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Quasi-progressions and Descending Waves

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1. INTRODUCTION AND DEFINITIONS

If A is a set of positive integers with positive upper uniform density, then A must contain arbitrarily large cubes, i.e., sets of the form

$$\langle a, y_1\ y_2, ..., y_m\rangle = \{a + \varepsilon_1 y_1 + \varepsilon_2 y_2 + \cdots + \varepsilon_m y_m : \varepsilon_j = 0 \text{ or } 1, 1 \leqslant j \leqslant m\}. \tag{$*$}$$

This fact is an essential step in Szemerédi's proof that any set with positive upper uniform density contains arbitrarily large arithmetic progressions.

In this paper we consider several other properties of a set of positive integers, each of which generalizes the notion of having arbitrarily long arithmetic progressions. We call these properties QP (having arbitrarily large "quasi-progressions"), CP (having arbitrarily large "combinatorial progressions"), and DW (having arbitrarily large "descending waves"). The definitions of these properties will appear at the end of this section.

Let us denote by AP and by C the properties of having arbitrarily long arithmetic pogressions and arbitrarily large cubes, respectively. Then, in Section 2, we will show that

$$AP \Rightarrow QP \Rightarrow CP \Rightarrow C \Rightarrow DW,$$

and that none of these implications is reversible.

By using Szemerédi's method for obtaining cubes, we can show that, if the sum of the reciprocals of the elements of a set A is infinite, then A has property C. While we cannot, at present, show that a set with infinite reciprocal sum must have properties CP or QP, we have a good excuse for the last failure: we show that Erdős' famous conjecture (that every set of positive integers with infinite reciprocal sum has property AP) is equivalent to the statement that every set with infinite reciprocal sum has property QP. These are done in Section 3.

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In Section 4 we consider descending waves. We obtain upper and lower bounds for $f(k)$, the smallest integer such that, if $\{1, 2, ..., f(k)\} = A \cup B$, then A or B contains a k-term descending wave. We also obtain an upper bound for

$$\max\{|S|: S \subseteq \{1, 2, ..., m\} \text{ and } S \text{ contains no } k\text{-term descending wave}\}.$$

(The upper bound is $c_k(\log m)^{k-2}$.)

We also consider in Section 4 how the growth rate of a sequene $\{a_n\}$ influences the presence of descending waves in the set $\{a_n\}$. We show that arbitrarily long descending waves must be present even in certain sets with rather large growth rates, but that sets $\{a_n\}$ with $a_{n+1}/a_n \geqslant 1+\varepsilon$ for all n have descending waves of bounded length.

We conclude the paper with some remarks and questions in Section 5.

We now define the properties QP, CP, and DW. A finite sequence $x_1 < x_2 < \cdots < x_k$ will be called a *k-term quasi-progression of diameter d* (abbreviated $k-QP(d)$) if

$$\text{Diam}\{x_{i+1} - x_i: 1 \leqslant i \leqslant k-1\} \leqslant d,$$

i.e.,

$$\exists N \text{ such that } N \leqslant x_{i+1} - x_i \leqslant N + d \text{ for } 1 \leqslant i \leqslant k-1.$$

A set of positive integers has property QP if, for some fixed d, the set contains a $k-QP(d)$ for each $k \geqslant 1$. Noting that a k-term arithmetic progression is just a $k-QP(0)$ we get immediately that $AP \Rightarrow QP$.

The sequence $x_1 < x_2 < \cdots < x_k$ will be called a *k-term combinatorial progression of order d* (abbreviated $k-CP(d)$) if

$$|\{[x_{i+1} - x_i]: 1 \leqslant i \leqslant k-1\}| \leqslant d.$$

(The integer function is present for the cases, mentioned below, when the x_i may not be integers.) A set of positive integers has property CP if, for some fixed d, the set contains a $k-CP(d)$ for each $k \geqslant 1$. Clearly, when the x_i are integers, a $k-QP(d)$ is a $k-CP(d+1)$ and so $QP \Rightarrow CP$.

Finally, a sequence $x_1 < x_2 < \cdots < x_k$, is called a *k-term descending wave* ($k-DW$) if the difference sequence is non-increasing, i.e.,

$$x_{j+1} - x_j \geqslant x_{j+2} - x_{j+1} \qquad \text{for} \quad 1 \leqslant j \leqslant k-2.$$

If a set of positive integers contains arbitrarily large descending waves then we say that it has property DW.

We observe that the definitions of k-$QP(d)$, $k-CP(d)$, and $k-DW$ can be applied to a sequence $x_1 < x_2 < \cdots < x_k$ even if the terms of this sequence are not integer valued. Thus a countable set of real numbers $R = \{r_1 < r_2 < \dots\}$ can be said to have properties QP, CP, or DW. However, it is easy to prove that, if R satisfies the reasonable condition that $r_{i+1} - r_i \geq 1$ for sufficiently large i, then R has property QP, CP, DW exactly when the corresponding set of integers $A = \{[r_i]\colon i \geq 1\}$ has the same property.

2. Relations between AP, QP, CP, C, DW

Let us restate our claim in the form of a theorem. The proof will occupy the remainder of this section.

