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Upper bounds for continuous seminorms and special properties of bilinear maps

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ABSTRACT

If E is a locally convex topological vector space, let $(P(E),\preccurlyeq)$ be the pre-ordered set of all continuous seminorms on E. We study, on the one hand, for θ an infinite cardinal those locally convex spaces E which have the θ-neighbourhood property introduced by E. Jordá, meaning that all sets M of continuous seminorms of cardinality $|M|\leqslant\theta$ have an upper bound in $P(E)$. On the other hand, we study bilinear maps $\beta : E_1\times E_2\to F$ between locally convex spaces which admit “product estimates” in the sense that for all $p_{i,j}\in P(F)$, $i,j=1,2,\ldots$, there exist $p_i\in P(E_1)$ and $q_j\in P(E_2)$ such that $p_{i,j}(\beta(x,y))\leqslant p_i(x)q_j(y)$ for all $(x,y)\in E_1\times E_2$. The relations between these concepts are explored, and examples given. The main applications concern spaces $C^r_c(M,E)$ of vector-valued test functions on manifolds.

1. Introduction

Primarily, this article is devoted to a strengthened continuity property for bilinear maps which arose recently in the study of convolution of vector-valued test functions. In addition, it describes relations between this notion and the countable neighbourhood property, and discusses further applications of the latter (and the θ-neighbourhood property).

Neighbourhood properties. For E a locally convex space, we obtain a pre-order $\preccurlyeq$ on the set $P(E)$ of all continuous seminorms on E by declaring $p\preccurlyeq q$ if $p\leqslant Cq$ pointwise for some $C>0$. The space E is said to have the *countable neighbourhood property* (or *cnp*, for short) if each countable set of continuous seminorms has an upper bound in $(P(E),\preccurlyeq)$ (see [8] and the references therein). Likewise, given an infinite cardinal number θ, the space E is said to have the *θ-neighbourhood property* (of *θ-np*, for short) if for each set M of continuous seminorms on E of cardinality $|M|\leqslant\theta$, there exists a continuous seminorm q on E such that $p\preccurlyeq q$ for all $p\in M$ (see [20, Definition 4.4]).

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Besides classical studies (see [5,8] and the references therein), the countable neighbourhood property also occurred more recently in the study of the tensor algebra $\mathcal{T}(E)$ of a locally convex space E. Topologize $\mathcal{T}(E) := \bigoplus_{n\in\mathbb{N}_0} \mathcal{T}^n(E)$ as the locally convex direct sum of the projective tensor powers $\mathcal{T}^0(E) := \mathbb{R}$, $\mathcal{T}^1(E) := E$, $\mathcal{T}^{n+1}(E) := \mathcal{T}^n(E) \otimes_\pi E$ of E. Answering a question by K.-H. Neeb [22, Problem VIII.5], it was shown that $\mathcal{T}(E)$ is a topological algebra (i.e., the bilinear tensor multiplication $\mathcal{T}(E) \times \mathcal{T}(E) \to \mathcal{T}(E)$ is jointly continuous) if and only if E has the cnp [13, Theorem B].

Product estimates. Following [3], a bilinear map $\beta : E_1 \times E_2 \to F$ between locally convex spaces is said to *admit product estimates* if, for each double sequence $(p_{i,j})_{i,j\in\mathbb{N}}$ of continuous seminorms $p_{i,j}$ on F, there exists a sequence $(p_i)_{i\in\mathbb{N}}$ of continuous seminorms on E_1 and a sequence $(q_j)_{j\in\mathbb{N}}$ of continuous seminorms on E_2 such that

$$(\forall i, j \in \mathbb{N},\ x \in E_1,\ y \in E_2) \quad p_{i,j}\big(\beta(x, y)\big) \leqslant p_i(x)q_j(y).$$

If β admits product estimates, then β is continuous.[1] However, a continuous bilinear map need not admit product estimates (see Section 5).[2] Thus, the existence of product estimates can be regarded as a *strengthened continuity property* for bilinear maps.

The concept of product estimates first arose in the study of convolution of vector-valued test functions. Consider the following setting (to which we shall return later):

1.1. Let $b : E_1 \times E_2 \to F$ be a continuous bilinear map between locally convex spaces such that $b \neq 0$. Let $r, s, t \in \mathbb{N}_0 \cup \{\infty\}$ with $t \leqslant r + s$. If $r = s = t = 0$, let G be a locally compact group; otherwise, let G be a Lie group. Let λ_G be a left Haar measure on G. If G is discrete, we need not impose any completeness assumptions on F. If G is metrizable and not discrete, we assume that F is sequentially complete. If G is not metrizable (and hence not discrete either), we assume that F is quasi-complete. These conditions ensure the existence of the integrals needed to define the convolution $\gamma *_b \eta : G \to F$ of $\gamma \in C^r_c(G, E_1)$ and $\eta \in C^s_c(G, E_2)$ via

$$(\gamma *_b \eta)(x) := \int_G b\big(\gamma(y), \eta\big(y^{-1}x\big)\big)\, d\lambda_G(y) \quad \text{for } x \in G. \tag{1}$$

Then $\gamma *_b \eta \in C^{r+s}_c(G, F)$ (see [3, Proposition 2.2]), whence

$$\beta_b : C^r_c(G, E_1) \times C^s_c(G, E_2) \to C^t_c(G, F), \quad (\gamma, \eta) \mapsto \gamma *_b \eta \tag{2}$$

makes sense.

For G compact, β_b is always continuous [3, Corollary 2.3]. If G is an infinite discrete group, then β_b is continuous if and only if G is countable and b admits product estimates [3, Proposition 6.1]. The main result of [3] reads:

Theorem A. *If G is neither discrete nor compact, then the convolution map β_b from* (2) *is continuous if and only if* (a), (b) *and* (c) *are satisfied*:

(a) *G is σ-compact*;
(b) *if $t = \infty$, then also $r = s = \infty$*; *and*
(c) *b admits product estimates.*

Structure of the article and main results. After some preliminaries (Section 2), we recall various examples of spaces with neighbourhood properties, and some permanence properties of the class of spaces possessing the θ-np (Section 3). In Section 4, we prove two simple, but essential results, which link the concepts discussed in this article: If E_1, E_2 and F are locally convex spaces and F or both of E_1 and E_2 have the cnp, then every continuous bilinear map $E_1 \times E_2 \to F$ admits product estimates (see Propositions 4.1 and 4.5). This immediately gives a large supply of mappings admitting product estimates. In Section 5, we describe two simple concrete examples of continuous bilinear maps for which it can be shown by hand that they do not admit product estimates. Section 6 provides basic background concerning the topology on spaces of vector-valued test functions, for later use. Sections 7 and 8 are devoted to the proofs of more difficult theorems. If M is a Hausdorff topological space, let $\theta(M)$ be the smallest cardinal of a cover of M by compact sets (the compact covering number of M). We show:

[1] If p is a continuous seminorm on F, set $p_{i,j} := p$ for all $i, j \in \mathbb{N}$, and find corresponding p_i, q_j. Then $p(\beta(x, y)) \leqslant p_1(x)q_1(y)$ for all $x \in E_1$, $y \in E_2$.

