256l^2\pi^{3/2}S_c^{9/2}}.
\end{aligned}
\tag{2.16}
$$

Near the critical point, the first term on the right is the critical temperature, the second and third terms vanish according to (2.11). So we get

$$T_{bh} - T_c = \frac{\left(12S_c^2 + 5l^2\pi\left(7\pi Q^2 - 4(1+b)S_c\right)\right)}{256l^2\pi^{3/2}S_c^{9/2}}(S - S_c)^3. \tag{2.17}$$

With the definition of the heat capacity, we further find $C_q \sim (T_{bh} - T_c)^{-2/3}$, namely the critical exponent is $-2/3$, which is the same as the one from the mean field theory.

2.2. The case of $k = 0$

For the case $k = 0$, the spacetime we are considered is a black brane. The Hawking temperature is

$$T_{bb} = -\frac{Q^2 - 4r_+^2(b + r_+(2a + 3r_+))}{16\pi r_+^3}. \tag{2.18}$$

Replacing the event horizon r_+ by the entropy S, (2.18) can be rewritten as

$$T_{bb} = \frac{-\pi^2 Q^2 + 4b\pi S + 8a\sqrt{\pi}S^{3/2} + 12S^2}{16\pi^{3/2}S^{3/2}}, \tag{2.19}$$

whereby, we can also observe the Van der Waals-like phase transition, which is presented in Fig. 3. Based on (2.11), we can also get the critical charge, critical entropy and critical temperature

$$Q_c = b/3, \tag{2.20}$$

$$S_c = \frac{b\pi}{6}, \tag{2.21}$$

$$T_c = \frac{3a + 2\sqrt{6}\sqrt{b}}{6\pi}. \tag{2.22}$$

It is obvious that these critical values depend on the parameters associated with the graviton mass. As a and b vanish, the Van der Waals-like phase transition disappears. That is, in Einstein gravity, this is no Van der Waals-like phase transition for an AdS-RN black brane.

To check Maxwell's equal area law, we should find the first order phase transition temperature $T_\star$ from the $F - T_{bb}$ plot, which is shown in Fig. 4. Reading from the horizontal coordinate of the junction, we find $T_\star = 0.4316$. Substituting this temperature into (2.19), we get $S_1 = 0.131235$, $S_3 = 1.32697$, where S_1, S_3 are the smallest and largest roots of equation $T_{bb} = T_\star$. Substituting these values to (2.15), we find the left equals 0.5161 and the right equals 0.5162. Obviously both the left and the right are equal nearly within our numerical accuracy, which indicates that Maxwell's equal area law is satisfied in this background.

Similarly we can also get the critical exponent for the second phase transition in (b) of Fig. 3. Writing the entropy as $S = S_c + \delta$ and expanding the temperature in small δ, we find

$$T_{bb} - T_c = \frac{\left(35\pi^2 Q^2 - 20b\pi S_c + 12S_c^2\right)}{256\pi^{3/2}S_c^{9/2}}(S - S_c)^3, \tag{2.23}$$

where we have used (2.11). This relation is similar to that of the black hole. That is to say, the critical exponent is $-2/3$ too.

3. Holographic entanglement entropy and its phase structure

Having understood the phase structure of a black hole and black brane in massive gravity from the viewpoint of thermodynamics, we will study entanglement entropy in both spacetimes. The main motivation is to check whether entanglement entropy has the similar phase structure as that of the thermal entropy.

For a given quantum field theory described by a density matrix ρ, entanglement entropy for a region A and its complement B is defined as

$$S_A = -Tr_A(\rho_A \ln \rho_A), \tag{3.1}$$

where $\rho_A = Tr_B(\rho)$ is the reduced density matrix. Usually it is not easy to calculate this quantity in field theory. But from the viewpoint of holography, Ref. [7] gave a very simple geometric description for computing S_A for static states in terms of the area of a bulk minimal surface anchored on ∂A, which states that

