The interaction of alternation points and poles in rational approximation

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Abstract

The interrelation of alternation points for the minimal error function and poles of best Chebyshev approximants is investigated if uniform approximation on the interval $[-1,1]$ by rational functions of degree $(n(s),m(s))$ is considered, $s \in \mathbb{N}$. In general, the alternation points need not to be uniformly distributed with respect to the equilibrium measure on $[-1,1]$, even not to be dense on the interval. We show that, at least for a subsequence $\mathbb{A} \subset \mathbb{N}$, the asymptotic behaviour of the alternation points to the degrees $(n(s),m(s))$, $s \in \mathbb{A}$, is completely determined by the location of the poles of the best approximants, and vice versa, if $m(s) \leq n(s)$ or $m(s) - n(s) = o(s/\log s)$ as $s \to \infty$.

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1. Introduction

Denote by $\mathcal{R}_{n,m}$ the collection of real rational functions with numerator in $\mathcal{P}_n$ and denominator in $\mathcal{P}_m$, respectively, where $\mathcal{P}_k$ is the set of algebraic polynomials of degree at most $k$, $k \in \mathbb{N}_0$. For each
pair of nonnegative integers \((n, m)\) there exists a unique function \(r^*_{n, m} \in \mathcal{R}_{n,m}\) that is the best Chebyshev approximation to \(f \in C[-1, 1]\) in the sense that
\[
\| f - r^*_{n, m} \| < \| f - r \| \quad \text{for all } r \in \mathcal{R}_{n,m}, \ r \neq r^*_{n, m},
\]
where \(\| \cdot \|\) denotes the sup norm on \([-1, 1]\). Writing \(r = p_n/q_m\) where \(p_n \in \mathcal{P}_n\) and \(q_m \in \mathcal{P}_m\) have no common factor, the defect of \(r\) is defined by
\[
\delta_{n,m}(r) := \min(n - \deg p_n, m - \deg q_m). \tag{1}
\]
Let us define
\[
d(r) = n + m + 1 - \delta_{n,m}(r) \tag{2}
\]
then \(d(r)\) is the dimension of the tangential space \(T(r)\) at the point \(r\) with respect to the coefficients of the numerator and denominator as parameter space. Moreover, \(T(r)\) is a Haar subspace. We write \(r^*_{n, m} = p^*_n/q^*_m\) with no common factors and define for abbreviation
\[
d_{n,m} := d(r^*_{n, m}).
\]
Then it is well-known that the best approximation \(r^*_{n, m}\) of \(f\) is characterized by the following equioscillation condition:
There exist \(d_{n,m} + 1\) points \(x^{(n,m)}_k\),
\[
-1 \leq x^{(n,m)}_0 < \cdots < x^{(n,m)}_{d_{n,m}} \leq 1,
\]
such that
\[
\lambda_{n,m} (-1)^k (f - r^*_{n, m})(x^{(n,m)}_k) = \| f - r^*_{n, m} \|, \quad 0 \leq k \leq d_{n,m}, \tag{3}
\]
where \(\lambda_{n,m} = +1\) or \(\lambda_{n,m} = -1\) is fixed. Such a set of points \(\{x^{(n,m)}_k\}\), called alternation set, is in general not unique. Therefore, in the following we denote by
\[
A_{n,m} = A_{n,m}(f) := \{x^{(n,m)}_k\}_{k=0}^{d_{n,m}}
\]
an arbitrary, but fixed alternation set for the best approximation \(r^*_{n, m}\) of \(f\) out of \(\mathcal{R}_{n,m}\).

Let \(v_{n,m}\) denote the normalized counting measure of \(A_{n,m}\), i.e.,
\[
v_{n,m}([\alpha, \beta]) := \frac{\# \{x^{(n,m)}_k : \alpha \leq x^{(n,m)}_k \leq \beta\}}{d_{n,m} + 1}. \tag{4}
\]
Kadec [6] has shown that there exists a subsequence \(A\) of \(\mathbb{N}\) such that
\[
v_{n,0} \rightharpoonup^* \mu \quad \text{as } n \in A, \ n \to \infty, \tag{5}
\]
where \(\mu\) is the equilibrium measure of \([-1, 1]\). For rational approximation, Borwein et al. [4] have proved that denseness in \([-1, 1]\) of a subsequence of alternation sets \(A_{n,m}\) holds whenever \(m = m(n)\) and \(n/m(n) \to \kappa > 1\) as \(n \to \infty\). Moreover, they have shown in the case \(\lim_{n \to \infty} m(n)/n = 0\) that there exists \(A \subset \mathbb{N}\) such that
\[
v_{n,m(n)} \rightharpoonup^* \mu \quad \text{as } n \in A, \ n \to \infty,
\]
a result of Kadec’s type. Kroó and Peherstorfer [7] have obtained lower bounds for the number of alternation points in any interval \([x, \beta]\) of \([-1, 1]\) in the case that \(m = m(n) < n\), again for some specified subsequence \(A\) of \(\mathbb{N}\) (comp. [2]).

In [4], Borwein et al. have shown that for \(m(n) = n + 1\) and any \(\varepsilon > 0\) there exists an \(f \in C[-1, 1]\) with the property that all extreme points of \(f - r_{n,m}^*\) lie in the subinterval \([-1, -1 + \varepsilon]\) for every \(n = 1, 2, \ldots\). This situation, or more generally \(m(n) = n + s\), \(s \in \mathbb{Z}\) fixed, was considered by Braess et al. [5]. Their results were based on the number \(v_{n,m}(e)\) of poles of the best approximants that lie outside an \(e\)-neighbourhood of \([-1, 1]\). To give a taste of their theorems we want to cite the following results:

(i) If \(\lim_{n \to \infty} v_{n,m}(e)/\log n = \infty\) for some fixed \(e > 0\), then the point set \(\bigcup_{n=0}^{\infty} A_{n,n}(f)\) is dense in \([-1, 1]\).