[2] If β is continuous, given $p_{i,j}$ as before we can still find continuous seminorms $P_{i,j}$ on E_1 and $Q_{i,j}$ on E_2 such that $p_{i,j}(\beta(x, y)) \leqslant P_{i,j}(x)Q_{i,j}(y)$. However, in general one cannot choose $P_{i,j}$ independently of j, nor $Q_{i,j}$ independently of i.

Theorem B. *Let E be a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, let M be a paracompact, locally compact, non-compact topological space*; *if $r > 0$, let M be a metrizable, non-compact, finite-dimensional C^r-manifold. Then*

$$\Psi_{c,E} : C^r_c(M) \times E \to C^r_c(M, E), \quad (\gamma, v) \mapsto \gamma v$$

is a hypocontinuous bilinear map. The map $\Psi_{c,E}$ is continuous if and only if E has the $\theta(M)$-neighbourhood property. If E is metrizable, then $\Psi_{c,E}$ is continuous if and only if E is normable.

Finally, we obtain a characterization of those (G, r, s, t, b) for which the convolution map β_b admits product estimates.

Theorem C. *Let (G, r, s, t, b) and $\beta_b : C^r_c(G, E_1) \times C^s_c(G, E_2) \to C^t_c(G, F)$ be as in 1.1. Then β_b has the following properties*:

- *If G is finite, then β_b is always continuous. Moreover, β_b admits product estimates if and only if b does.*
- *If G is an infinite discrete group, then β_b admits product estimates if and only if β_b is continuous, which holds if and only if G is countable and b admits product estimates.*
- *If G is an infinite compact group, then β_b is always continuous. Moreover, β_b admits product estimates if and only if the conditions* (b) *and* (c) *from Theorem* A *are satisfied.*
- *If G is neither compact nor discrete, then β_b admits product estimates if and only if β_b is continuous, which holds if and only if* (a), (b) *and* (c) *from Theorem* A *are satisfied.*

For example, consider a compact, non-discrete Lie group G. Then the convolution map $C^0(G) \times C^\infty(G) \to C^\infty(G)$ is continuous but does not admit product estimates.

In the final section, we show by example that product estimates can also be available for non-degenerate bilinear maps on locally convex spaces which do not admit continuous norms.

2. Preliminaries and basic facts

Generalities. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By a *locally convex space*, we mean a Hausdorff locally convex real topological vector space. A map between topological spaces is called a *topological embedding* if it is a homeomorphism onto its image. If E is a vector space and p a seminorm on E, define $B^p_r(x) := \{y \in E\colon\ p(y - x) < r\}$ and $\overline{B}^p_r(x) := \{y \in E\colon\ p(y - x) \leqslant r\}$ for $r > 0$ and $x \in E$. Let $E_p := E/p^{-1}(0)$ be the associated normed space, with the norm $\|.\|_p$ given by $\|x + p^{-1}(0)\|_p := p(x)$. If X is a set and $\gamma : X \to E$ a map, we define $\|\gamma\|_{p,\infty} := \sup_{x\in X} p(\gamma(x))$. If $(E, \|.\|)$ is a normed space and $p = \|.\|$, we write $\|\gamma\|_\infty$ instead of $\|\gamma\|_{p,\infty}$.

Facts concerning direct sums. If $(E_i)_{i\in I}$ is a family of locally convex spaces, we equip the direct sum $E := \bigoplus_{i\in I} E_i$ with the locally convex direct sum topology [6]. We identify E_i with its canonical image in E.

Remark 2.1. If $U_i \subseteq E_i$ is a 0-neighbourhood for $i \in I$, then the convex hull $U := \operatorname{conv}(\bigcup_{i\in I} U_i)$ is a 0-neighbourhood in E, and a basis of 0-neighbourhoods is obtained in this way (as is well known). If I is countable, then the 'boxes' $\bigoplus_{i\in I} U_i := E \cap \prod_{i\in I} U_i$ form a basis of 0-neighbourhoods in E (cf. [19]). It is clear from this that the topology on E is defined by the seminorms $q : E \to [0, \infty[$ taking $x = (x_i)_{i\in I}$ to $\sum_{i\in I} q_i(x_i)$, for q_i ranging through the set of continuous seminorms on E_i (because $B^q_1(0) = \operatorname{conv}(\bigcup_{i\in I} B^{q_i}_1(0))$). If I is countable, we can take the seminorms $q(x) := \max\{q_i(x_i)\colon\ i \in I\}$ instead (because $B^q_1(0) = \bigoplus_{i\in I} B^{q_i}_1(0)$).

Some types of locally convex spaces. If E is a topological vector space, we write E_{lcx} for E, equipped with the finest among those (not necessarily Hausdorff) locally convex vector topologies which are coarser than the original topology (see, e.g., [13]). A topological space X is called a *k_ω-space* if $X = \varinjlim K_n$ as a topological space, for a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact Hausdorff spaces with continuous inclusion maps $K_n \to K_{n+1}$ (see [9,14]). We write $\mathbb{R}^{(\mathbb{N})}$ for the space of finitely supported real sequences, equipped with the finest locally convex vector topology. Thus $\mathbb{R}^{(\mathbb{N})} = \bigoplus_{n\in\mathbb{N}} \mathbb{R}$.

Hypocontinuity. As a special case of more general concepts, we call a bilinear map $\beta : E_1 \times E_2 \to F$ between locally convex spaces *hypocontinuous* in its first argument (resp., in its second argument) if it is separately continuous and the restriction $\beta|_{B\times E_2} : B \times E_2 \to F$ is continuous for each bounded subset $B \subseteq E_1$ (resp., $\beta|_{E_1\times B}$ is continuous for each bounded subset $B \subseteq E_2$). If β is hypocontinuous in both arguments, it will be called *hypocontinuous*.[3]

[3] See, e.g., [12, Proposition 16.8] for the equivalence of this definition with more classical ones (cf. also Proposition 4 in [6, Chapter III, §5, no. 3]).

3. Spaces with neighbourhood properties

We recall basic examples of spaces with the θ-neighbourhood property, and some permanence properties of the class of such spaces.

Proposition 3.1.