THEOREM 1. *$AP \Rightarrow QP \Rightarrow CP \Rightarrow C \Rightarrow DW$, and none of these implications is reversible.*

Proof. We have already seen that $AP \Rightarrow QP \Rightarrow CP$. The implication $C \Rightarrow DW$ is also easy to see: If a set A contains the m-cube $(*)$ above, where we may assume that $y_1 \geq y_2 \geq \cdots \geq y_m$, then A also contains the $(m+1)-DW$

$$a,\ a + y_1 + y_2, \dots, a + y_1 + y_2 + \cdots + y_m.$$

We proceed to prove $CP \Rightarrow C$. This easily follows from the statement: For all $m, d \geq 1$, there exists $r = r(d, m)$ such that, if $x_1, x_2, \dots, x_r$ is an $r - CP(d)$, then $\{x_1, x_2, \dots, x_r\}$ contains an m-cube. The proof of this is by induction on m. For $m = 1$ we let $r = 2$. Any $2 - CP(d)$ is a 1-cube. For $m + 1$ we take $r = r(d, m+1) = t \cdot r(d, m)$, where t is determined presently. Let $r' = r(d, m)$ and let $x_1, x_2, \dots, x_{tr'}$ be an $r - CP(d)$. Each block,

$$x_{kr'+1}, x_{kr'+2}, \dots, x_{(k+1)r'} \qquad \text{for} \quad 0 \leq k \leq t-1,$$

is an r'-$CP(d)$ with the same set $\{f_1, f_2, \dots, f_d\}$ of possible differences. By the inductive hypothesis, each of these blocks contains an m-cube. Any generator y_i of an m-cube in a block is of the form

$$f_{i_1} + f_{i_2} + \cdots + f_{i_j},$$

where $j < r'$. Hence there are less than $(d+1)^{r'}$ different generators and so less than $(d+1)^{r'm}$ m-tuples of generators. Hence, if $t = (d+1)^{r'm}$, then two of the blocks in $x_1, x_2, \dots, x_{tr'}$ will have m-cubes with the same set of generators. If these two are $\langle x_i, y_1, y_2, ..., y_m\rangle$ and $\langle x_j, y_1, y_2, ..., y_m\rangle$ with $x_i < x_j$, then

$$\langle x_i, y_1, y_2, ..., y_m, x_j - x_i\rangle$$

is an $(m+1)$-cube in $\{x_1, x_2, ..., x_r\}$.

We now proceed to show that none of the reverse implications hold. First $DW \not\Rightarrow C$. Rearrange the sequence $1, 2, 4, 8, ...,$ of powers of two, forming a sequence $d_0, d_1, d_2, ...$ which has arbitrarily long decreasing blocks (e.g., 1, 4, 2, 32, 16, 8, 512, 256, ...). Next define

$$a_0 = 1, \qquad a_{i+1} = a_i + d_i$$

and let $A = \{a_i : i \geqslant 0\}$. Clearly A has DW. If A contains a 2-cube $\langle b, y_1, y_2\rangle$, then $y_1 = b + y_1 - b = a_j - a_i = d_i + d_{i+1} + \cdots + d_{j-1}$ and $y_1 = b + y_1 + y_2 - (b + y_2) = a_t - a_s = d_s + d_{s+1} + \cdots + d_{t-1}$. Sums of distinct powers of two are unique and so $j = t$ which contradicts $a_t > a_j$.

Proof that $C \not\Rightarrow CP$. Let A be the set of all positive integers whose decimal representation uses only zeros and ones, i.e.,

$$A = \left\{k : k = \sum_{i=0}^{N} \varepsilon_i 10^i,\ \varepsilon_i = 0 \text{ or } 1,\ N \geqslant 0,\ k > 0\right\}.$$

It is clear from this definition that A has property C. Let $b_1 < b_2 < b_3 < \cdots < b_n$ be an increasing sequence in A and suppose $b_{i+1} - b_i = b_{r+1} - b_r$, where $i < r$. It follows that there exists j, $i < j < r$, such that $b_{j+1} - b_j > b_{i+1} - b_i$. Now assume that A has property CP for order d. If $b_1 < b_2 < b_3 < \cdots < b_n$ is an n-$CP(d)$ in A, then by assumption $b_{i+1} - b_i$ can take on at most d different values. But if n is very large, say $n = d^{(d+1)}$, we can find d^d indices i with the same $b_{i+1} - b_i$ and in between these indices $d^{(d-1)}$ other indices j with a larger $b_{j+1} - b_j$, and so on. In this way we get more than d differences appearing in the sequence $\{b_i\}$ contrary to our supposition.