(ii) If \(\lim_{n \to \infty} v_{n,m}(e)/n = 1\) for each \(e > 0\), then there exists a subsequence \(A\) of \(\mathbb{N}\) such that

\[
v_{n,n} \rightharpoonup \mu \quad \text{as} \quad n \in A, \quad n \to \infty.
\]

Hence, all these results show that there is a relation between the alternation points and the poles of the rational approximants \(r_{n,m}^*\). This idea was followed up in [3] where weak*-convergence results were obtained between the counting measures of the alternation sets \(v_{n,m}\), the counting measures of the poles and the equilibrium measure \(\mu\) of \([-1, 1]\). To be precise, let \(f\) be not a rational function and let \(n\) and \(m(n)\) satisfy

\[
m(n) \leq n, \quad m(n) \leq m(n + 1) \leq m(n) + 1.
\]

Moreover, let

\[
q_{m(n)}^*(x)q_{m(n+1)}^*(x) = b_n \prod_{i=1}^{l_n} (x - y_i)
\]

be the product of the denominators of \(r_{n,m(n)}^*\) and \(r_{n+1,m(n+1)}^*\), then

\[
\tau_n(A) := \frac{\# \{y_i : y_i \in A\}}{l_n} \quad (A \subset \mathbb{C})
\]

denotes the normalized counting measure of all finite poles of \(r_{n,m(n)}^*\) and \(r_{n+1,m(n+1)}^*\), counted with their multiplicities. Then in [3] it was proved that there exists a subsequence \(A \subset \mathbb{N}\) such that

\[
v_{n,m(n)} - \tau_n \rightharpoonup (1 - \alpha_n) \mu \rightharpoonup 0 \quad \text{as} \quad n \to \infty, \quad n \in A
\]

in the weak*-topology where

\[
\alpha_n = \frac{l_n}{d_{n,m(n)} + 1}
\]

and \(\overline{\tau}_n\) denotes the balayage measure of \(\tau_n\) onto \([-1, 1]\).

The purpose of the present paper is to obtain a convergence result of type (7), where the restriction \(m(n) \leq n\) in (6) is avoided. We point out that this condition was essentially used in the proofs in [3]. Hence, this restriction implied that a lot of known examples (cf. [8,10]) were outside this case.
2. Main result

In the following, we assume that the pairs

$$(n(s), m(s)) \in \mathbb{N}_0 \times \mathbb{N}_0$$

depend on parameters $s \in \mathbb{N}$. Let

$$E_{n(s), m(s)} := \inf_{r \in \mathbb{R}} \| f - r \| = \| f - r^*_{n(s), m(s)} \|$$

and define for abbreviation

$$E_s := E_{n(s), m(s)}, \quad r^*_s := r^*_{n(s), m(s)}, \quad p^*_s := p^*_{n(s)}, \quad q^*_s := q^*_{m(s)}, \quad d(s) := d_{n(s), m(s)}$$

and

$$x^{(s)}_k := x^{(n(s), m(s))}_k, \quad k = 0, 1, \ldots, d(s).$$

Again we use the normalized counting measure for the alternation sets \(\{x^{(s)}_k\}_{k=0}^{d(s)}\), namely

$$v_s(A) := \frac{\#\{x^{(s)}_k : x^{(s)}_k \in A\}}{d(s) + 1} \quad (A \in \mathbb{C}). \quad (8)$$

Moreover, we need the normalized counting measure $\tau_s$ of the union of the finite poles of $r^*_s$, and $r^*_{s+1}$. As above, all poles are counted with their multiplicities.

An important role is played by the balayage measure $\hat{\tau}_s$ of $\tau_s$ onto $[-1, 1]$. $\hat{\tau}_s$ is the unique measure supported on $[-1, 1]$ for which $\| \hat{\tau}_s \| = \| \tau_s \|$ and

$$U^{\hat{\tau}_s}(z) = U^{\tau_s}(z) + c, \quad z \in [-1, 1],$$

where

$$c = \int G(t, \infty) \, d\tau_s(t)$$

and $G(z, a)$ denotes Green’s function of $\Omega = \overline{\mathbb{C}} \setminus [-1, 1]$ with pole at $a \in \Omega$ (cf. [9, p.116]). Furthermore, $\hat{\tau}_s$ has the following properties:

(a) $U^{\hat{\tau}_s}(z) \leq U^{\tau}(z) + c, \quad z \in \mathbb{C}$.
(b) If $h$ is continuous on $\overline{\mathbb{C}}$ and harmonic in $\overline{\mathbb{C}} \setminus [-1, 1]$, then $\int \ h \, d\tau_s = \int \ h \, d\hat{\tau}_s$.

Our main result can be formulated in the following statement.