$$S_A = \frac{\text{Area}(\gamma)}{4}, \tag{3.2}$$

where γ is the codimension-2 minimal surface with boundary condition $\partial\gamma = \partial A$. Here we will still use (3.2) to calculate entanglement entropy and study the corresponding phase structure for preliminary study.
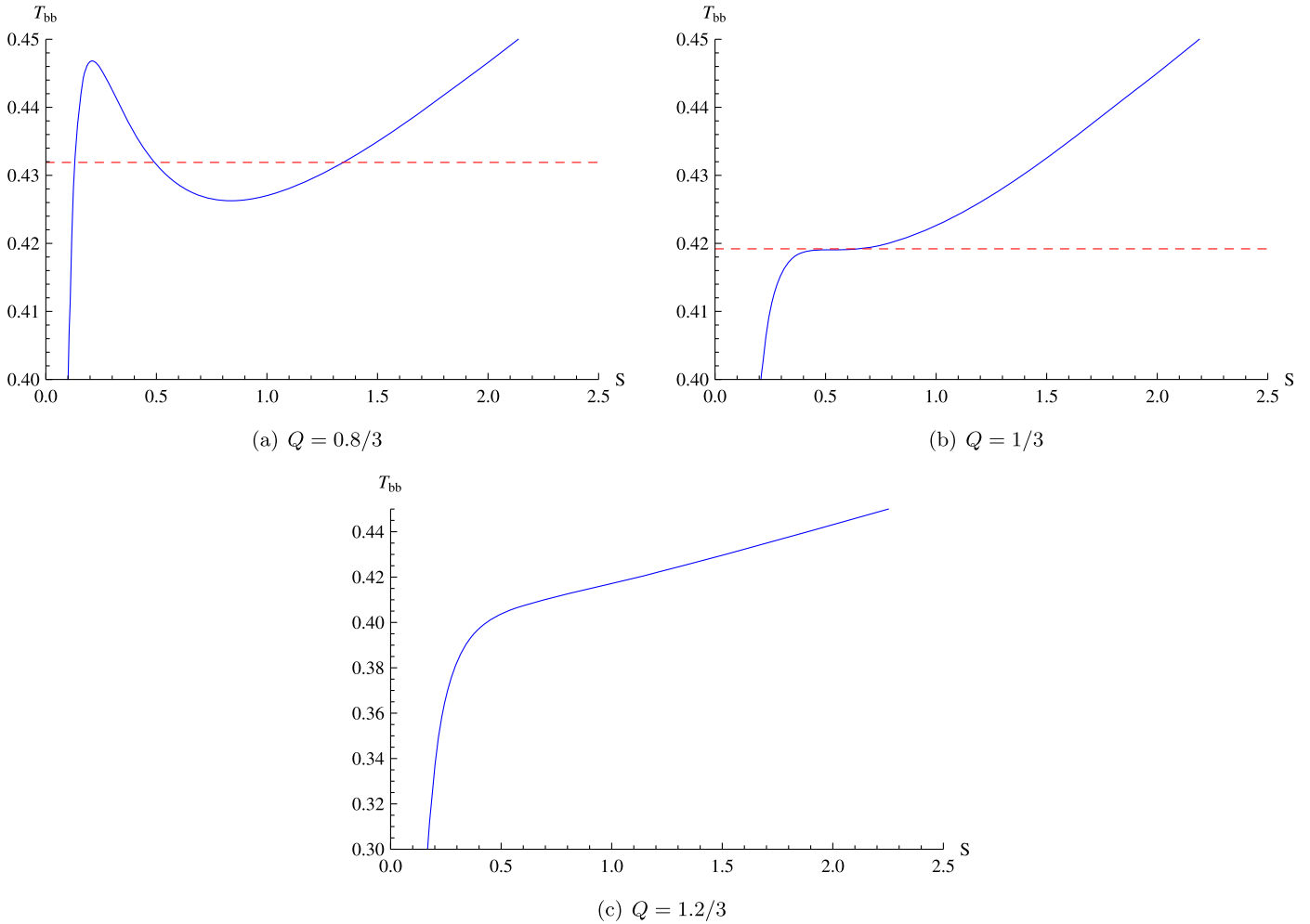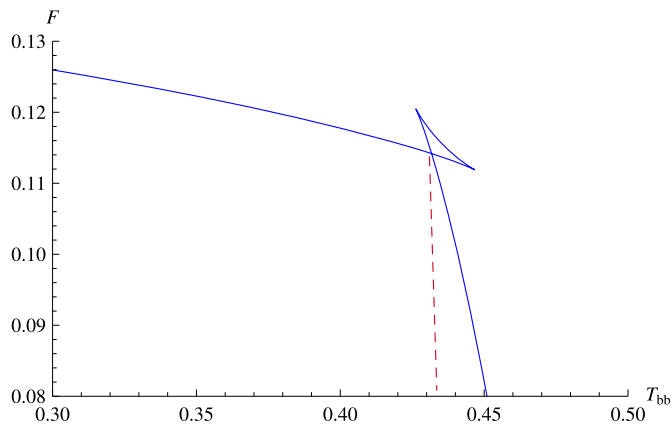
Fig. 3. Relation between the entropy and temperature for a black brane in a fixed charge ensemble. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.