(a) *A metrizable locally convex space has the cnp if and only if it is normable. Every normable space satisfies the θ-np for each infinite cardinal θ.*
(b) *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of locally convex spaces that have the cnp. Then also the locally convex direct sum $\bigoplus_{n\in\mathbb{N}} E_n$ has the cnp.*
(c) *Let E be a locally convex space that has the θ-np for some infinite cardinal θ. Then also each vector subspace $F\subseteq E$ has the θ-np.*
(d) *Let θ be an infinite cardinal and $E_1,\ldots,E_n$ be locally convex spaces that have the θ-np. Then also $E_1\times\cdots\times E_n$ has the θ-np.*
(e) *If a locally convex space E is a k_ω-space or $E=F_{\mathrm{lcx}}$ for a topological vector space F which is a k_ω-space, then E has the cnp.*
(f) *$\mathbb{R}^{(\mathbb{N})}$ has the cnp.*
(g) *For each infinite cardinal θ, there exists a locally convex space E that has the θ-np but does not have the θ'-np for any $\theta'>\theta$.*
(h) *If a locally convex space E has the θ-np for an infinite cardinal θ, then also E/F has the θ-np, for every closed vector subspace $F\subseteq E$.*
(i) *Let E be the locally convex direct limit of a countable direct system of locally convex spaces having the cnp. Then also E has the cnp. In particular, every LB-space has the cnp.*
(j) *Every DF-space (and every gDF-space) has the cnp.*

Proof. (a) See [5, 1.1(i)] and [20, p. 285].

(b) See [8, p. 223].

(c) See [20, p. 285].

(d) Let $(p_j)_{j\in J}$ be a family of continuous seminorms on $E:=E_1\times\cdots\times E_n$, indexed by a set J of cardinality $|J|\leqslant\theta$. Then there exist continuous seminorms $p_{i,j}$ on E_i for $i\in\{1,\ldots,n\}$ with $p_j(x)\leqslant\max\{p_{1,j}(x_1),\ldots,p_{n,j}(x_n)\}$ for all $x=(x_1,\ldots,x_n)\in E$. Since E_i has the θ-np, there exists a continuous seminorm P_i on E_i such that $P_{i,j}\preccurlyeq P_i$ for all $j\in J$, and thus $P_{i,j}\leqslant C_{i,j}P_i$ with suitable $C_{i,j}>0$. Then $p(x):=\max\{P_1(x_1),\ldots,P_n(x_n)\}$ defines a continuous seminorm p on E such that $p_j\preccurlyeq p$ for all $j\in J$ (as $p_j\leqslant C_jp$ with $C_j:=\max\{C_{1,j},\ldots,C_{n,j}\}$).

(e) See [13, Corollary 8.1].

(f) Since $\mathbb{R}^{(\mathbb{N})}=\bigoplus_{n\in\mathbb{N}}\mathbb{R}$, the assertion follows from (a) and (b).

(g) Let X be a set of cardinality $|X|>\theta$ and $E:=\ell^\infty(X)$ be the space of all bounded real-valued functions on X, equipped with the (unusual!) topology defined by the seminorms

$$\|.\|_Y : E\to[0,\infty[,\quad \gamma\mapsto\sup\bigl\{\bigl|\gamma(y)\bigr|:\ y\in Y\bigr\},$$

for Y ranging through the subsets of X of cardinality $|Y|\leqslant\theta$. It can be shown that E has the asserted properties (see [13, Example 8.2]).[4]

(h) Let $\pi:E\to E/F$, $x\mapsto x+F$. If J is a set of cardinality $\leqslant\theta$ and $(q_j)_{j\in J}$ a family of continuous seminorms on E/F, then the $q_j\circ\pi$ are continuous seminorms on E, whence there exists a continuous seminorm p on E and $c_j>0$ such that $q_j\circ\pi\leqslant c_jp$ for all $j\in J$. Let $q:E/F\to[0,\infty[$ be the Minkowski functional of $\pi(B_1^p(0))$. Now $B_1^p(0)\subseteq c_jB_1^{q_j\circ\pi}(0)$ and thus also $B_1^p(0)+F\subseteq c_jB_1^{q_j\circ\pi}(0)$. Hence $B_1^q(0)\subseteq c_jB_1^{q_j}(0)$ and thus $q_j\leqslant c_jq$.

(i) See [8, p. 223] for the first claim. With (a), the final assertion follows.

(j) See [18, Satz 1.1(i)]. □

4. Bilinear maps with product estimates

The following results provide links between the cnp and product estimates.

Proposition 4.1. *Let E_1, E_2 and F be locally convex spaces and $\beta:E_1\times E_2\to F$ be a continuous bilinear map. If E_1 and E_2 have the countable neighbourhood property, then β satisfies product estimates.*

Proof. Let $p_{i,j}$ be continuous seminorms on F for $i,j\in\mathbb{N}$. Since β is continuous bilinear, for each $(i,j)\in\mathbb{N}^2$ there exists a continuous seminorm $P_{i,j}$ on E_1 and a continuous seminorm $Q_{i,j}$ on E_2 such that $p_{i,j}(\beta(x,y))\leqslant P_{i,j}(x)Q_{i,j}(y)$ for all $(x,y)\in E_1\times E_2$. Because E_1 has the cnp, there exists a continuous seminorm p on E_1 such that $P_{i,j}\preccurlyeq p$ for all $i,j\in\mathbb{N}$.

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[4] That E has the θ-np if $\theta=2^{\aleph_0}$ was also mentioned in [20, p. 285].

Likewise, there exists a continuous seminorm q on E_2 such that $Q_{i,j} \preccurlyeq q$ for all $i, j \in \mathbb{N}$. Thus, for $i, j \in \mathbb{N}$ there are $r_{i,j}, s_{i,j} \in]0,\infty[$ such that $P_{i,j} \leqslant r_{i,j} p$ and $Q_{i,j} \leqslant s_{i,j} q$. For $i \in \mathbb{N}$, let a_i be the maximum of $1, r_{i,1}s_{i,1}, \ldots, r_{i,i}s_{i,i}$, and define $p_i := a_i p$. For $j \in \mathbb{N}$, let b_j be the maximum of $1, r_{1,j}s_{1,j}, \ldots, r_{j-1,j}s_{j-1,j}$, and define $q_j := b_j q$. Let $i, j \in \mathbb{N}$. If $i \geqslant j$, then

$$p_{i,j}\big(\beta(x,y)\big) \leqslant P_{i,j}(x)Q_{i,j}(y) \leqslant r_{i,j}s_{i,j}p(x)q(y) \leqslant a_i p(x)q(y) \leqslant p_i(x)q_j(y)$$

for all $x \in E_1$ and $y \in E_2$. If $i < j$, then $p_{i,j}(\beta(x,y)) \leqslant r_{i,j}s_{i,j}p(x)q(y) \leqslant b_j p(x)q(y) \leqslant p_i(x)q_j(y)$. Thus β satisfies product estimates. □

Combining Propositions 3.1(a) and 4.1, we obtain:

Corollary 4.2. *If $(E_1, \|.\|_1)$ and $(E_2, \|.\|_2)$ are normed spaces, then every continuous bilinear map $\beta : E_1 \times E_2 \to F$ to a locally convex space F admits product estimates.*

Corollary 4.3. *Every bilinear map from $\mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{(\mathbb{N})}$ to a locally convex space admits product estimates.*

Proof. It is well known that $\mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{(\mathbb{N})} = \varinjlim \mathbb{R}^n \times \mathbb{R}^n$ as a topological space (cf. [4] and [17, Theorem 4.1]). Since bilinear maps on $\mathbb{R}^n \times \mathbb{R}^n$ are always continuous, it follows that every bilinear map β from $\mathbb{R}^{(\mathbb{N})} \times \mathbb{R}^{(\mathbb{N})}$ to a locally convex space is continuous. Combining Propositions 3.1(f) and 4.1, we deduce that β admits product estimates. □

Remark 4.4. The condition described in Proposition 4.1 is sufficient, but not necessary for product estimates. For example, consider the convolution map $\beta : C^\infty(K) \times C^\infty(K) \to C^\infty(K)$ on a non-discrete, compact Lie group K. Then β satisfies product estimates (by Theorem C). However, $C^\infty(K)$ is a non-normable, metrizable space, and therefore does not have the cnp (see Proposition 3.1(a)).