**Theorem.** Let $f$ be not a rational function and let $(n(s), m(s)) \in \mathbb{N}_0 \times \mathbb{N}_0, s \in \mathbb{N}$, be a strictly increasing sequence with

$$n(s) \leq n(s + 1) \leq n(s) + 1; \quad m(s) \leq m(s + 1) \leq m(s) + 1.$$  \quad (9)
Moreover, let $\kappa(s)$, $s \in \mathbb{N}$, be a sequence in $\mathbb{N}$ with
\[
\lim_{s \to \infty} \frac{\kappa(s) \log s}{s} = 0 \quad (10)
\]
such that the degrees $n(s)$ of the numerators and the degrees $m(s)$ of the denominators of the rational approximants $r_{m(s),n(s)}^*$ satisfy
\[
m(s) \leq n(s) + \kappa(s). \quad (11)
\]
Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that
\[
\nu_s - \alpha_s \hat{\nu}_s - (1 - \alpha_s) \mu \rightarrow^* 0 \quad \text{as } s \to \infty, \ s \in \Lambda,
\]
where
\[
\alpha_s = \frac{\deg q_s^* + \deg q_{s+1}^*}{d(s) + 1}.
\]
If $m(s) \geq n(s) - \kappa$, $\kappa \in \mathbb{N}$ fixed, then it is possible that in the theorem
\[
\liminf_{s \to \infty, s \in \Lambda} \alpha_s \geq 1.
\]
In this case, generally no connection between the alternation sets and the equilibrium measure $\mu$ can be expected without additional conditions on the poles of the best approximants, as shown by results of Braess et al. [5].

It is possible to formulate the result of the theorem in a more concise form. Let
\[
R_s = r_{s+1}^* - r_s^* = \frac{p}{q},
\]
where $p$ and $q$ have no common divisor. Then the degree of $p/q$ is defined by
\[
\deg p/q = \max(\deg p, \deg q).
\]
In our situation we have
\[
\deg R_s = \max(d(s), \deg q_s^* + \deg q_{s+1}^*),
\]
since all finite zeros of $R_s$ are in $(-1, 1)$ and all poles of $R_s$ are outside $[-1, 1]$. Then the number of zeros, resp. poles, of $R_s$ in the complex plane $\mathbb{C}$ is $\deg R_s$ where all zeros and poles are counted with their multiplicity.

Now, we define the normalized zero counting measure $\sigma_{\text{zero},s}$ and the normalized pole counting measure $\sigma_{\text{pole},s}$ of $R_s$ in $\mathbb{C}$, i.e., for $A \in \mathbb{C}$
\[
\sigma_{\text{zero},s}(A) = \frac{\# \{\text{zeros of } R_s \text{ in } A\}}{\deg R_s}
\]
resp.
\[
\sigma_{\text{pole},s}(A) = \frac{\# \{\text{poles of } R_s \text{ in } A\}}{\deg R_s}.
\]
Then
\[ \sigma_{\text{zero},s}(\overline{C}) = \sigma_{\text{pole},s}(\overline{C}) = 1 \]
and we obtain the following result.

**Corollary.** Under the conditions of the above theorem there exists \( A \subset \mathbb{N} \) such that
\[ \widehat{\sigma}_{\text{zero},s} - \widehat{\sigma}_{\text{pole},s} \to 0 \quad \text{as} \quad s \to \infty, \quad s \in A, \]
where \( \widehat{\sigma}_{\text{zero},s} \) and \( \widehat{\sigma}_{\text{pole},s} \) are the balayage measures of the normalized zero (resp. pole) counting measures of \( R_s = r_s^* - r_{s+1}^* \) onto the interval \([-1, 1]\).

Let us illustrate the corollary in Kadec’s case, i.e., \((n(s), m(s)) = (s, 0)\). Then \( R_s = p_{s+1}^* - p_s^* \) and \( p_s^*, p_{s+1}^* \) are the best approximating polynomials to \( f \) with respect to \( P_s \), resp. \( P_{s+1} \). \( R_s \) has a pole of multiplicity \( s + 1 \) at \( \infty \) if \( p_s^* \neq p_{s+1}^* \). Since \( \hat{\sigma}_{\infty} = \mu, \hat{\sigma}_{\text{zero},s} = \sigma_{\text{zero},s} \) and the zeros of \( R_s \) separate the alternation points of \( f - p_s^* \), we have
\[ \lim_{s \in A, s \to \infty} \sigma_{\text{zero},s} = \lim_{s \in A, s \to \infty} \nu_s, 0 = \mu \]
for a subsequence \( A \) of \( \mathbb{N} \), that is Kadec’s result (5).

### 3. Proof

In the following, we denote by \( c, c_1, c_2, \ldots \) positive constants, independent of \( s \) and \( f \), which may be different at different occurrences.

First, let us note that \( m(s + 1) \leq m(s) + 1 \) implies for all \( s \in \mathbb{N} \)
\[ m(s) \leq c_1 + s, \quad n(s) \leq c_2 + s. \]
Hence, for all \( s \in \mathbb{N} \) we obtain
\[ l_s \leq \deg q_s^* + \deg q_{s+1}^* \leq 2s + c. \]
Furthermore, \( d(s) = n(s) + m(s) + 1 - \delta_{n(s),m(s)}(r_s^*) \) together with the condition (9) for the strictly increasing sequence \((n(s), m(s)), s \in \mathbb{N}\), yields
\[ \frac{s}{2} \leq \max(n(s), m(s)) + 1 \leq d(s) \leq 2s + c. \]
We may furthermore assume without loss of generality that \( \lim_{s \to \infty} \kappa(s) = \infty \) and \( \kappa(s) \geq 2 \) for all \( s \in \mathbb{N} \).