3.1. Phase transition of entanglement entropy in a finite volume system

For a black hole, the space on the boundary is a round. The volume of the space is finite in this case. To avoid the entanglement entropy to be affected by the surface that wraps the horizon, we will choose a small region as A. To be more precise, as done in [15,16], we choose the region A to be a spherical cap on the

Fig. 4. Relation between the free energy and temperature for a black brane at $Q = 0.8/3$. The intersection point between the red dotted line and horizontal ordinate is the temperature of first order phase transition.

boundary delimited by $\theta \leq \theta_0$. In this case the area can be written as

$$A = 2\pi \int_0^{\theta_0} \mathcal{L}(r(\theta), \theta) d\theta, \quad \mathcal{L} = r \sin\theta \sqrt{\frac{(r')^2}{f(r)} + r^2}, \tag{3.3}$$

in which $r' = dr/d\theta$. Imagining θ as time, and treating $\mathcal{L}$ as the Lagrangian, one can get the equation of motion for $r(\theta)$ by making use of the Euler–Lagrange equation. It seems to be impossible to get the analytical solution of $r(\theta)$, so we will solve it numerically with the boundary conditions $r'(0) = 0$, $r(0) = r_0$. With the numerical result of $r(\theta)$, we can easily obtain the area and further entanglement entropy. Note that the area is divergent, so it should be regularized by subtracting off the area in pure AdS with the same entangling surface on the boundary. In Einstein gravity, one can often get an analytical solution of the minimal surface in the pure AdS. In our massive gravity, we find that the spacetime for the pure AdS is given by[4]

$$dS^2 = -(1 + \frac{r^2}{l^2} + ar + b)dt^2 + \frac{1}{(1 + \frac{r^2}{l^2} + ar + b)}dr^2$$