Next $CP \not\Rightarrow QP$. We will make use of a remarkable sequence of zeros and ones, $\{z_1, z_2, z_3, ...\}$, which has the property that there do not exist five adjacent blocks of equal composition. (This means that, for any $a \geqslant 0$ and $d \geqslant 1$, not all of the five numbers

$$\sum_{i=1}^{d} z_{a+kd+i}, \qquad 0 \leqslant k \leqslant 4,$$

are the same.) The existence of such a sequence is due to J. Justin [4]. For each $t \geqslant 1$ let $S(t)$ be the following set of t positive integers,

$$S(t) = \{5^t + t + z_1, 5^t + 2t + z_1 + z_2, ..., 5^t + t^2 + z_1 + z_2 + \cdots + z_t\}.$$

Let $B(t)$ be the set $tS(t) = \{tx: x \in S(t)\}$. One easily checks that the first member of $S(t+1)$ (resp. $B(t+1)$) is more than twice as large as the greatest member of $S(t)$ (resp. $B(t)$). Each $B(t)$ is a t-$CP(2)$ since a difference is

$$\begin{aligned} &t5^t + (k+1)t^2 + t(z_1 + \cdots + z_{k+1}) - (t5^t + kt^2 + t(z_1 + \cdots + z_k)) \\ &\quad = t^2 + tz_{k+1} = t^2 \text{ or } t^2 + t. \end{aligned}$$

We define $A = B(1) \cup B(2) \cup \cdots$. Clearly A has property CP. Now suppose that A has property QP for diameter d. Let $t_0 > d$. Let $P = \{b_1 < b_2 < b_3 < \cdots < b_n\}$ be an n-$QP(d)$ in A. Suppose $\{b_i\}$ intersects $B(t_1)$, $B(t_2)$, and $B(t_3)$, where $t_0 \leqslant t_1 < t_2 < t_3$. Let b_i be a member of $P \cap B(t_1)$ and b_j be the largest member of $P \cap B(t_2)$. Then $(b_{j+1} - b_j) - (b_{i+1} - b_i) > b_j - (b_{i+1} - b_i) \geqslant b_i > t_1 5^{t_1} > d$ and this contradicts P being a $QP(d)$. Hence, if n is sufficiently large, we may assume that P contains six terms, $b_i, b_{i+1}, ..., b_{i+5}$ in some $B(t)$, where $t \geqslant t_0$. Now, for suitable $u > v > w$,

$$\begin{aligned} &|b_{j+2} - b_{j+1} - (b_{j+1} - b_j)| \\ &\quad = |t5^t + ut^2 + t(z_1 + \cdots + z_u) - 2(t5^t + vt^2 + t(z_1 + \cdots + z_v)) \\ &\qquad + t5^t + wt^2 + t(z_1 + \cdots + z_w)| \\ &\quad = |(u + w - 2v)t^2 + t((z_{v+1} + \cdots + z_u) - (z_{w+1} + \cdots + z_v))| \leqslant d < t. \end{aligned}$$

If follows that $u - v = v - w$ and $z_{v+1} + \cdots + z_u = z_{w+1} + \cdots + z_v$ so that the above six members of P determine five adjacent blocks of $\{z_i\}$ which have the same composition, a contradiction.

Finally we show that $QP \not\Rightarrow AP$. Here let $A = S(1) \cup S(2) \cup S(3) \cup ...$, where $S(t)$ is defined above. Clearly A has QP since each $S(t)$ is a t-$QP(1)$. An argument similar to the above would show that, if an arithmetic progression, $P = \{b_1 < b_2 < b_3 < \cdots < b_n\}$, in A were sufficiently long, then P would have to contain six terms in some $S(t)$. This, in turn, would again produce five adjacent blocks of $\{z_i\}$ which have the same composition.

3. Sets with Infinite Reciprocal Sum

Here we prove the two results on sets with infinite reciprocal sum mentioned in the Introduction. (Some other results on sets with infinite reciprocal sum can be found in [1].)

THEOREM 2. *The following two statements are equivalent:*

1. *(Erdős' conjecture). If A is any set of positive integers such that the sum of the reciprocals of the elements of A is infinite, then A has property AP.*

2. *If A is any set of positive integers such that the sum of the reciprocals of the elements of A is infinite, then A has property QP.*

Proof. Clearly $1 \Rightarrow 2$. We show that "not 1" implies "not 2." Assume that A is a set of positive integers with $\sum_{i \in A} 1/i = \infty$ and which contains no k-term arithmetic progression for a fixed k. We will construct a set B with infinite reciprocal sum which does not have property QP. We note that, for each $n \geqslant 1$, $g \geqslant 0$, the set $nA + g = \{na_i + g \mid i \geqslant 1\}$ does not contain any $k-QP(n-1)$. For otherwise we have elements $a_1, a_2, ..., a_k$ in A with

$$N \leqslant (na_{j+1} + g) - (na_j + g) \leqslant N + (n-1), \qquad 1 \leqslant j \leqslant k-1,$$

which implies that all $a_{j+1} - a_j$ are equal, contrary to assumption.

We construct finite sets $B_1, B_2, B_3, ...$ as follows. Let B_1 consist of enough terms of A so that $\sum_{i \in B_1} 1/i > 1$. Having chosen $B_1, B_2, ..., B_{n-1}$, we let $g \geqslant 3 \cdot \max(B_{n-1})$ and B_n consist of enough terms of $nA + g$ so that

$$\sum_{i \in B_n} 1/i > 1.$$

We set $B = B_1 \cup B_2 \cup B_3 \cup \cdots$, and note that B_n does not contain any k-$QP(n-1)$ and that B has infinite reciprocal sum. We need only show that, for each $d \geqslant 0$, B does not contain arbitrarily long $QP(d)$. Let $S = \{b_1, b_2, ..., b_t\}$ be a t-$QP(d)$ in B where, for some $N \geqslant 1$,

$$N \leqslant b_{j+1} - b_j \leqslant N + d, \qquad 1 \leqslant j \leqslant t-1.$$