It is well-known that there exists a subsequence \( A \subset \mathbb{N} \) such that
\[ A_s := \frac{E_s + E_{s+1}}{E_s - E_{s+1}} \leq s^2 \quad \text{for} \quad s \in A, s \to \infty \]
(12)
(cf. [1, Lemma 7.3.3, p. 243]). Especially, for \( s \in A \) we have \( r_s^* \neq r_{s+1}^* \) and by (3) that
\[ (-1)^k (r_{s+1}^* - r_s^*) (x_k^{(s)}) \geq E_s - E_{s+1} \]
(13)
for $0 \leq k \leq d(s)$, where we have used without loss of generality that in (3) the number $n(s), m(s) = 1$. Let

$$R_s = r_{s+1}^* - r_s^* = \frac{p_{s+1}^* q_s^* - p_s^* q_{s+1}^*}{q_s^* q_{s+1}^*} = \frac{P_s}{Q_s},$$

(14)

then (13) implies that for $s \in A$

$$(-1)^k R_s(x_k^{(s)}) \geq E_s - E_{s+1}, \quad 0 \leq k \leq d(s).$$

(15)

Since $r_s^* \neq r_{s+1}^*$ then, because of

$$n(s + 1) \leq n(s) + 1 \quad \text{and} \quad m(s + 1) \leq m(s) + 1,$$

it turns out that $P_s \in \mathcal{P}_{d(s)}$; i.e. (15) implies that all zeros of $P_s$ and $R_s$ are in $(-1, 1)$.

We may assume that $q_s^*, q_{s+1}^*$ are monic polynomials, then

$$Q_s(x) = \prod_{i=1}^{l_s} (x - y_i^{(s)})$$

(16)

and all zeros of $Q_s$ are outside $[-1, 1]$ and they are real or occur in conjugate pairs counted with their multiplicities. Next, define

$$\kappa_s = \deg Q_s - \deg P_s,$$

(17)

then with $\delta = \delta_{n(s), m(s)}(r_s^*)$ we have

$$\kappa_s \leq m(s) - \delta + m(s + 1) - (n(s) + m(s) + 1 - \delta)$$

or

$$\kappa_s \leq m(s) - n(s) \leq \kappa(s).$$

(18)

If $\kappa_s \leq 0$, then the polynomial $Q_s$ can be reconstructed by interpolation at the alternation points $x_k^{(s)}$, $0 \leq k \leq d(s)$. If $\kappa_s > 0$, we need additional interpolation points for doing this. First, we consider

$$Z_1^{(s)} := \{y_i^{(s)} : |y_i^{(s)}| > \kappa(s)\}$$

and we assume that $y_i^{(s)}$ are ordered in such a way that

$$|y_i^{(s)}| \geq |y_{i+1}^{(s)}|$$

and

$$Z_1^{(s)} = \{y_i^{(s)} : 1 \leq i \leq l_{1,s}\}.$$

Then

$$l_{1,s} \leq l_s$$

and we define

$$R_{1,s}(z) := \frac{P_s(z)}{Q_s(z)} \prod_{j=1}^{l_{1,s}} (z - y_j^{(s)}) = \frac{P_s(z)}{Q_{1,s}(z)},$$
where

\[ Q_{1,s}(z) = \prod_{i=l_{1,s}+1}^{l_s} (z - \gamma_i^{(s)}). \]  

(19)

Note that \( Q_{1,s}(z) \equiv Q_s(z) \), if \( Z_1^{(s)} = \emptyset \).

From (15) we obtain

\[ (-1)^k \varepsilon_1 R_{1,s}(x_k^{(s)}) \geq (E_s - E_{s+1}) \prod_{j=1}^{l_{1,s}} (|y_j^{(s)}| - 1), \quad 0 \leq k \leq d(s), \]

(20)

where \( \varepsilon_1 = +1, \) resp. \( \varepsilon_1 = -1, \) if the number of negative real points in \( Z_1^{(s)} \) is even, resp. odd (counted with their multiplicities).

If \( \deg Q_{1,s} = l_s - l_{1,s} > d(s) \), then we define

\[ \kappa_{1,s} := \deg Q_{1,s} - d(s) = l_s - l_{1,s} - d(s) \]

(21)

and fix the points

\[ \xi_j^{(s)} := -js, \quad 1 \leq j \leq \kappa_{1,s}. \]

(22)

Then condition (10) implies that there exists \( s_0 \in \mathbb{N}, s_0 \geq 2, \) such that all points \( \xi_j^{(s)} < -\kappa(s) \) for all \( s \geq s_0 \).

We consider now

\[ R_{2,s}(z) = R_{1,s}(z) \prod_{j=1}^{\kappa_{1,s}} (z - \xi_j^{(s)}) = \frac{P_{2,s}(z)}{Q_{1,s}(z)}, \]

where

\[ P_{2,s}(z) = P_2(z) \prod_{j=1}^{\kappa_{1,s}} (z - \xi_j^{(s)}) \]

(23)

and we set \( P_{2,s} = P_s \) if \( \kappa_{1,s} \leq 0 \).

Now, we can reconstruct the polynomial \( Q_{1,s}(z) \) by interpolation at the points

\[ \{x_k^{(s)}\}_{k=0}^{d(s)} \cup \{\xi_j^{(s)}\}_{j=1}^{\kappa_{1,s}}. \]

In the case \( \kappa_{1,s} \leq 0 \) the second set \( \{\xi_j^{(s)}\}_{j=1}^{\kappa_{1,s}} = \emptyset \). Let

\[ w(z) = \prod_{k=0}^{d(s)} (z - x_k^{(s)}) \prod_{j=1}^{\kappa_{1,s}} (z - \xi_j^{(s)}), \]

then we obtain by Lagrange’s interpolation formula

\[ Q_{1,s}(z) = \sum_{k=0}^{d(s)} \frac{Q_{1,s}(x_k^{(s)}) w(z)}{(z - x_k^{(s)}) w'(x_k^{(s)})} + \sum_{j=1}^{\kappa_{1,s}} \frac{Q_{1,s}(\xi_j^{(s)}) w(z)}{(z - \xi_j^{(s)}) w'(\xi_j^{(s)})}, \]
where the last sum is defined as 0 if $\kappa_{1,s} \leq 0$.