$$+ r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4}$$

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[4] Note that in this case, the space is not strictly a pure AdS, but for convenience we also call it pure AdS.
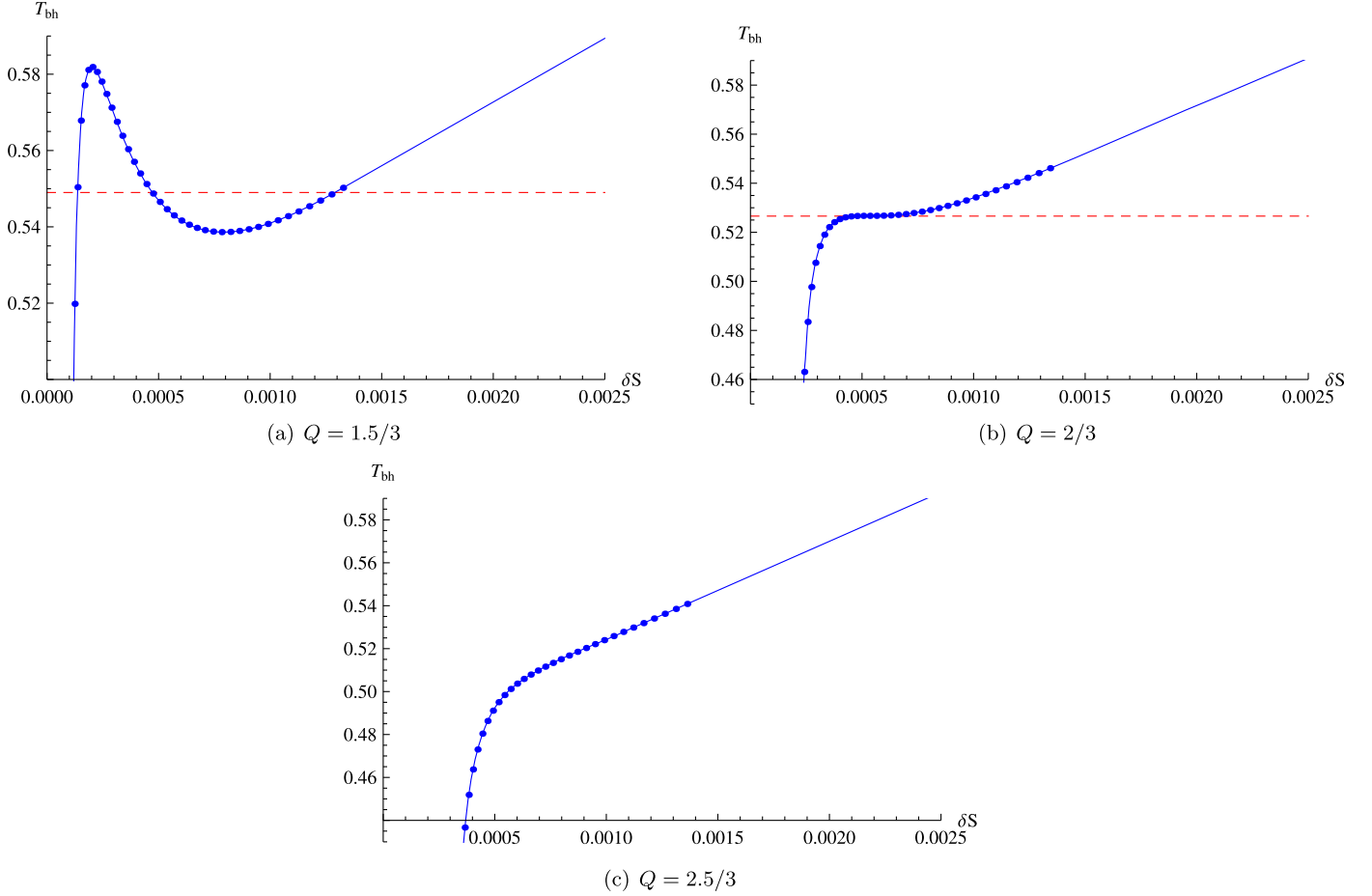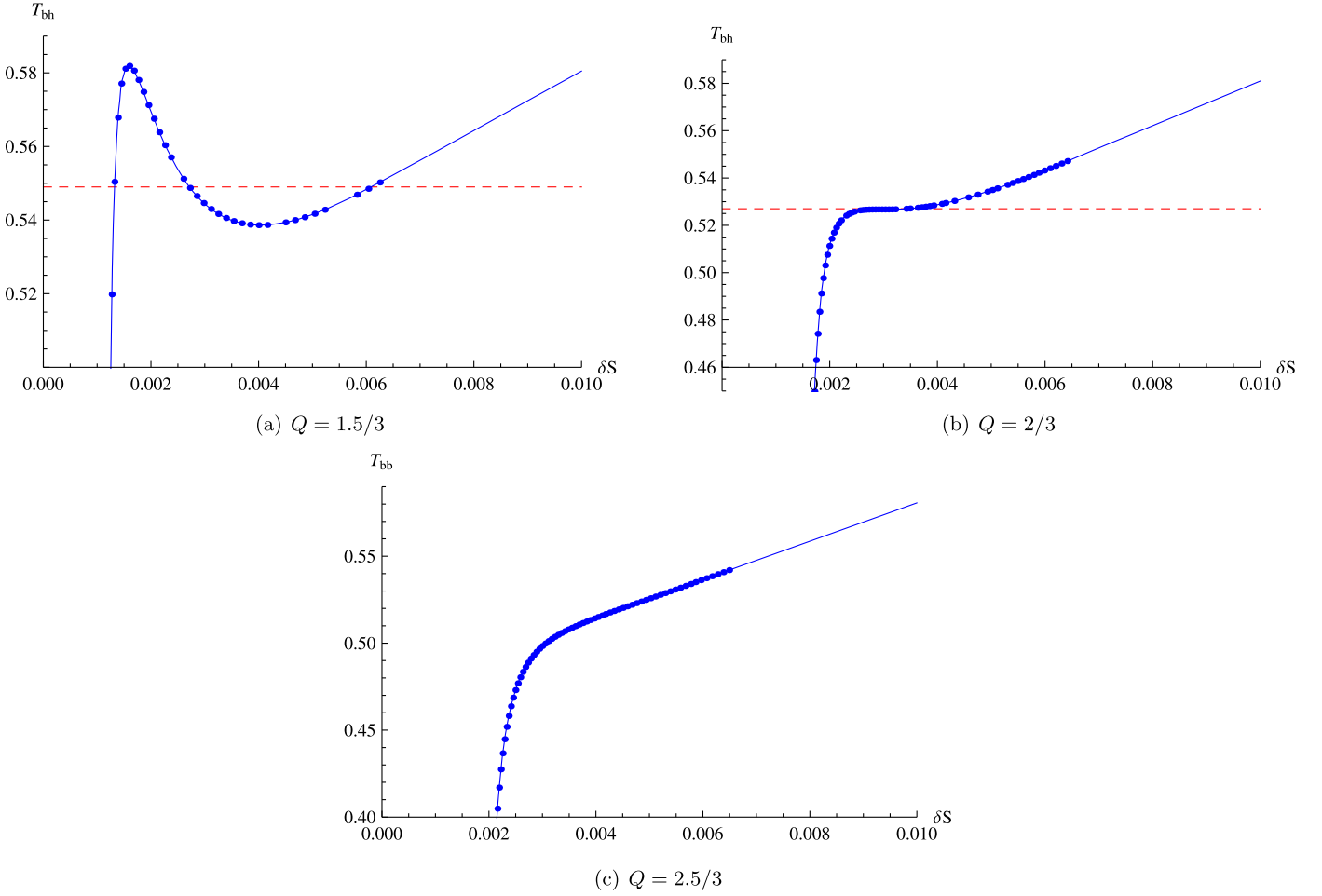
Fig. 5. Relation between the entanglement entropy and temperature for a black hole in a fixed charge ensemble. The boundary is located at $\theta_0 = 0.1$. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.