If $i \geqslant 2$ and b_i, b_{i+1} belong to different sets B_n, then, for $j < i$,

$$N + d \geqslant b_{i+1} - b_i \geqslant 2b_i \geqslant 2(b_{j+1} - b_j) \geqslant 2N.$$

It follows that, if S intersects with $h+1$ different sets B_n, then we would obtain $N + d \geqslant 2^h N$, which implies $h \leqslant \log_2(d+1)$. Hence, if B has property QP for diameter d then no t-$QP(d)$ in B can meet more than $\log_2(d+1)+1$ different sets B_n. Hence, by choosing t sufficiently large, we may assure that S has at least k consecutive terms in some B_n where $n \geqslant d+1$. But these k terms are a k-$QP(d)$ which is a k-$QP(n-1)$ in B_n. This contradiction completes the proof.

THEOREM 3. *If A is a set of positive integers with infinite reciprocal sum, then A has property C (and therefore also property DW).*

Proof. It is shown in [3, p. 19] that, if

$$\alpha = 2 + \sqrt{3}, \qquad \lambda(k) = \alpha \cdot n^{1-(1/2^k)},$$

$A = \{a_1 < a_2 < \cdots < a_t\}$, $A \subseteq \{1, 2, ..., n\}$, and $t \geqslant \lambda(k)$, then A contains a k-cube. Thus, if $A = \{a_1, a_2, a_3, ...\}$ is any set of positive integers which does not contain any k-cube, we get, for $n \geqslant 1$, $A(n) < \lambda(k)$, where $A(n) = |A \cap \{1, 2, ..., n\}|$. Hence

$$n = A(a_n) < \alpha \cdot a_n^{1-(1/2^k)}$$

so that $a_n \geqslant cn^{1+\varepsilon}$, where c and ε are positive constants. This implies that $\sum_n 1/a_n < \infty$.

4. Descending Waves

We shall approach the problem of descending waves from several points of view. Our first is analogous to a result of van der Waerden. Let $f(k)$ be the smallest positive integer such that, if $\{1, 2, ..., f(k)\}$ is 2-colored, then there must be a monochromatic k-DW. Our first result bounds $f(k)$ above and below.

Theorem 4. $k^2 - k + 1 \leqslant f(k) \leqslant k^3/3 - 4k/3 + 3$.

Proof. For the lower bound, we need only observe that the 2-coloring

$$\underbrace{0\,0...0}_{k-1}\;\underbrace{1\,1...1}_{k-1}\;\underbrace{0\,0...0}_{k-2}\;\underbrace{1\,1...1}_{k-2}...\underbrace{0\,0}_{2}\;\underbrace{1\,1}_{2}\;\underbrace{0}_{1}\;\underbrace{1}_{1}$$

of $\{1, 2, ..., k^2 - k\}$ has no monochromatic k-DW.

For the upper bound we first prove a simple lemma: If $B_1, B_2, ..., B_t$ are consecutive blocks of integers (i.e., $0 \leqslant b_1 < b_2 < \cdots < b_{t+1}$, $B_i = [b_i + 1, b_{i+1}]$), $|B_1| \geqslant |B_2| \geqslant \cdots \geqslant |B_t|$, $t \geqslant s^2 - s + 1$, and $x_i \in B_i$ for $1 \leqslant i \leqslant t$, then the set $\{x_1, x_2, ..., x_t\}$ contains an s-DW $\{x_{i_1}, x_{i_2}, ..., x_{i_s}\}$ with $x_{i_s} = x_t$ and $x_{i_s} - x_{i_{s-1}} > |B_{t-1}|$.

To see this, just let $i_j = t - (s-j)(s-j+1)$ for $1 \leqslant j \leqslant s$. Then $x_{i_1}, x_{i_2}, ..., x_{i_s}$ is an s-DW with the last term $= x_t$. This is easily shown by the following calculations:

$$i_{j+1} - i_j = 2(s-j);$$

$$\begin{aligned} x_{i_{j+1}} - x_{i_j} &> |B_{i_j+1}| + |B_{i_j+2}| + \cdots + |B_{i_{j+1}-1}| \geqslant (2(s-j)-1)\,|B_{i_{j+1}}| \\ &\geqslant |B_{i_{j+1}}| + |B_{i_{j+1}+1}| + |B_{i_{j+1}+2}| + \cdots + |B_{i_{j+2}}| \geqslant x_{i_{j+2}} - x_{i_{j+1}}. \end{aligned}$$

Since $i_{s-1} = t-2$, we obtain $x_{i_s} - x_{i_{s-1}} > |B_{t-1}|$.