The next result was stimulated by a remark of C. Bargetz.[5]

Proposition 4.5. *Let E_1, E_2 and F be locally convex spaces. If F has the countable neighbourhood property, then every continuous bilinear map $\beta : E_1 \times E_2 \to F$ admits product estimates.*

Proof. If $p_{i,j}$ are continuous seminorms on F for $i, j \in \mathbb{N}$, then the cnp of F provides a continuous seminorm P on F and real numbers $C_{i,j} > 0$ such that $p_{i,j} \leqslant C_{i,j} P$ for all $i, j \in \mathbb{N}$. Since β is continuous, there exist continuous seminorms p on E_1 and q on E_2 such that $P(\beta(x,y)) \leqslant p(x)q(y)$ for all $x \in E_1$ and $y \in E_2$. By the lemma in [4], there are $c_i > 0$ for $i \in \mathbb{N}$ such that $c_i c_j \leqslant 1/C_{i,j}$ for all $i, j \in \mathbb{N}$, and thus $C_{i,j} \leqslant \frac{1}{c_i c_j}$. Define $p_i := \frac{1}{c_i} p$ and $q_j := \frac{1}{c_j} q$. Then $p_{i,j}(\beta(x,y)) \leqslant C_{i,j} P(\beta(x,y)) \leqslant C_{i,j} p(x)q(y) \leqslant \frac{1}{c_i c_j} p(x)q(y) \leqslant p_i(x)q_j(y)$ for all $x \in E_1$ and $y \in E_2$, as required. □

For later use, let us record some obvious facts:

Lemma 4.6. *Let E_1, E_2, F, X_1, X_2 and Y be locally convex spaces, $\beta : E_1 \times E_2 \to F$ be a continuous bilinear map and $\lambda_1 : X_1 \to E_1$, $\lambda_2 : X_2 \to E_2$ and $\Lambda : F \to Y$ be continuous linear maps.*

(a) *If β admits product estimates, then also $\Lambda \circ \beta$ and $\beta \circ (\lambda_1 \times \lambda_2)$ admit product estimates.*
(b) *If Λ is a topological embedding, then β admits product estimates if and only if $\Lambda \circ \beta$ admits product estimates.*

5. Bilinear maps without product estimates

We give two elementary examples of continuous bilinear maps not admitting product estimates. Further examples are provided by Theorem C.

Example 5.1. We endow the direct power $A := \mathbb{R}^{\mathbb{N}}$ with the product topology (of pointwise convergence), which makes it a Fréchet space and can be defined using the seminorms

$$\|.\|_n : \mathbb{R}^{\mathbb{N}} \to [0,\infty[, \quad \big\|(x_i)_{i\in\mathbb{N}}\big\|_n := \max\big\{|x_i|\colon 1 \leqslant i \leqslant n\big\}$$

[5] In a conversation from May 11, 2012, C. Bargetz explained to the author that β_b in Theorem A is continuous if $G = \mathbb{R}^n$, $r = s = t = \infty$ and F is a quasi-complete DF-space, as a consequence of a result on topological tensor products by L. Schwartz and a result from his thesis [1]. Since every DF-space has the cnp (cf. [18, Satz 1.1(i)]), Proposition 4.5 shows that Bargetz' hypotheses are subsumed by Theorem A.

for $n \in \mathbb{N}$. Let $\beta : A \times A \to A$, $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \mapsto (x_i y_i)_{i\in\mathbb{N}}$ be pointwise multiplication. Then β is a bilinear map and continuous, as $\|\beta(x, y)\|_n \leqslant \|x\|_n \|y\|_n$ for all $n \in \mathbb{N}$ and $x, y \in A$. The map β (which turns A into a non-unital associative topological algebra) does not satisfy product estimates.

To see this, consider the continuous seminorms $p_{i,j} := \|.\|_{i+j}$ on A. Let $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ be any sequences of continuous seminorms on A. Then $p_1 \leqslant r\|.\|_n$ for some $r > 0$ and some $n \in \mathbb{N}$. Let $e_{n+1} = (0, \ldots, 1, 0, \ldots) \in A$ be the sequence with a single non-zero entry 1 at position $n + 1$. Then

$$p_{1,n}\big(\beta(e_{n+1}, e_{n+1})\big) = p_{1,n}(e_{n+1}) = \|e_{n+1}\|_{n+1} = 1.$$

However, $p_1(e_{n+1})q_n(e_{n+1}) \leqslant r\|e_{n+1}\|_n q_n(e_{n+1}) = 0q_n(e_{n+1}) = 0$. Therefore $p_{1,n}(\beta(e_{n+1}, e_{n+1})) > p_1(e_{n+1})q_n(e_{n+1})$, and β cannot admit product estimates.

Example 5.2. Consider the Fréchet space $A := C^\infty[0,1] := C^\infty([0,1],\mathbb{R})$, whose vector topology is defined by the seminorms

$$\|.\|_{C^k} : C^\infty[0,1] \to [0,\infty[, \quad \|\gamma\|_{C^k} := \max\big\{\big\|\gamma^{(j)}\big\|_\infty\colon\ 0 \leqslant j \leqslant k\big\}$$

for $k \in \mathbb{N}_0$. The Leibniz rule for derivatives of products implies that the bilinear pointwise multiplication map $\beta : C^\infty[0,1] \times C^\infty[0,1] \to C^\infty[0,1]$, $\beta(\gamma,\eta) := \gamma\cdot\eta$ with $(\gamma\cdot\eta)(x) := \gamma(x)\eta(x)$ is continuous (since $\|\beta(\gamma,\eta)\|_{C^k} \leqslant 2^k\|\gamma\|_{C^k}\|\eta\|_{C^k}$), as is well known. We now show that β does not satisfy product estimates. To see this, let $p_{i,j} := \|.\|_{C^{i+j}}$ for $i, j \in \mathbb{N}$. Suppose that there exist continuous seminorms p_i and q_i on A for $i \in \mathbb{N}$, such that

$$p_{i,j}\big(\beta(\gamma,\eta)\big) \leqslant p_i(\gamma)q_j(\eta) \quad \text{for all } i, j \in \mathbb{N}.$$