It is obvious that the parameters related to the graviton mass affect the structure of the pure AdS. The analytical result for the pure AdS in Einstein gravity thus cannot be used and we obtain the corresponding minimal surface for the pure AdS in our massive gravity by numeric strategy too. We label the regularized entanglement entropy as δS. For the numerical computation, we choose $\theta_0 = 0.1$ and $\theta_0 = 0.16$. The UV cutoff in the dual field theory is set to be $r(0.099)$ and $r(0.159)$ by holography respectively. To compare with the phase transition of thermal entropy, we will study the relation between the entanglement entropy and Hawking temperature, which is regarded as the temperature of the dual field theory. The numeric results for $\theta_0 = 0.1$ and $\theta_0 = 0.16$ are shown in Fig. 5 and Fig. 6 respectively. We can see that for a given charge, the relation between the entanglement entropy and temperature is similar to that between the black hole entropy and temperature. That is, there is a critical charge and below it the first order phase transition occurs. In this case there is an unstable hole interpolating between a small and a large stable hole. As the temperature grows to the critical temperature $T_\star$, the small black hole jumps to the large black hole. As the charge grows to the critical charge, the small hole and the large hole merge so that the unstable hole shrinks into an inflection point where the second order phase transition occurs. For a large enough charge, a large stable black hole forms and the entanglement entropy grows monotonically as the temperature increases. Comparing Fig. 5 with Fig. 6, we find the profile for different sizes of entangling surface seems to be the same, which indicates that the phase structure of entanglement entropy does not depend on the size of entangling surface.

For the first order phase transition of the entanglement entropy, we now check whether Maxwell's equal area law holds. For the case $\theta_0 = 0.1$, we first construct an Interpolating Function of the temperature $T(\delta S, Q)$ using the data obtained numerically.[5] At the first order phase transition point, we find the smallest and largest roots for the equation $T = T_\star$ are $\delta S_1 = 0.000136338$, $\delta S_3 = 0.00128997$. Substituting these values into the equal area law

$$T_\star(\delta S_3 - \delta S_1) = \int_{\delta S_1}^{\delta S_3} T(\delta S, Q)d\delta S, \tag{3.5}$$

we find the left equals 0.0006333 and the right equals 0.0006337. Obviously both the left and the right are equal within our numerical accuracy. Similarly, for the case $\theta_0 = 0.16$, we find the left equals 0.002625 and the right equals 0.002627, which also indicates that the equal area law is satisfied.

Now let's turn to the critical exponent of the second order phase transition of the entanglement entropy. Similar to the definition of specific heat capacity in thermodynamics, we also define an analogous specific heat capacity for entanglement entropy

$$\mathcal{C}_Q = T_{bh} \frac{\partial \delta S}{\partial T_{bh}} \Big|_Q. \tag{3.6}$$

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[5] Though the entanglement entropy resembles the black hole entropy, they have different function dependence of temperature.

(a) $Q = 1.5/3$

(b) $Q = 2/3$

(c) $Q = 2.5/3$

Fig. 6. Relation between the entanglement entropy and temperature for a black black hole in a fixed charge ensemble. The boundary is located at $\theta_0 = 0.16$. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.
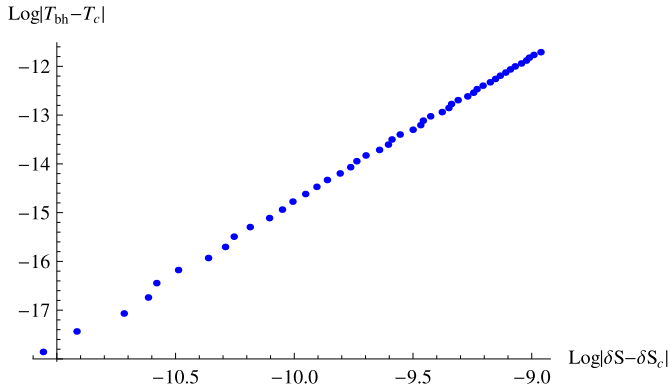

Fig. 7. Plot of $\log | T_{bh} - T_c |$ versus $\log | \delta S - \delta S_c |$ based on 45 points closest to the inflection point in Fig. 5(b).

Provided a similar relation as that in (2.17) is satisfied, then with (3.6) we can get the critical exponent of phase transition of entanglement entropy. Here we are interested in the logarithm of the quantities $T_{bh} - T_c$, $\delta S - \delta S_c$, in which T_c is the critical temperature in (2.14) and δS_c is the critical entropy obtained numerically. Taking the case $\theta = 0.1$ as an example, the linear relation is shown in Fig. 7 and the analytical relation can be fitted as

$$\log | T_{bh} - T_c | = 15.0607 + 2.983 \log | \delta S - \delta S_c |. \tag{3.7}$$

Obviously the slope is about 3, therefore the critical exponent for the second order phase transition of entanglement entropy agrees with that of the thermal entropy.