Then for $z \neq x_{k}^{(s)}, \xi_{j}^{(s)}$

$$\frac{Q_{1,s}(z)}{w(z)} = \sum_{k=0}^{d(s)} \frac{Q_{1,s}(x_{k}^{(s)})}{(z - x_{k}^{(s)})w'(x_{k}^{(s)})} + \sum_{j=1}^{\kappa_{1,s}} \frac{Q_{1,s}(\xi_{j}^{(s)})}{(z - \xi_{j}^{(s)})w'(\xi_{j}^{(s)})}.$$

Define for abbreviation

$$\frac{1}{\beta_{k}} = w'(x_{k}^{(s)}), \ 0 \leq k \leq d(s)$$

and

$$\frac{1}{\alpha_{j}} = w'(\xi_{j}^{(s)}), \ 1 \leq j \leq \kappa_{1,s}.$$

Then for $z \neq x_{k}^{(s)}, \xi_{j}^{(s)}$

$$\left| \frac{Q_{1,s}(z)}{w(z)} \right| \leq D(z) \left( \sum_{k=0}^{d(s)} |\beta_{k} Q_{1,s}(x_{k}^{(s)})| + \sum_{j=1}^{\kappa_{1,s}} |\alpha_{j} Q_{1,s}(\xi_{j}^{(s)})| \right)$$

$$= D(z)(S_{1} + S_{2}), \quad (26)$$

where $S_{1}$, resp. $S_{2}$ is the first, resp. second sum on the right-hand side and

$$D(z) := \max \left( \max_{0 \leq k \leq d(s)} |z - x_{k}^{(s)}|^{-1}, \max_{1 \leq j \leq \kappa_{1,s}} |z - \xi_{j}^{(s)}|^{-1} \right). \quad (27)$$

Next, we have to obtain upper bounds for the two sums on the right-hand side of (26).

Let $a_{s}$ be the leading coefficient of $P_{s}$ and let us consider the Chebyshev approximation of $P_{2,s}(z)$ with respect to $\mathcal{P}_{d(s)+\kappa_{1,s}-1}$ and the weight function $1/Q_{1,s}(z)$ at the points $\{x_{k}^{(s)}\}_{k=0}^{d(s)}$ with interpolation conditions at the points $\{\xi_{j}^{(s)}\}_{j=1}^{\kappa_{1,s}}$. Let $\rho$ denote the minimal deviation for this weighted approximation problem. Then de la Vallée-Poussin’s Theorem, together with (20) and (23), implies

$$\rho \geq (E_{s} - E_{s+1}) B_{s} C_{s}, \quad (28)$$

where

$$B_{s} := \prod_{j=1}^{l_{1,s}} (|y_{j}^{(s)}| - 1), \quad C_{s} := \prod_{j=1}^{\kappa_{1,s}} (|\xi_{j}^{(s)}| - 1). \quad (29)$$

Moreover, the minimal deviation $\rho$ can be calculated by the following well-known Lemma for Chebyshev approximation with interpolation conditions.

**Lemma.**

$$\rho = \frac{|a_{s}|}{\sum_{k=0}^{d(s)} |\beta_{k} Q_{1,s}(x_{k}^{(s)})|},$$

where $\beta_{k}$ is defined by (24).
Hence, this lemma and (28) imply
\[
\sum_{k=0}^{d(s)} |\beta_k Q_{1,s}(x_k^{(s)})| \leq \frac{|a_s|}{(E_s - E_{s+1}) B_s C_s}.
\]  

(30)

For estimating $|a_s|$, we use a method of [2]. Define the function
\[
h(z) := \log |R_s(z)| - \sum_{j=1}^{l_s} G(z, y_j^{(s)}) + (d(s) - l_s) G(z, \infty)
\]
then $h$ is subharmonic in $\mathbb{C}$. By the maximum principle we get together with \( \lim_{z \to \infty} (G(\infty, z) + \log \frac{1}{2} - \log |z|) = 0 \) that
\[
h(\infty) = \log |a_s| - \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) + (d(s) - l_s) \log \frac{1}{2}
\leq \max_{-1 \leq x \leq 1} \log |R_s(x)| = \log \|R_s\|
\leq \log(E_s + E_{s+1})
\]
or
\[
\log |a_s| \leq \log(E_s + E_{s+1}) + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) - (d(s) - l_s) \log \frac{1}{2}.
\]

Inserting in (30), it follows that
\[
\log S_1 = \log \sum_{k=0}^{d(s)} |\beta_k Q_{1,s}(x_k^{(s)})| 
\leq \log \frac{A_s}{B_s C_s} + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) + (d(s) - l_s) \log \frac{1}{2}
\]  

(31)

is an upper bound for the first sum in (26).