We derive a contradiction. After increasing p_1, we may assume that $p_1 = r\|.\|_{C^k}$ for some $r > 0$ and some $k \in \mathbb{N}_0$. Let $h \in A$ be a function whose restriction to $[\frac14, \frac34]$ is identically 1. For each $\gamma \in A$ with support $\operatorname{supp}(\gamma) \subseteq [\frac14, \frac34]$, we then have

$$\|\gamma\|_{C^{k+1}} = \|\gamma\cdot h\|_{C^{k+1}} = p_{1,k}(\gamma\cdot h) \leqslant p_1(\gamma)q_k(h) \leqslant K\|\gamma\|_{C^k}$$

with $K := rq_k(h)$. Let $g \in C_c^\infty(\mathbb{R})$ with $g(0) \neq 0$ and $\operatorname{supp}(g) \subseteq [-\frac14, \frac14]$. Then $g^{(j)} \neq 0$ for all $j \in \mathbb{N}_0$ (because otherwise g would be a polynomial and hence not compactly supported, contradiction). For $t \in]0,1]$, define a map $g_t \in A$ via $g_t(x) := t^k g((x-\frac12)/t)$. Then $g_t^{(j)}(x) = t^{k-j}g^{(j)}((x-\frac12)/t)$ for each $j \in \mathbb{N}_0$, entailing that $S := \sup\{\|g_t\|_{C^k}\colon\ t \in]0,1]\} < \infty$ and $\|g_t\|_{C^{k+1}} \geqslant \|g_t^{(k+1)}\|_\infty = t^{-1}\|g^{(k+1)}\|_\infty \to \infty$ as $t \to 0$. This contradicts the estimate $\|g_t\|_{C^{k+1}} \leqslant K\|g_t\|_{C^k} \leqslant KS$.

6. Spaces of vector-valued test functions

In this section, we compile preliminaries concerning spaces of vector-valued test functions, for later use. The proofs can be found in [3].

The manifolds considered in this article are finite-dimensional, smooth and metrizable (but not necessarily σ-compact).[6] The Lie groups considered are finite-dimensional, real Lie groups.

Vector-valued C^r-maps on manifolds. If $r \in \mathbb{N}_0 \cup \{\infty\}$, $U \subseteq \mathbb{R}^n$ is open and E a locally convex space, then a map $\gamma : U \to E$ is called C^r if the partial derivatives $\partial^\alpha\gamma : U \to E$ exist and are continuous, for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| := \alpha_1 + \cdots + \alpha_n \leqslant r$. If $V \subseteq \mathbb{R}^n$ is open and $\tau : V \to U$ a C^r-map, then also $\gamma \circ \tau$ is C^r (as a special case of infinite-dimensional calculus as in [21,16,10], or [15]). It therefore makes sense to consider C^r-maps from manifolds to locally convex spaces. If M is a manifold and $\gamma : M \to E$ a C^1-map to a locally convex space, we write $d\gamma$ for the second component of the tangent map $T\gamma : TM \to TE \cong E \times E$. If $X : M \to TM$ is a smooth vector field on M and γ as before, we write

$$X.\gamma := d\gamma \circ X.$$

The topology on $C_c^r(M,E)$. Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and E be a locally convex space. If $r = 0$, let M be a (Hausdorff) locally compact space, and equip the space $C^0(M,E) := C(M,E)$ of continuous E-valued functions on M with the compact-open topology. If $r > 0$, let M be a C^r-manifold. Set $d^0\gamma := \gamma$, $T^0M := M$, $T^kM := T(T^{k-1}M)$ and $d^k\gamma := d(d^{k-1}\gamma) : T^kM \to E$ for $k \in \mathbb{N}$ with $k \leqslant r$. Equip $C^r(M,E)$ with the initial topology with respect to the maps $d^k : C^r(M,E) \to C(T^kM,E)$ for $k \in \mathbb{N}_0$ with $k \leqslant r$, where $C(T^k(M),E)$ is equipped with the compact-open topology. Returning to $r \in \mathbb{N}_0 \cup \{\infty\}$, endow the space $C_K^r(M,E) := \{\gamma \in C^r(M,E)\colon\ \operatorname{supp}(\gamma) \subseteq K\}$ with the topology induced by $C^r(M,E)$, for each compact subset K of M. Let $\mathcal{K}(M)$ be the set of compact subsets of M. Equip $C_c^r(M,E) := \bigcup_{K\in\mathcal{K}(M)} C_K^r(M,E)$ with the locally convex direct limit topology. Then $C_c^r(M,E)$ is Hausdorff (because the inclusion map $C_c^r(M,E) \to C^r(M,E)$ is continuous). As usual, we abbreviate $C^r(M) := C^r(M,\mathbb{R})$, $C_K^r(M) := C_K^r(M,\mathbb{R})$ and $C_c^r(M) := C_c^r(M,\mathbb{R})$. The following fact is well known (see, e.g., [11, Proposition 4.4]):

[6] Recall that a manifold is metrizable if and only if it is paracompact, as follows, e.g., from [2, Theorem II.4.1].

Lemma 6.1. *If $U \subseteq \mathbb{R}^n$ is open, $K \subseteq U$ compact and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the topology on $C^r_K(U, E)$ arises from the seminorms $\|.\|_{k,p}$ defined via*

$$\|\gamma\|_{k,p} := \max\bigl\{\bigl\|\partial^\alpha \gamma\bigr\|_{p,\infty}\colon \alpha \in \mathbb{N}_0^n,\ |\alpha| \leqslant k\bigr\},$$

for all $k \in \mathbb{N}_0$ with $k \leqslant r$ and continuous seminorms p on E.

In the next three lemmas (which are Lemmas 1.3, 1.14 and 1.15 from [3], respectively), we let E be a locally convex space and $r \in \mathbb{N}_0 \cup \{\infty\}$. If $r = 0$, we let M be a locally compact space. If $r > 0$, then M is a manifold.

Lemma 6.2. *Let $(h_j)_{j\in J}$ be a family of functions $h_j \in C^r_c(M)$ whose supports $K_j := \operatorname{supp}(h_j)$ form a locally finite family. Then the map*

$$\Phi : C^r_c(M, E) \to \bigoplus_{j\in J} C^r_{K_j}(M, E), \quad \gamma \mapsto (h_j \cdot \gamma)_{j\in J}$$

is continuous and linear. If $(h_j)_{j\in J}$ is a partition of unity (i.e., $h_j \geqslant 0$ and $\sum_{j\in J} h_j = 1$ pointwise), then Φ is a topological embedding.