3.2. Phase transition of entanglement entropy in an infinite volume system

As have seen in Section 2.2, for a black brane in massive gravity, there exists a Van der Waals-like phase transition, which does't exist in Einstein gravity. So we will study entanglement entropy in this spacetime to check whether the phase structure of entanglement entropy resembles that of the thermal entropy.

To study the entanglement entropy, we should choose a proper region for A. Here we choose a rectangular strip parametrized by the boundary coordinates (x_1, x_2) with the assumption that this infinite strip is invariant under the x_2-direction. In order to make the notation as simple as possible, we would like to rename this exceptional coordinate x_1 as x and employ it to parameterize the minimal area surface such that the entanglement entropy in this background can be written as

$$A = L \int_{-\ell/2}^{\ell/2} r \sqrt{\frac{(r')^2}{f(r)} + r^2} dx, \tag{3.8}$$

(a) $Q = 0.8/3$
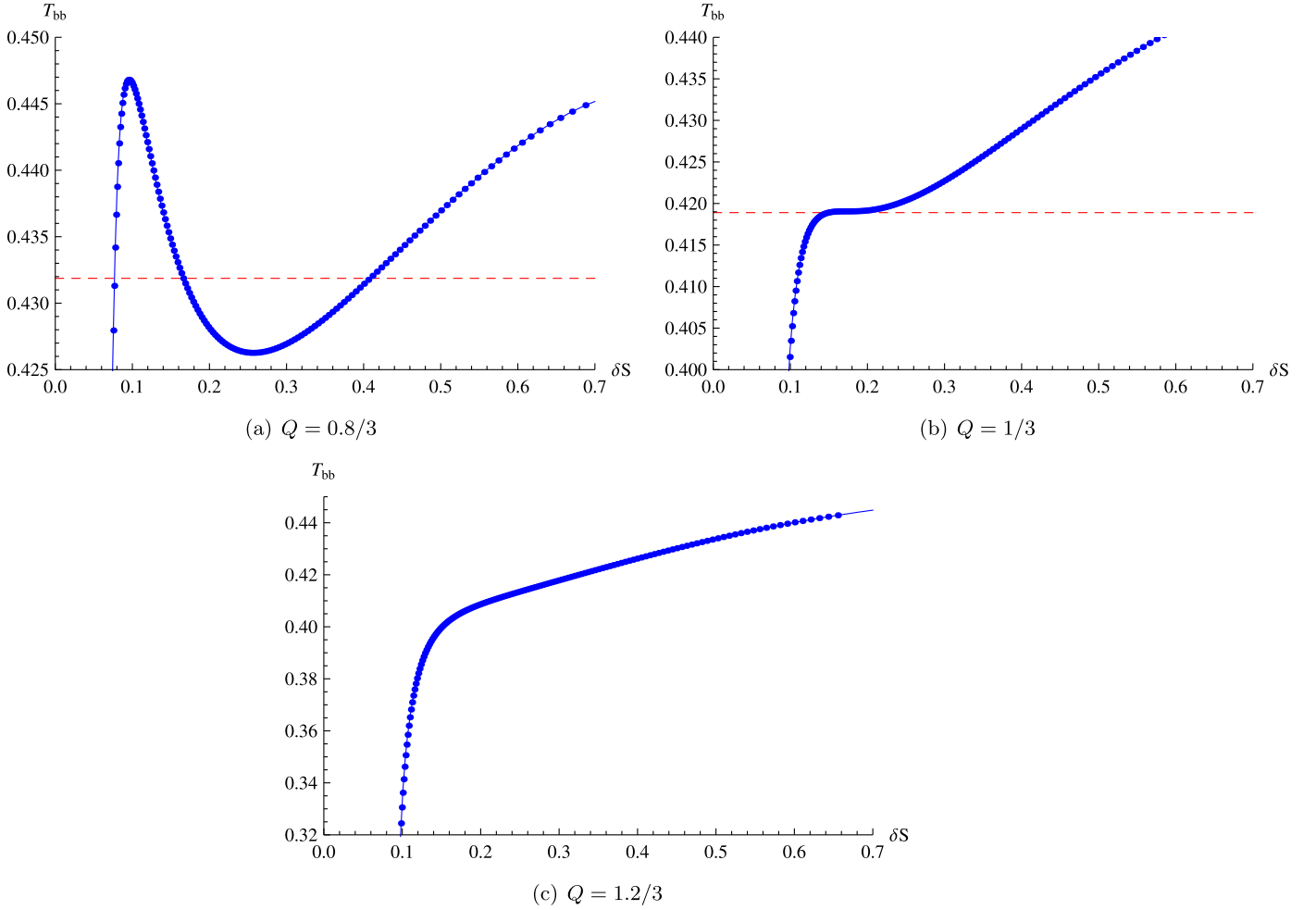

(b) $Q = 1/3$

(c) $Q = 1.2/3$

Fig. 8. Relation between the entanglement entropy and temperature for a black black brane in a fixed charge ensemble. The boundary separation is $\ell = 1.2$. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.