Concerning the second sum we have to consider only the case $\kappa_{1,s} \geq 1$. Since
\[
\frac{\kappa_{1,s} + d(s)}{js} \kappa(s) \leq \frac{\kappa(s) + d(s)}{s} \kappa(s) \leq \frac{\kappa(s) + c + 2s}{s} \kappa(s) \leq c_1 \kappa(s)
\]
and
\[
\frac{d(s) + 1}{js} \leq \frac{d(s) + 1}{s} \leq c_2,
\]
we obtain for \( s \geq s_0 \geq 2 \)

\[
|z_j Q_{1,s}(\zeta_j^{(s)})| = \prod_{l=l_{1,s}+1}^{l_{s}} |\zeta_j^{(s)} - y_i^{(s)}| \prod_{i=1}^{\kappa_{1,s}} |\zeta_j^{(s)} - \zeta_i^{(s)}| \prod_{k=0}^{d(s)} |\zeta_j^{(s)} - x_k^{(s)}| \\
\leq \frac{[js + \kappa(s)]^{l_{s} - l_{1,s}}}{s^{\kappa_{1,s} - 1}[js - 1]^{d(s)+1}} \\
= \frac{(js)^{\kappa_{1,s} - 1}(1 + \kappa(s)/js)^{\kappa_{1,s} + d(s)}}{s^{\kappa_{1,s} - 1}(1 - 1/js)^{d(s)+1}} \\
\leq \kappa_{1,s}^{-1} \exp \left( \frac{\kappa_{1,s} + d(s)}{js} \kappa(s) + 2 \frac{d(s) + 1}{js} \right) \\
\leq c_3 \kappa(s)^{\kappa(s)}.
\]

Therefore,

\[
\log S_2 = \log \sum_{j=1}^{\kappa_{1,s}} |z_j Q_{1,s}(\zeta_j^{(s)})| \leq c \kappa(s) \log \kappa(s) \\
= o(s) \text{ as } s \to \infty. \tag{32}
\]

Back to (26), let \( D_s \) be the constant

\[
D_s := \prod_{j=1}^{\kappa_{1,s}} |\zeta_j^{(s)}|.
\]

Then

\[
D_s \left| \frac{Q_{1,s}(z)}{w(z)} \right| = Q_{1,s}(z) \prod_{j=1}^{\kappa_{1,s}} |\zeta_j^{(s)}| \prod_{k=0}^{d(s)} \frac{1}{|z - x_k^{(s)}|} \\
\leq D(z)(D_s S_1 + D_s S_2).
\]

Since for \(|z| \leq 2\) and all \( s \)

\[
\prod_{j=1}^{\kappa_{1,s}} \frac{|\zeta_j^{(s)}|}{|z - \zeta_j^{(s)}|} \geq \prod_{j=1}^{\kappa_{1,s}} \frac{|js|}{|js| + 2} \geq \left( 1 + \frac{2}{s} \right)^{-\kappa(s)} \\
\geq e^{-2\kappa(s)/s} \geq c_1,
\]

we obtain for \(|z| \leq 2\)

\[
\log \frac{Q_{1,s}(z)}{\prod_{k=0}^{d(s)} |z - x_k^{(s)}|} \leq c + \log D(z) + \log(D_s S_1 + D_s S_2).
\]

Next, we use for \( \alpha, \beta > 0 \) that

\[
\log(\alpha + \beta) \leq \log(2\alpha) + \log(2\beta).
\]
Thus, we can write
\[ \log \frac{Q_{1,s}(z)}{d(s)} \leq c + \log D(z) + \log(D_s S_1) + \log(D_s S_2). \] (33)

Now, by (29) and (31) we obtain for \( s \geq s_0, |z| \leq 2 \) that
\[
\log D_s S_1 \leq \log \frac{A_s}{B_s C_s} + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) + \sum_{j=1}^{\kappa_{1,s}} \log |\xi_j^{(s)}| - (d(s) - l_s) \log \frac{1}{2}
\]
\[
= \log A_s - \sum_{j=1}^{l_{1,s}} \log(|y_j^{(s)}| - 1) - \sum_{j=1}^{\kappa_{1,s}} \log(|\xi_j^{(s)}| - 1)
\]
\[
+ \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) + \sum_{j=1}^{\kappa_{1,s}} \log |\xi_j^{(s)}| - (d(s) - l_s) \log \frac{1}{2}
\]
\[
\leq c + \log A_s + \sum_{j=1}^{l_{1,s}} (G(\infty, y_j^{(s)}) - \log |z - y_j^{(s)}|)
\]
\[
+ 3 \frac{l_{1,s}}{\kappa(s) - 1} + \sum_{j=l_{1,s}+1}^{l_s} G(\infty, y_j^{(s)}) - (d(s) - l_s) \log \frac{1}{2},
\] (34)

since
\[
\frac{\sum_{j=1}^{\kappa_{1,s}} \log |\xi_j^{(s)}|}{|\xi_j^{(s)}| - 1} \leq \kappa_{1,s} \log \frac{1}{1 - 1/s} \leq c \kappa_{1,s} \frac{1}{s} \leq c
\]

and
\[
\sum_{j=1}^{l_{1,s}} \log \frac{|z - y_j^{(s)}|}{|y_j^{(s)}| - 1} \leq \sum_{j=1}^{l_{1,s}} \log \frac{|y_j^{(s)}| + 2}{|y_j^{(s)}| - 1} = \sum_{j=1}^{l_{1,s}} \log \left( 1 + \frac{3}{|y_j^{(s)}| - 1} \right)
\]
\[
\leq \sum_{j=1}^{l_{1,s}} \frac{3}{|y_j^{(s)}| - 1} \leq 3 \frac{l_{1,s}}{\kappa(s) - 1}.
\]

Note, that
\[
\frac{l_{1,s}}{\kappa(s) - 1} \leq \frac{l_s}{\kappa(s) - 1} \leq 2s + c
\]

hence with the Landau symbol \( o(s) \) we can write
\[
3 \frac{l_{1,s}}{\kappa(s) - 1} + \log A_s = o(s) \text{ as } s \to \infty,
\] (35)

where we have used that \( \kappa(s) \to \infty \) as \( s \to \infty \).
Summarizing, we have

\[
\log D_s S_1 \leq o(s) + \sum_{j=1}^{l_1,s} \left( G(\infty, y_j^{(s)}) + \log \frac{1}{|z - y_j^{(s)}|} \right) + \sum_{j=l_{1,s}+1}^{l_s} G(\infty, y_j^{(s)}) + (d(s) - l_s) \log \frac{1}{2}.
\]

(36)

Furthermore,

\[
D_s = \prod_{j=1}^{k_{1,s}} |z_j^{(s)}| = \prod_{j=1}^{k_{1,s}} (j s) = s^{k_{1,s}} k_{1,s}! \leq s^{\kappa(s)} \kappa(s)^{k(s)}
\]

or

\[
\log D_s \leq \kappa(s) (\log s + \log \kappa(s)) = o(s) \text{ as } s \to \infty
\]

(37)

since \( \kappa(s) = o(s/\log s) \).