Next we suppose that $n \geqslant k^3/3 - 4k/3 + 3 = d$ and that $\{1, 2, ..., n\}$ is 2-colored such that there is no monochromatic k-DW. We partition the first d integers of this set into consecutive blocks of decreasing order, $B_1, B_2, ..., B_t$, where $t = k^2 - 3k + 4$, as follows: $|B_1| = k$; $|B_2| = |B_3| = k - 1; ...; |B_t| = 1$. Here, in general, there will be $2j$ blocks of length $k - j$ for $1 \leqslant j \leqslant k - 2$ (only one block, the first, of length k and one block, the last, of length one.) Hence the number of blocks is $1 + 2 + 4 + 6 + \cdots + 2(k-2) + 1 = t$. Also, the number of consecutive integers contained in the union of all these blocks is, as stated,

$$k + 1 + \sum_{j=1}^{k-2} 2j(k-j) = \frac{k^3}{3} - \frac{4k}{3} + 3.$$

If B_u is of length $k - s$ $(s \geqslant 1)$ then $u > 1 + 2 + 4 + \cdots + 2(s-1) = s^2 - s + 1$. It follows from the assumption about the coloring and the above lemma that no block of our partition can be monochromatic. For, supposing B_u to be the first monochromatic block (say all 1's), if $u = 1$, then the k integers of B_1 form a k-DW. On the other hand, if $1 < u \leqslant t$, then each block which comes before B_u must contain an integer colored 1 and, if $|B_u| = k - s$, the lemma implies that there is an s-DW, $x_1, x_2, ..., x_s$, of integers colored 1 such that $x_s \in B_{u-1}$. Let $x_{s+1}, x_{s+2}, ..., x_k$ be the k-s elements of B_u. From the construction used in the proof of the lemma, we see that $x_s - x_{s-1} > |B_{u-2}| \geqslant |B_{u-1}| \geqslant x_{s+1} - x_s$. Hence $x_1, x_2, ..., x_k$ is a monochromatic k-DW contrary to assumption. The theorem is proved by observing that B_t is necessarily monochromatic.

If one defines $f(k)$ requiring a monochromatic *strict* descending wave (i.e., the differences form a strictly decreasing sequence $d_1 > d_2 > \cdots > d_{k-1}$), then the above method will yield lower and upper bounds $c_1 k^3$ and $c_2 k^4$, respectively.

Further, if we consider the above method but use intervals each of length k, then we obtain the result: If $\{1, 2, ..., k^3 - 3k^2 + 4k\}$ is 2-colored, then there are either k consecutive monochrome integers or there is a monochromatic k-DW.

Next we proceed to find an upper bound on the order of a subset of $\{1, 2, ..., n\}$ which has no k-DW.

THEOREM 5. *Let $S \subseteq \{1, 2, ..., 2^n\}$ and suppose that S contains no k-DW where $3 \leqslant k \leqslant n + 2$. Then*

$$|S| \leqslant 2^{k-2} \binom{n}{k-2}.$$

Proof. Since descending waves are invariant under translation we may assume that $\min(S)=1$. We begin an induction at $k=3$ by observing that, if S contains no 3-DW, then each interval $I_t=\{2^t+1, ..., 2^{t+1}\}$, $0 \leqslant t \leqslant n-1$, contains no more than one element of S (for, if $a, b \in I_t$, $a<b$, then $\{1, a, b\}$ would be a 3-DW). Hence,

$$|S| \leqslant n+1 \leqslant 2n = 2\binom{n}{1},$$

provided that $n \geqslant 1$ (i.e., $k=3 \leqslant n+2$).

Now fix $k \geqslant 3$ and let $S \subseteq \{1, 2, ..., 2^n\}$ be a set which contains no $(k+1)$-DW. Then, as before, $I_t \cap S$ cannot contain any k-DW (since adjoining 1 to such a DW would give a $(k+1)$-DW in S). Thus by the induction hypothesis we have, for $k \leqslant t+2$,

$$|S \cap \{2^t+1, 2^t+2, ..., 2^{t+1}\}| \leqslant 2^{k-2}\binom{t}{k-2}.$$

For $k+1 \leqslant n+2$ we write

$$\{1, ..., 2^n\} = \{1, ..., 2^{k-2}\} \cup \bigcup_{t=k-2}^{n-1} \{2^t+1, ..., 2^{t+1}\}.$$

Thus we obtain

$$\begin{aligned} |S| &\leqslant 2^{k-2} + \sum_{t=k-2}^{n-1} 2^{k-2}\binom{t}{k-2} \\ &= 2^{k-2}\left(1+\binom{n}{k-1}\right) \leqslant 2^{k-1}\binom{n}{k-1}. \end{aligned}$$

Three corollaries follow from Theorem 5.

COROLLARY 1. *If $m \geqslant 2^{k-2}$ and S is a subset of $\{1, 2, ..., m\}$ which contains no k-DW, then*

$$|S| \leqslant \frac{2^{k-1}}{(k-2)!}(\log_2 m)^{k-2}.$$

COROLLARY 2. *If an infinite sequence $S=\{a_1<a_2<a_3<...\}$ contains no k-DW, then there is a constant $c>1$ (in fact, $c=2^{((k-2)!/2^{k-1})^{1/k-2}}$) such that, for $a_t \geqslant 2^{k-2}$,*

$$a_t \geqslant c^{t^{1/(k-2)}}.$$

Hence, if for each $\varepsilon>0$

$$a_n < e^{n^\varepsilon}$$

for all sufficiently large n, then $\{a_n\}$ has DW. For example, if

$$a_n \leqslant e^{n^{1/\log\log n}}$$

then $\{a_n\}$ has DW. Consequently, if $\{a_n\}$ is a sequence such that $a_n \leqslant p(n)$ for infinitely many n, where $p(x)$ is a fixed polynomial, then $\{a_n\}$ has property DW. This last remark gives a proof, independent of Theorem 3, that $\sum_A 1/a = \infty$ implies that A contains arbitrarily long descending waves.