in which L is length of the minimal surface along the x_2-direction which is set to be 1 here, ℓ is the boundary separation connected by two boundary points: $(t_0, -\ell/2)$, $(t_0, \ell/2)$, $r' = dr/dx$ and

$$f(r) = -\frac{2M}{r} + \frac{Q^2}{4r^2} + \frac{r^2}{l^2} + ar + b. \quad (3.9)$$

The integrand in (3.8) can be thought as the Lagrangian $\mathcal{L}$ of a fictitious system with x the proper time. Since the Lagrangian does not depend explicitly on x, there is an associated conserved quantity, which is useful for us to get the following equation of motion of $r(x)$

$$4r(x)^2 f(r)^2 - 2f(r)r'(x)^2 + r(x)r'(x)^2 f'(r) - 2f(r)rr''(x) = 0. \quad (3.10)$$

To solve $r(x)$, we use the boundary conditions

$$r(0) = r_0, r'(0) = 0. \quad (3.11)$$

Substituting the numerical result of $r(x)$ into (3.8), we can get the area and further the entanglement entropy. We are also interested in the regularized entanglement entropy $\delta S \equiv S - S_0$, where S_0 is entanglement entropy in the pure AdS space of our massive gravity, which is not amenable to analytical calculation but can be obtained by our numerical strategy. In what follows, we are interested in the cases $\ell = 1.2$, $\ell = 2.2$. The UV cutoff is set to be $r(1.19)$ and $r(2.19)$. The numerical results are shown in Fig. 8.

and Fig. 9. We can see that the behavior of the curves in Fig. 8 or Fig. 9 is reminiscent of that in Fig. 3. Namely in the T_{bb}–δS phase space, the phase structure of entanglement entropy depends on the charge of the black brane. As the charge grows from small to large enough, the first order phase transition and second order phase transition occur successively, and a stable black brane forms lastly.

We can also check whether Maxwell's equal area law holds for the first order phase transition in (a) of Fig. 8 and Fig. 9. For the case $\ell = 1.2$, the smallest and largest roots for the equation $T(\delta S, Q) = T_\star$ are $\delta S_1 = 0.167805$, $\delta S_3 = 0.403158$, where $T(\delta S, Q)$ is an Interpolating Function and $T_\star$ is the critical temperature of the first order phase transition. Substituting these values into (3.5), we find the left equals 0.10080 and the right equals 0.10076. Similarly, for the case $\ell = 2.2$, the left and right of (3.5) are found to be 0.20568 and 0.20565 respectively. It is obvious that the equal area law is satisfied and its validity is independent of the boundary separations.