Lemma 6.3. *For each $0 \neq v \in E$, the map $\Phi_v : C^r_c(M) \to C^r_c(M, E)$, $\Phi_v(\gamma) := \gamma v$ is linear and a topological embedding (where $(\gamma v)(x) := \gamma(x)v$).*

Lemma 6.4. *The map $\Psi_{K,E} : C^r_K(M) \times E \to C^r_K(M, E)$, $(\gamma, v) \mapsto \gamma v$ is continuous, for each compact subset $K \subseteq M$.*

Definition 6.5. Let G be a Lie group, with identity element 1, and $K \subseteq G$ be a compact subset. Let $\mathcal{B}$ be a basis of the tangent space $T_1(G)$, and E be a locally convex space. For $v \in \mathcal{B}$, let $\mathcal{L}_v$ be the left-invariant vector field on G given by $\mathcal{L}_v(g) := T_1(L_g)(v)$, and $\mathcal{R}_v$ the right-invariant vector field $\mathcal{R}_v(g) := T_1(R_g)(v)$ (where $L_g, R_g : G \to G$, $L_g(x) := gx$, $R_g(x) := xg$). Let

$$\mathcal{F}_L := \{\mathcal{L}_v\colon v \in \mathcal{B}\} \quad \text{and} \quad \mathcal{F}_R := \{\mathcal{R}_v\colon v \in \mathcal{B}\}.$$

Given $r \in \mathbb{N}_0 \cup \{\infty\}$, $k, \ell \in \mathbb{N}_0$ with $k + \ell \leqslant r$, and a continuous seminorm p on E, we define $\|\gamma\|^L_{k,p}$ (resp., $\|\gamma\|^R_{k,p}$) for $\gamma \in C^r_K(G, E)$ as the maximum of the numbers

$$\|X_j \ldots X_1.\gamma\|_{p,\infty},$$

for $j \in \{0, \ldots, k\}$ and $X_1, \ldots, X_j \in \mathcal{F}_L$ (resp., $X_1, \ldots, X_j \in \mathcal{F}_R$). Define $\|\gamma\|^{R,L}_{k,\ell,p}$ as the maximum of the numbers

$$\|X_i \ldots X_1.Y_j \ldots Y_1.\gamma\|_{p,\infty},$$

for $i \in \{0, \ldots, k\}$, $j \in \{0, \ldots, \ell\}$ and $X_1, \ldots, X_i \in \mathcal{F}_R$, $Y_1, \ldots, Y_j \in \mathcal{F}_L$. Then $\|.\|^L_{k,p}$, $\|.\|^R_{k,p}$ and $\|.\|^{R,L}_{k,\ell,p}$ are seminorms on $C^r_K(G, E)$. If $E = \mathbb{R}$ and $p = |.|$, we relax notation and write $\|.\|^R_k$ instead of $\|.\|^R_{k,p}$.

In the situation of Definition 6.5, we have the following (see [3, Lemma 1.8]):

Lemma 6.6. *For each $t \in \mathbb{N}_0 \cup \{\infty\}$, compact set $K \subseteq G$ and locally convex space E, the topology on $C^t_K(G, E)$ coincides with the topologies defined by each of the following families of seminorms:*

(a) *The family of the seminorms $\|.\|^L_{j,p}$, for $j \in \mathbb{N}_0$ such that $j \leqslant t$ and continuous seminorms p on E.*
(b) *The family of the seminorms $\|.\|^R_{j,p}$, for $j \in \mathbb{N}_0$ such that $j \leqslant t$ and continuous seminorms p on E.*

If $t < \infty$ and $t = k + \ell$, then the topology on $C^t_K(G, E)$ is also defined by the seminorms $\|.\|^{R,L}_{k,\ell,p}$, for continuous seminorms p on E.

To enable uniform notation in the proofs for Lie groups and locally compact groups, we write $\|.\|^L_{0,p} := \|.\|^R_{0,p} := \|.\|^{R,L}_{0,0,p} := \|.\|_{p,\infty}$ if p is a continuous seminorm on E and G a locally compact group. We also write $\|.\|^R_0 := \|.\|_\infty$. For example, Lemma 6.6 then remains valid for locally compact groups G.

The following fact (covered by [3, Lemma 2.6]) will be used repeatedly:

Lemma 6.7. *Let (G, r, s, t, b) be as in 1.1, $K \subseteq G$ be compact, $\gamma \in C^r_K(G, E_1)$, $\eta \in C^s_c(G, E_2)$ and q, p_1, p_2 be continuous seminorms on F, E_1 and E_2, respectively, such that $q(b(x, y)) \leqslant p_1(x)p_2(y)$ for all $(x, y) \in E_1 \times E_2$. Let $k, \ell \in \mathbb{N}_0$ with $k \leqslant r$ and $\ell \leqslant s$. Then*

$$\|\gamma *_b \eta\|^{R,L}_{k,\ell,q} \leqslant \|\gamma\|^R_{k,p_1} \|\eta\|^L_{\ell,p_2} \lambda_G(K).$$

7. Proof of Theorem B

First, we briefly discuss the compact covering number.

Lemma 7.1. *Let M be a paracompact, locally compact, non-compact topological space. Then the following hold*:

(a) *M is σ-compact if and only if $\theta(M) = \aleph_0$.*
(b) *$\theta(M) = |J|$ for every locally finite cover $(V_j)_{j\in J}$ of M by relatively compact, open, non-empty sets.*
(c) *M can be expressed as a topological sum (disjoint union) of open, σ-compact, non-empty subsets U_j, $j \in J$. For any such, $\theta(M) = \max\{|J|, \aleph_0\}$.*
(d) *If M is a manifold, then $\theta(M)$ is the maximum of $\aleph_0$ and the number of connected components of M.*

Proof. (a) By definition, M is σ-compact if and only if $\theta(M) \leqslant \aleph_0$; and as M is assumed non-compact, this is equivalent to $\theta(M) = \aleph_0$.

(b) We have $\theta(M) \leqslant |J|$ by minimality, as $(\overline{V_j})_{j\in J}$ is a compact cover. For the converse, let $(K_a)_{a\in A}$ be a cover of M by compact sets, with $|A| = \theta(M)$. Then $J_a := \{j \in J\colon\ K_a \cap V_j \neq \emptyset\}$ is finite, for each $a \in A$. Hence $|J| = |\bigcup_{a\in A} J_a| \leqslant |A|\aleph_0 = |A| = \theta(M)$ and thus $|J| = \theta(M)$.

(c) The first assertion is well known [7, Theorem 5.1.27]. Each U_j admits a countable, locally finite cover $(V_{j,i})_{i\in I_j}$ by relatively compact, open, non-empty sets. Let $L := \coprod_{j\in J} I_j$ be the disjoint union of the sets I_j. Then $(V_{j,i})_{(j,i)\in L}$ is a locally finite, relatively compact open cover of M. Moreover, J or one of the sets I_j is infinite. Hence $\theta(M) = |L| = \max\{|J|, \aleph_0\}$.