COROLLARY 3. *Define $g(\varepsilon, k)$ to be the smallest n such that $A \subseteq \{1, 2, ..., n\}$ and $|A| > \varepsilon n$ imply that A has a k-DW. Then for $k \geqslant 4$ and $\varepsilon \leqslant 0.9$ we have*

$$\frac{k^2-k}{2\varepsilon} \leqslant g(\varepsilon, k) \leqslant \left(\frac{6e}{\varepsilon}\right)^{k-2}.$$

Proof. The left-hand inequality follows by taking the set colored "1" in the construction at the beginning of the proof of Theorem 4 as a subset of $\{1, 2, ..., [(k^2-k)/2\varepsilon]\}$. For the right-hand inequality we proceed as follows: Let $n = [(6e/\varepsilon)^{k-2}]$, $A \subseteq \{1, 2, ..., n\}$, $|A| > \varepsilon n$, and suppose that A has no k-DW. From Corollary 1 above we get

$$\varepsilon_n < |A| \leqslant \frac{2^{k-1}}{(k-2)!} (\log_2 n)^{k-2},$$

$$\frac{n}{(\log_2 n)^{k-2}} < \frac{2^{k-1}}{\varepsilon(k-2)!},$$

which, using $k^k e^{-k} \sqrt{2\pi k}\, e^{1/(12k+1)} \leqslant k!$, implies

$$\sqrt{2\pi(k-2)}\,(1-(\varepsilon/6e)^{k-2}) < \frac{2}{3}\left(\frac{2\varepsilon}{6} \log_2 \frac{6e}{\varepsilon}\right)^{k-2}.$$

But this inequality is false if $k \geqslant 4$ and $\varepsilon \leqslant 0.9$.

For $k = 3$ and any ε, the beginning of the proof of Theorem 5 shows that if $\varepsilon 2^t \geqslant t+1$, then $g(\varepsilon, 3) \leqslant 2^t$.

We shall consider below the existence of descending waves contained in sequences $\{a_1 < a_2 < a_3 < ...\}$ where the a_n are real numbers and $a_{n+1} - a_n \geqslant 1$ for all large n (see Section 1). The remarks following Corollary 2 above show that if a_n increases slowly then $\{a_n\}$ has DW. On the other hand, the next theorem shows that a_n cannot grow as an exponential and still have that property

THEOREM 6. *For each real $\varepsilon > 0$, let $k(\varepsilon)$ be the maximum, over all sequences $A = \{a_1 < a_2 < a_3 < \dots\}$ with $a_{n+1}/a_n \geqslant 1+\varepsilon$ for all n, of the length of the longest descending wave in A. Then*

$$[1/\varepsilon] + 1 \leqslant k(\varepsilon) < (1/\varepsilon) + 2.$$

Proof. Let $0 < b_0 < b_1 < \cdots < b_t$ be a DW in such a sequence A. Then $b_t = (b_t - b_{t-1}) + \cdots + (b_1 - b_0) + b_0 \geqslant t(b_t - b_{t-1}) + b_0$, so that

$$1 \geqslant t\left(1 - \frac{b_{t-1}}{b_t}\right) + \frac{b_0}{b_t} \geqslant t\left(1 - \frac{1}{1+\varepsilon}\right) + \frac{b_0}{b_t} > t\left(\frac{\varepsilon}{1+\varepsilon}\right).$$

Therefore $t < 1 + 1/\varepsilon$ whence $k(\varepsilon) < 1/\varepsilon + 2$.

For the lower bound, given $\varepsilon < 1$, define $a_i = i$ for $i = 1, 2, \dots, t$, where $t = [1/\varepsilon]$ and $a_{t+k} = t(1+\varepsilon)^k$ for $k \geqslant 1$. Then A satisfies the condition of the theorem and $1, 2, \dots, t, t(1+\varepsilon)$ is a DW in A of length $[1/\varepsilon] + 1$. Hence $k(\varepsilon) \geqslant [1/\varepsilon] + 1$.

The case where $\{a_n\}$ is an exponential sequence is special:

THEOREM 7. *Let $p(\varepsilon)$ be the length of the longest DW in the sequence $a_n = c^n$, where $c = 1 + \varepsilon$. Then there exist constants A and B such that*

$$A/\sqrt{\varepsilon} \leqslant p(\varepsilon) \leqslant B/\sqrt{\varepsilon}.$$

Proof. For the lower bound consider the sequence (with $t+2$ terms)

$$1, c^{t+1}, c^{(t+1)+t}, \dots, c^{(t+1)+t+(t-1)+\cdots+1}.$$

This is a DW if and only if, for each s, $1 \leqslant s \leqslant t$, we have

$$2 \geqslant c^s + \frac{1}{c^{s+1}}.$$

These inequalities all hold if and only if

$$2 \geqslant c^t + \frac{1}{c^{t+1}}$$

and this inequality is equivalent to $c^t \leqslant 1 + \sqrt{\varepsilon/c}$. (For $t = 1$ this inequality requires that $\varepsilon \leqslant \frac{1}{2}(\sqrt{5} - 1) \sim 0.61$ and that ε be smaller for larger values of t.) Hence, for given ε, say $\varepsilon < 0.6$, we may take $t = [\log(1 + \sqrt{\varepsilon/c})/\log(c)]$. This last quantity is asymptotic with $1/\sqrt{\varepsilon}$. For $\varepsilon < 0.6$ we may let $A = 0.787$.