We can also get the critical exponent of the heat capacity in the neighborhood of the critical point for the second order phase transition, which is shown in (b) of Fig. 8 or Fig. 9. Taking the case $\ell = 1.2$ as an example. According to the numeric result of the entanglement entropy-temperature curve, we first construct an Interpolating Function $T(\delta S, Q)$. At the critical temperature, we obtain the critical entropy δS_c by solving the equation $T(\delta S, Q) = T_c$, where T_c is defined in (2.18). The relation between $\log |T_{bb} - T_c|$
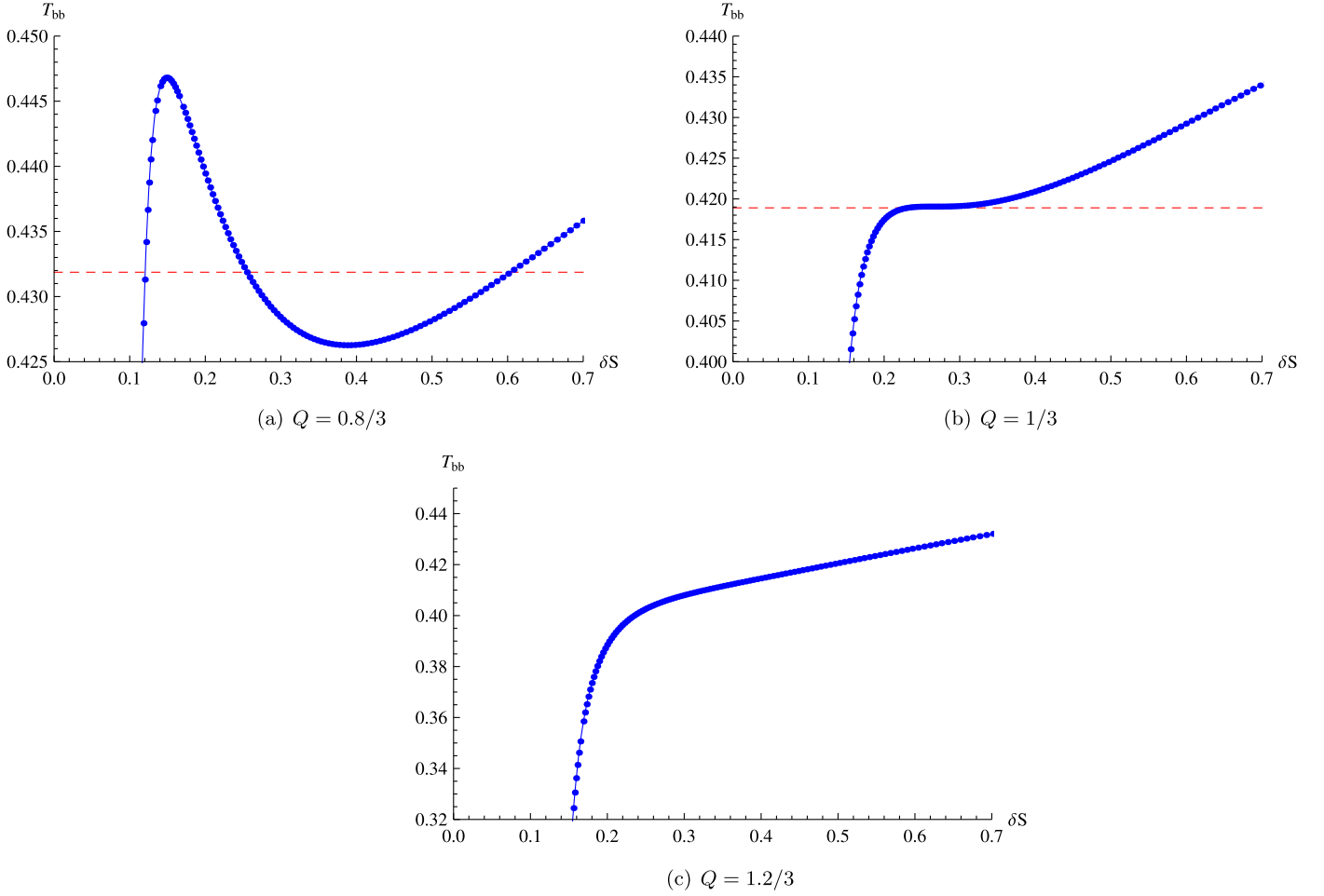
Fig. 9. Relation between the entanglement entropy and temperature for a black black brane in a fixed charge ensemble. The boundary separation is $\ell = 2.2$. The red dotted lines in (a) and (b) correspond to the locations of first order phase transition and second order phase transition.
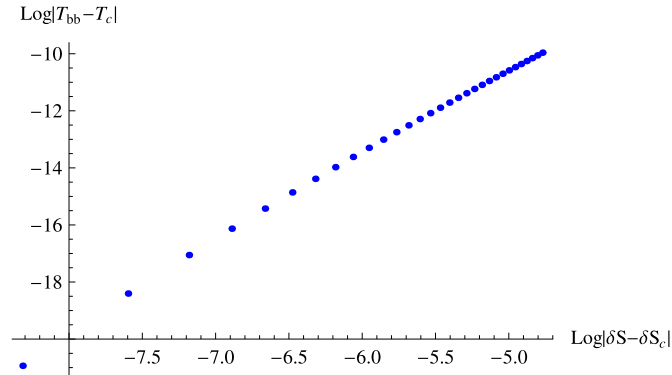

Fig. 10. Plot of $\log |T_{bh} - T_c|$ versus $\log |\delta S - \delta S_c|$ based on 31 points closest to the inflection point in Fig. 8(b).

and $\log |\delta S - \delta S_c|$ is plotted in Fig. 8. This curve can be fitted as (see Fig. 10)

$$\log |T_{bh} - T_c| = 6.29083 + 3.01454 \log |\delta S - \delta S_c|. \qquad (3.12)$$

It is obvious that the slope equals to 3 nearly. In other words, for entanglement entropy in quantum system with infinite volume, the critical exponent of the heat capacity also agrees with that of the thermal entropy.

4. Conclusions

In this paper, we have studied the phase structure of holographic entanglement entropy in massive gravity. Our result shows that the phase structure of holographic entanglement entropy is the same as that of the thermal entropy regardless of the volume of a spacetime on the boundary. So one is suggested to use the behavior of holographic entanglement entropy to identify the thermal phase transition. But maybe more importantly, one should be encouraged by these accumulated evidences to find a general proof for such a coincidence between the holographic entanglement entropy and thermal entropy.

Acknowledgements

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