(d) Apply (c) to the partition of M into its connected components. $\square$

Proof of Theorem B. If $\gamma \in C^r_c(M)$, let $K := \mathrm{supp}(\gamma)$. Because $\Psi_{K,E}$ from Lemma 6.4 is continuous, also $\Psi_{c,E}(\gamma, .) = \Psi_{K,E}(\gamma, .)$ is continuous. For each $v \in E$, the linear map $\Psi_{c,E}(., v) = \Phi_v$ is continuous, by Lemma 6.3. Hence $\Psi_{c,E}$ is separately continuous. As is clear, $\Psi_{c,E}$ is bilinear. For each bounded set $B \subseteq C^r_c(M)$, there exists a compact set $K \subseteq M$ such that $B \subseteq C^r_K(M)$ (see, e.g., [3, Lemma 1.16(c)]). Hence $\Psi_{c,E}|_{B\times E} = \Psi_{K,E}|_{B\times E}$ is continuous and thus $\Psi_{c,E}$ is hypocontinuous in the first argument. Since each $C^r_K(M)$ is a Fréchet space and hence barrelled, $C^r_c(M) = \varinjlim C^r_K(M)$ is a locally convex direct limit of barrelled spaces and hence barrelled [23, II.7.2]. The separately continuous bilinear map $\Psi_{c,E}$ on $C^r_c(M) \times E$ is therefore hypocontinuous in the second argument [23, III.5.2]. Hence $\Psi_{c,E}$ is hypocontinuous.

We let $(U_j)_{j\in J}$ be a locally finite cover of M by relatively compact, open sets U_j. Then $|J| = \theta(M)$ (see Lemma 7.1(b)). Let $(h_j)_{j\in J}$ be a C^r-partition of unity subordinate to $(U_j)_{j\in J}$, in the sense that $K_j := \mathrm{supp}(h_j) \subseteq U_j$. Then also those U_j with $h_j \neq 0$ form a cover. We may therefore assume that $h_j \neq 0$ for all $j \in J$.

Now suppose that $\Psi_{c,E}$ is continuous. Let p_j be a continuous seminorm on E, for each $j \in J$. Let U be the set of all $\gamma \in C^r_c(M, E)$ such that $\|h_j\gamma\|_{p_j,\infty} \leqslant 1$ for all $j \in J$. Because

$$\Phi : C^r_c(M, E) \to \bigoplus_{j\in J} C^r_{K_j}(M, E), \quad \gamma \mapsto (h_j\gamma)_{j\in J}$$

is continuous (see Lemma 6.2), U is a 0-neighbourhood. Hence, there are 0-neighbourhoods $V \subseteq C^r_c(M)$ and $W \subseteq E$ such that $\Psi_{c,E}(V \times W) \subseteq U$. After shrinking W, we may assume that $W = \overline{B}^q_1(0)$ for some continuous seminorm q on E. For each $j \in J$, we have $\varepsilon_j h_j \in V$ for some $\varepsilon_j > 0$. Hence $\Psi_{c,E}(\varepsilon_j h_j, w) \in U$ for each $w \in W$ and thus $1 \geqslant \|\varepsilon_j h_j w\|_{p_j,\infty} = \varepsilon_j p_j(w)\|h_j\|_\infty$. So, abbreviating $C_j := 1/(\varepsilon_j\|h_j\|_\infty)$, we have $p_j(w) \leqslant C_j$ for all $w \in \overline{B}^q_1(0)$ and thus $p_j \leqslant C_j q$. Hence $p_j \preccurlyeq q$ for all j and thus E has the $\theta(M)$-np.

Conversely, let E have the $\theta(M)$-np. If $r \geqslant 1$, we can cover each $\overline{U_j}$ with finitely many chart domains $W_{j,i}$ and replace U_j by $U_j \cap W_{j,i}$, without increasing the cardinality of the family (since $|J|\aleph_0 = |J|$). We may therefore assume that each U_j is the domain of a chart $\phi_j : U_j \to V_j \subseteq \mathbb{R}^n$. Let $U \subseteq C^r_c(M, E)$ be a 0-neighbourhood. Because Φ just defined is a topological embedding, after shrinking U we may assume that there are continuous seminorms p_j on E and $k_j \in \mathbb{N}_0$ such that $k_j \leqslant r$ and

$$U = \left\{\gamma \in C^r_c(M, E)\colon \sum_{j\in J} \left\|(h_j\gamma) \circ \phi_j^{-1}\right\|_{k_j,p_j} < 1\right\}$$

(see Remark 2.1, Lemma 6.1 and [11, Lemma 3.7]). By the $\theta(M)$-np, there exists a continuous seminorm q on E and a family $(C_j)_{j\in J}$ of real numbers $C_j > 0$ such that $p_j \leqslant C_j q$ for each $j \in J$. Then

$$V := \left\{\gamma \in C^r_c(M)\colon \sum_{j\in J} C_j\left\|(h_j\gamma) \circ \phi_j^{-1}\right\|_{k_j} < 1\right\}$$

is a 0-neighbourhood in $C^r_c(M)$ and $\Psi_{c,E}(V \times \overline{B}^q_1(0)) \subseteq U$ because $\|(h_j\gamma v) \circ \phi_j^{-1}\|_{k_j,p_j} = p_j(v)\|(h_j\gamma) \circ \phi_j^{-1}\|_{k_j} \leqslant C_j q(v)\|(h_j\gamma) \circ \phi_j^{-1}\|_{k_j} \leqslant C_j\|(h_j\gamma) \circ \phi_j^{-1}\|_{k_j}$, with sum < 1. Hence $\Theta_{c,E}$ is continuous at $(0, 0)$ and hence continuous.

If E is normable, then E has the $\theta(M)$-np (see Proposition 3.1(a)), whence $\Psi_{c,E}$ is continuous. If E is metrizable and $\Psi_{c,E}$ is continuous, then E has the $\theta(M)$-np, and thus E has the cnp. Hence E is normable (by Proposition 3.1(a)). $\square$

8. Proof of Theorem C

Lemma 8.1. *Let (G, r, s, t, b) and β_b be as in 1.1. If β_b admits product estimates, then also b admits product estimates.*

Proof. Let $K \subseteq G$ be a compact identity neighbourhood. If the map $\beta_b : C^r_c(G, E_1) \times C^s_c(G, E_2) \to C^t_c(G, F)$ admits product estimates, then also the convolution map $\theta : C^r_K(G, E_1) \times C^s_K(G, E_2) \to C^t_{KK}(G, F)$ admits product estimates, being obtained via restriction and co-restriction from β_b (see Lemma 4.6(a) and (b)).

If G is discrete, we simply take $K := \{1\}$, in which case b can be identified with θ and hence admits product estimates – as required.