For the upper bound we proceed as follows. First note that if $c^{r_1}, c^{r_2}, c^{r_3}$ is a 3-DW, then $r_3 - r_2 < r_2 - r_1$. Let $R = 1/\sqrt{\varepsilon}$ and let $a_1, a_2, ..., a_k$ be a DW in $\{c^s\}$. Letting $t = [R] + 1$ we get $a_t - a_{t-1} < a_t/R$. Write $a_t = c^{r_1}$ and $a_{t+1} = c^{r_2}$. Clearly $c^{r_2 - r_1} < (1/R) + 1$ so that

$$r_2 - r_1 < \frac{\log((1/R)+1)}{\log c} \sim R.$$

It follows that k is less than, approximately, twice R.

The sequences $a_n = \exp(n^\varepsilon)$ of Corollary 2, as we shall see below, all have property DW even though they are upper bounds for sequences which do not have DW. More precisely, we prove the following.

THEOREM 8. *For any $\varepsilon > 0$, there exists a sequence $A = \{a_n\}$ of positive integers such that A does not have property DW and, for all large n, $a_n < \exp(n^\varepsilon)$. (Compare the remarks following Corollary 2.)*

Proof. The sequence $\{2^n\}$ does not contain a 3-DW and $2^n < \exp(n^\varepsilon)$ for all $\varepsilon \geqslant 1$. Let $N > 1$ and put

$$A = A^N = \{2^{i(1)} + 2^{i(2)} + \cdots + 2^{i(N)}: i(1) > i(2) > \cdots > i(N) \geqslant 0\}.$$

We first prove that if $A = \{a_n\}$ then $a_n < \exp(n^\delta)$ for all large n, where $\delta > 1/N$. Let $a_n = 2^{i(1)} + 2^{i(2)} + \cdots + 2^{i(N)} > 2^{i(1)}$. Then

$$n = A(a_n) > A(2^{i(1)}) = \binom{i(1)-1}{N} > C(i(1))^N.$$

Hence $i(1) < Dn^{1/N}$ and

$$a_n < 2^{i(1)+1} = 2 \cdot 2^{i(1)} < 2 \cdot 2^{Dn^{1/N}} < e^{n^\delta}$$

for suitable constants C, D, and all large n. Thus we choose N such that $1/N < \varepsilon$. We can assume inductively that A^{N-1} does not have property DW. Suppose A^N has DW and let $a_1, a_2, ..., a_w$ be a descending wave in A^N. Write

$$a_t = \sum_{s=1}^{N} 2^{i(s,t)} \qquad (t = 1, 2, ..., w).$$

Note that $i(1, t) \leqslant i(1, t+1)$. If equality holds for arbitrarily long blocks, then these blocks determine long DWs in A^{N-1} contrary to the inductive hypothesis. Hence we may assume (by taking w large enough) that

$i(1, t) < i(1, t+1)$ occurs at least $N+2$ times. Clearly then we have $i(1, t+1) - N > i(1, 2)$ for some t. Then

$$\begin{aligned} a_{t+1} - a_t &> 2^{i(1,t+1)} - (2^{i(1,t)} + 2^{i(1,t)-1} + \cdots + 2^{i(1,t)-N+1}) \\ &= 2^{i(1,t+1)} - (2^{i(1,t)+1} - 2^{i(1,t)-N+1}) \\ &\geqslant 2^{i(1,t)-N+1} > 2^{i(1,2)+1} > a_2 - a_1. \end{aligned}$$

This proves that A^N does not have property DW.

The sets A^N above have rather irregular growth. The next theorem shows that if the growth pattern is sufficiently regular, then the sequence has property DW. First note the following remarks: Let $\alpha_i = 1 + (1/2^i)$. Then for all $i \geqslant 1$ we have $(1/\alpha_i) + \alpha_{i+1} < 2$. If ε is small, then there exists a maximal $q(\varepsilon)$ such that, for $i = 1, 2, ..., q(\varepsilon)$, $(1/a) + b \leqslant 2$ whenever $\alpha_i - \varepsilon \leqslant a \leqslant \alpha_i$ and $\alpha_{i+1} - \varepsilon \leqslant b \leqslant \alpha_i$. It is clear that $q(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$.

THEOREM 9. *Let $B = \{b_1 < b_2 < b_3 < \cdots\}$ be a set with the following property: For each $t > 1$ there exist integers i, k, s such that*

(i) $b_{j+1}/b_j \leqslant \alpha_{s+t}$ *for* $i \leqslant j \leqslant i+k$,

(ii) $b_{i+k}/b_i \geqslant \prod \alpha_r$ *where the product is taken over* $s+1 \leqslant r \leqslant s+t$,

and

(iii) $s+t \leqslant q(\varepsilon)$, *where* $\varepsilon = 2\max\{(b_{j+1}/b_j) - 1 \colon i \leqslant j \leqslant i+k\}$.