Now we consider \( z \) the level curves

\[
\Gamma_{1/s} : = \{ z \in \mathbb{C} : G(z, \infty) = \log(1 + \frac{1}{s}) \}
\]

of Green’s function \( G(z, \infty) \) which are ellipses with foci at 1 and \(-1\) and major semi-axis \( a = \frac{1}{2}(s + \frac{1}{s} + (s + \frac{1}{s})^{-1}) \). Hence,

\[
\text{dist}(\Gamma_{1/s}, [-1, 1]) \geq 1/(4s^2) \text{ for } s \in \mathbb{N}
\]

and therefore

\( D(z) \leq 4s^2 \text{ for } s \geq s_0. \)

Then (32) – (37) imply for \( z \in \Gamma_{1/s} \) that

\[
\log \frac{Q_{1,s}(z)}{d(s) \prod_{k=0}^{l_{1,s}} |z - x_k^{(s)}|} \leq o(s) + \sum_{j=1}^{l_{1,s}} (G(\infty, y_j^{(s)}) - \log |z - y_j^{(s)}|) + \sum_{j=l_{1,s}+1}^{l_s} G(\infty, y_j^{(s)}) - (d(s) - l_s) \log \frac{1}{2}
\]

or

\[
\log \left| a_s \frac{Q_s(z)}{P_s(z)} \right| \leq o(s) + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) - (d(s) - l_s) \log \frac{1}{2}.
\]

The last inequality can be written with the logarithmic potentials \( U^{\tau_s} \) and \( U^{\tau_s} \) as

\[
U^{\tau_s}(z) - a_s U^{\tau_s}(z) \leq \frac{1}{d(s) + 1} \left( o(s) + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) \right) - \left( 1 - \frac{l_s + 1}{d(s) + 1} \right) \log \frac{1}{2}.
\]
Moreover, we use
\[ U^\mu(z) = -G(z, \infty) - \log \frac{1}{2} \]
and obtain for \( z \in \Gamma_{1/s} \)
\[ U^\nu(z) - \alpha_s U^{\tau_s}(z) - (1 - \alpha_s) U^\mu(z) \leq \frac{1}{d(s) + 1} \left( o(s) + \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}) + \log \frac{1}{2} \right) + (1 - \alpha_s) G(z, \infty). \]
Hence, for \( z \in \Gamma_{1/s} \)
\[ U^\nu(z) - \alpha_s U^{\tau_s}(z) - (1 - \alpha_s) U^\mu(z) \leq o(1) + \frac{1}{d(s) + 1} \sum_{j=1}^{l_s} G(\infty, y_j^{(s)}). \]
Next, we decompose the measure \( \tau_s \) into
\[ \tau_s = \tau_{s,1} + \tau_{s,2}, \]
where
\[ \tau_{s,1} := \tau_s|_{\text{ext } \Gamma_{1/s}} \]
and \( \text{ext } \Gamma_{1/s} := \{ z \in \mathbb{C} : G(z, \infty) > \log(1 + 1/s) \} \). Let \( \tau_{s,1}^* \) denote the balayage measure of \( \tau_{s,1} \) onto \( \Gamma_{1/s} \). Since \( \nu_s \) and \( \tau_s \) are probability measures and the numbers \( \alpha_s \) are bounded, we may assume that we have chosen \( \Lambda \) in such a way that
\[ \nu_s - \alpha_s (\tau_{s,1}^* + \tau_{s,2}) - (1 - \alpha_s) \mu \xrightarrow{s \to \infty} \sigma \text{ as } s \in \Lambda, \]
where \( \sigma \) is a signed measure on \([-1, 1]\) with \( \sigma([-1, 1]) = 0 \).
We want to show that \( \sigma = 0 \). To this end, let us consider the point sets
\[ T_1 := \{ y_i : y_i \in \text{ ext } \Gamma_{1/s} \} \]
and
\[ T_2 = \{ y_i \}_{i=1}^{l_s} \setminus T_1. \]
Then, for \( z \in \Gamma_{1/s} \)
\[ U^{\tau_{s,1}}(z) + U^{\tau_{s,2}}(z) = U^{\tau_{s,1}}(z) + U^{\tau_{s,2}}(x) + \frac{1}{l_s} \sum_{y_i \in T_1} G_s(y_i, \infty), \]
where \( G_s(z, \infty) \) is Green’s function of \( \Gamma_{1/s} \) with pole at \( \infty \). Since \( G_s(z, \infty) = G(z, \infty) - \log (1 + 1/s) \) we obtain for \( z \in \Gamma_{1/s} \) and \( s \geq 2 \)
\[ U^\nu(z) - \alpha_s (U^{\tau_{s,1}}(z) + U^{\tau_{s,2}}(z)) - (1 - \alpha_s) U^\mu(z) \leq o(1) \text{ for } s \in \Lambda, \]
\[ s \to \infty. \] (39)
If we assume \( \sigma \neq 0 \) in (38) then the maximum principle and Carleson’s theorem (cf. [9]) imply that
\[ \max_{z \in T_s} U^\sigma(z) > 0 \]
for some $x > 0$, since $U^\sigma(z)$ is subharmonic in $\mathbb{C}\setminus I$ and $U^\sigma(\infty) = 0$. Applying the maximum principle again, we get
\[
\max_{z \in I_{1/s}} U^\sigma(z) > \max_{z \in I_s} U^\sigma(z)
\]
for $1/s < x$. This contradicts the inequality (39). Hence $\sigma = 0$.