Then B has property DW.

Proof. Let $t > 1$. Take i, k, and s to satisfy the above three properties. Let $a_1 = b_{i+n(1)}$, where $n(1)$ is the largest integer >0 such that $b_{i+n(1)}/b_i \leqslant \alpha_{s+1}$. Next take $a_2 = b_{i+n(2)}$, where $n(2)$ is the largest $>n(1)$ such that $b_{i+n(2)}/b_{i+n(1)} \leqslant \alpha_{s+2}$. Continue in this manner forming the subsequence $\{a_1, a_2, ..., a_t\}$ of B. Condition (i) assures us that each $n(g)$ exists as long as $n(g) \leqslant k$. But condition (ii) implies that $n(t) \leqslant k$ for, otherwise,

$$\begin{aligned} b_{i+k}/b_i < b_{i+n(t)}/b_i &= (b_{i+n(1)}/b_i)(b_{i+n(2)}/b_{i+n(1)}) \cdots (b_{i+n(t)}/b_{i+n(t-1)}) \\ &\leqslant \textstyle\prod_{s+1 \leqslant r \leqslant s+t} \alpha_r. \end{aligned}$$

Now we show $a_1, a_2, ..., a_t$ is a DW. For this we need only show that $2a_{j+1} \geqslant a_j + a_{j+2}$ for each $j = 1, 2, ..., t-2$. This is equivalent to

$$\frac{1}{a_{j+1}/a_j} + \frac{a_{j+2}}{a_{j+1}} \leqslant 2.$$

By the remarks preceding the theorem and condition (iii), it will suffice to show that

$$\alpha_{s+j} - \varepsilon \leqslant b_{i+n(j)}/b_{i+n(j-1)} \leqslant \alpha_{s+j}$$

for each $j = 1, 2, ..., t$ (where $n_0 = 0$). This follows easily:

$$\begin{aligned} \alpha_{s+j} - \frac{b_{i+n(j)}}{b_{i+n(j-1)}} &< \frac{b_{i+n(j)+1}}{b_{i+n(j-1)}} - \frac{b_{i+n(j)}}{b_{i+n(j-1)}} \\ &= \frac{b_{i+n(j)}}{b_{i+n(j-1)}} \left(\frac{b_{i+n(j)+1}}{b_{i+n(j)}} - 1 \right) \\ &\leqslant \alpha_{s+j}(0.5\,\varepsilon) < \varepsilon. \end{aligned}$$

COROLLARY 4. *If $b_{i+1}/b_i \to 1$ $(i \to \infty)$, then B has property DW.*

Proof. We show that B satisfies the conditions of the Theorem. Let $t > 1$. Choose $s = 0$. Next choose i large enough so that (i) holds (for *all* $j \geqslant i$) and $t \leqslant q(\varepsilon)$, where $\varepsilon = 2 \sup_{j \geqslant i} \{(b_{j+1}/b_j) - 1\}$. Finally choose k so that $b_{i+k}/b_i \geqslant \prod_{1 \leqslant r \leqslant t} \alpha_r$.

From Corollary 4 it follows that a sequence of the form $a_n = \exp(n^\varepsilon)$ has DW. Conditions which are both necessary and sufficient for a set to possess property DW appear to be difficult to state.

5. REMARKS

It would be nice to prove either Theorem 2, replacing QP by CP, or Theorem 3, replacing C by CP. (It is, of course, very unlikely that we would prove both of these modified theorems as that would give us Erdös' Conjecture.)

It is known that the sequence of squares $\{n^2\}$ does not have property AP. Does it have property QP, CP, or C?

Professors Joel Spencer and Noga Alon have announced that, in Theorem 4, $ck^3 \leqslant f(k)$ for a suitable constant c. They have, evidently, also significantly improved both bounds for $g(\varepsilon, k)$ in Corollary 3: For suitable constants c and d (depending only on ε)

$$c^{k^{1/2}} \leqslant g(\varepsilon, k) \leqslant d^{k^{1/2} \log k}.$$

Besides descending waves, one can also consider ascending waves. A sequence $\{a_1 < \cdots < a_k\}$ will be a k-AW if $a_{i+1} - a_i \leqslant a_{i+2} - a_{i+1}$ for $1 \leqslant i \leqslant k-2$. There are arbitrarily large finite sets with no 3-AW and yet every infinite set has property AW, in fact, any infinite set will contain an infinite subsequence which is an ascending wave. We can show that for each k there is a smallest number $h(k)$ such that for $n \geqslant h(k)$ any set $\{a_1 < \cdots < a_n\}$ of integers has a k-AW or a k-DW. In fact, P. Erdős and Gy. Szekeres [2] showed that, if $f(k, t)$ denotes the smallest positive integer such that any set $\{a_1 < a_2 < \cdots < a_{f(k,t)}\}$ contains either a k-AW or a t-DW, then $f(k, t) = f(k-1, t) + f(k, t-1) - 1$ $(k \geqslant 2, t \geqslant 2)$. This immediately gives

$$f(k, t) = \binom{k+t-4}{k-2} + 1 \qquad (k \geqslant 2,\ t \geqslant 2),$$

so that $h(k) = f(k, k) \sim 4^{k-2}/\sqrt{\pi k}$.

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