Our final step is to show that
\[
\tau^*_s,1 + \tau^*_s,2 \xrightarrow[*]{\ast} \tilde{\tau}_s \text{ as } s \in A, \ s \to \infty
\]
which is equivalent to
\[
\lim_{s \to \infty} \int g \, d(\tau^*_s,1 + \tau^*_s,2) = \lim_{s \to \infty} \int g \, d\tilde{\tau}_s
\]
for all continuous functions $g$ on $\mathbb{C}$ with compact support.

Let $h$ be the harmonic extension of $g|_{[-1,1]}$ onto $\mathbb{C}$. Since $g$ and $h$ are uniformly continuous, there exists for $\varepsilon > 0$ a number $\delta > 0$ such that
\[
|g(z) - g(z')| < \varepsilon \quad \text{and} \quad |h(z) - h(z')| < \varepsilon
\]
for all $z, z'$ with $|z - z'| < \delta$.

Let $\ast \in \text{ext } I_{1/s}$. Then there exists a point $x_\ast \in I$ with $|x - x_\ast| < 1/s$ and consequently for all such $z$
\[
|g(z) - h(z)| \leq |g(z) - g(x_\ast)| + |g(x_\ast) - h(z)|
\]
\[
= |g(z) - g(x_\ast)| + |h(x_\ast) - h(z)|
\]
\[
\leq 2 \varepsilon
\]
if $1/s < \delta$. Hence, for $s > 1/\delta$
\[
\int g \, d\tilde{\tau}_s = \int h \, d\tau_s = \int h \, d(\tau^*_s,1 + \tau^*_s,2)
\]
\[
= \int g \, d(\tau^*_s,1 + \tau^*_s,2) + \int (h - g) \, d(\tau^*_s,1 + \tau^*_s,2)
\]
and therefore
\[
\left| \int g \, d\tilde{\tau}_s - \int g \, d(\tau^*_s,1 + \tau^*_s,2) \right| \leq 2 \varepsilon
\]
Summarizing, we have obtained
\[
\nu_s - (1 - x_s)\tilde{\tau}_s - x_s \mu \xrightarrow[*]{\ast} 0 \text{ as } s \in A, \ s \to \infty
\]
and the theorem is proved. □

Concerning the proof of the Corollary, we consider first the case $\deg R_s = d(s)$. Then $R_s$ has $l_s = \deg q^*_s + \deg q^*_s + 1$ finite poles and $d(s) - l_s$ poles at $\infty$. Hence
\[
\hat{\sigma}_{\text{pole},s} = \frac{l_s}{d(s)} \tilde{\tau}_s + \frac{d(s) - l_s}{d(s)} \mu
\]
\[
= \frac{d(s) + 1}{d(s)} \nu_s \tilde{\tau}_s + \left( 1 - \nu_s \frac{d(s) + 1}{d(s)} \right) \mu.
\]
Then we obtain with $d(s) \geq s/2$ and
\[
\lim_{s \to \infty} (\sigma_{\text{zero},s} - v_s) = \lim_{s \to \infty} (\sigma_{\text{zero},s} - v_s) = 0
\]
that
\[
\lim_{s \in A, s \to \infty} (\sigma_{\text{zero},s} - \sigma_{\text{pole},s}) = \lim_{s \in A, s \to \infty} (v_s - \alpha_s \tau_s - (1 - \alpha_s) \mu) = 0.
\]

In the case that $\deg R_s = l_s$, the rational function $R_s$ has $d(s)$ finite zeros on $(-1, 1)$ and $l_s - d(s)$ zeros at infinity. Let us denote by $\sigma_{1,s}$ the normalized measure associated with the finite zeros on $(-1, 1)$. Then
\[
\hat{\sigma}_{\text{zero},s} = \frac{d(s)}{l_s} \sigma_{1,s} + \left(1 - \frac{d(s)}{l_s}\right) \mu
\]
and therefore
\[
\hat{\sigma}_{\text{zero},s} - \hat{\sigma}_{\text{pole},s} = \frac{d(s)}{l_s} \sigma_{1,s} + \left(1 - \frac{d(s)}{l_s}\right) \mu - \tau_s
\]
\[
= \frac{d(s)}{l_s} \left(\sigma_{1,s} - \frac{l_s}{d(s)} \tau_s - \left(1 - \frac{l_s}{d(s)}\right) \mu\right).
\]
Since $d(s) \leq l_s$ and $\lim_{s \to \infty} (\sigma_{1,s} - v_s) = 0$, we obtain
\[
\lim_{s \in A, s \to \infty} (\hat{\sigma}_{\text{zero},s} - \hat{\sigma}_{\text{pole},s}) = \lim_{s \in A, s \to \infty} (v_s - \alpha_s \tau_s - (1 - \alpha_s) \mu) = 0
\]
and the convergence result for the zero measures $\sigma_{\text{zero},s}$ and pole measures $\sigma_{\text{pole},s}$ holds.

References