For general G, choose a non-zero function $h \in C^0_K(G)$ such that $h \geqslant 0$; if G is a Lie group, we assume that h is smooth. After replacing h with the function $y \mapsto h(y) + h(y^{-1})$ if necessary, we may assume that $h(y) = h(y^{-1})$ for all $y \in G$. Also, after replacing h with a positive multiple if necessary, we may assume that $\int_G h(y)^2\, d\lambda_G(y) = 1$. Then $\phi_1 : E_1 \to C^r_K(G, E_1)$, $u \mapsto hu$ and $\phi_2 : E_2 \to C^r_K(G, E_2)$, $v \mapsto hv$ are continuous linear maps, and also $\varepsilon : C^t_{KK}(G, F) \to F$, $\gamma \mapsto \gamma(1)$ is continuous linear. Hence $\varepsilon \circ \theta \circ (\phi_1 \times \phi_2) : E_1 \times E_2 \to F$ admits product estimates, by Lemma 4.6(a). But this map takes $(u, v) \in E_1 \times E_2$ to

$$(hu * hv)(1) = b(u, v) \int_G h(y)h\bigl(y^{-1}\bigr)\, d\lambda_G(y) = b(u, v)$$

and thus coincides with b. Hence b admits product estimates. □

The next lemma, as well as Lemmas 8.5 and 8.6, are relevant only for the study of convolution on Lie groups. Readers exclusively interested in the case that G is a locally compact group and $r = s = t = 0$ can skip them.

Lemma 8.2. *Let (G, r, s, t, b) and β_b be as in 1.1. If β_b admits product estimates, $t = \infty$ and G is not discrete, then also $r = s = \infty$.*

Proof. Because $\beta := \beta_b$ admits product estimates, it is continuous. Hence, if G is not compact, then $r = s = \infty$ by Theorem A. It remains to show that β does not admit product estimates if G is compact, $r < \infty$, and $s = \infty$ (the same then follows for $r = \infty$, $s < \infty$ using [3, Lemmas 1.13 and 23.4]). As a tool, let $\theta : C^r(G) \times C^s(G) \to C^t(G)$ be the convolution of scalar-valued functions. Pick $u \in E_1$, $v \in E_2$ such that $w := b(u, v) \neq 0$. Let $\Phi_u : C^r(G) \to C^r(G, E_1)$, $\Phi_v : C^s(G) \to C^s(G, E_2)$ and $\Phi_w : C^t(G) \to C^t(G, F)$ be the topological embeddings taking γ to γu, γv and γw, respectively (see Lemma 6.3). In view of Lemma 4.6(b), if we can show that θ does not admit product estimates, then $\Phi_w \circ \theta = \beta \circ (\Phi_u \times \Phi_v)$ will not admit product estimates either (Lemma 4.6(b)). Hence also β does not admit product estimates (Lemma 4.6(a)). We may therefore assume that $E_1 = E_2 = F = \mathbb{R}$ and $\beta = \theta$. Consider the continuous seminorms $P_{i,j} := \|.\|^L_i$ on $C^\infty(G)$ for $i, j \in \mathbb{N}$. If β would admit product estimates (which will lead to a contradiction), we could find continuous seminorms P_i on $C^r(G)$ and Q_j on $C^\infty(G)$ such that $P_{i,j}(\gamma * \eta) \leqslant P_i(\gamma)Q_j(\eta)$. After increasing P_i and Q_j if necessary, we may assume that $P_i = a_i\|.\|^L_r$ and $Q_j = c_j\|.\|^L_{s_j}$ with suitable $a_i, c_j > 0$ and $s_j \in \mathbb{N}_0$ (see Lemma 6.6). Thus

$$\|\gamma * \eta\|^L_i \leqslant a_i c_j \|\gamma\|^L_r \|\eta\|^L_{s_j}$$

for all $i, j \in \mathbb{N}$ and $(\gamma, \eta) \in C^r(G) \times C^\infty(G)$. In particular, with $j := 1$ and $\ell := s_1 \in \mathbb{N}_0$, we obtain

$$\|\gamma * \eta\|^L_i \leqslant a_i c_1 \|\gamma\|^L_r \|\eta\|^L_\ell$$

for all $i \in \mathbb{N}$ and $(\gamma, \eta) \in C^r(G) \times C^\infty(G)$. Hence β_b would be continuous as a map $(C^r(G), \|.\|^L_r) \times (C^\infty(G), \|.\|^L_\ell) \to C^\infty(G)$, using the usual Fréchet topology on the right-hand side, but only the indicated norms on the left. This is impossible, as recorded in [3, Lemma 5.1]. □

Lemma 8.3. *Let (G, r, s, t, b) and β_b be as in 1.1. Assume that G is σ-compact and b admits product estimates. Moreover, assume that $t < \infty$ or $r = s = t = \infty$. Then also β_b admits product estimates.*

The proof will be based on three lemmas:

Lemma 8.4. *Let $(E_i)_{i\in\mathbb{N}}$ and $(F_j)_{j\in\mathbb{N}}$ be sequences of locally convex spaces, H be a locally convex space and $\beta_{i,j} : E_i \times F_j \to H$ be continuous bilinear maps for $i, j \in \mathbb{N}$. Assume that, for every double sequence $(P_{\sigma,\tau})_{\sigma,\tau\in\mathbb{N}}$ of continuous seminorms on H, there are continuous seminorms $P_{i,\sigma}$ on E_i, continuous seminorms $Q_{j,\tau}$ on F_j and numbers $C_{i,j,\sigma,\tau} > 0$, such that*

$$P_{\sigma,\tau}\bigl(\beta_{i,j}(x, y)\bigr) \leqslant C_{i,j,\sigma,\tau} P_{i,\sigma}(x) Q_{j,\tau}(y)$$

for all $i, j, \sigma, \tau \in \mathbb{N}$ and all $x \in E_i$ and $y \in F_j$. Then the bilinear map $\beta : (\bigoplus_{i\in\mathbb{N}} E_i) \times (\bigoplus_{j\in\mathbb{N}} F_j) \to H$ taking $((x_i)_{i\in\mathbb{N}}, (y_j)_{j\in\mathbb{N}})$ to $\sum_{i,j\in\mathbb{N}} \beta_{i,j}(x_i, y_j)$ admits product estimates.

Proof. The map $b:\mathbb{R}^{(\mathbb{N})}\times\mathbb{R}^{(\mathbb{N})}\to\mathbb{R}^{(\mathbb{N}\times\mathbb{N})}$, $b((u_i)_{i\in\mathbb{N}},(v_j)_{j\in\mathbb{N}}):=(u_iv_j)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ admits product estimates, by Corollary 4.3. For all $\sigma,\tau\in\mathbb{N}$,

$$p_{\sigma,\tau}(w):=\sum_{i,j\in\mathbb{N}}C_{i,j,\sigma,\tau}|w_{i,j}|$$

defines a continuous seminorm on $\mathbb{R}^{(\mathbb{N}\times\mathbb{N})}$ (see Remark 2.1). Hence, there exist continuous seminorms p_σ and q_τ on $\mathbb{R}^{(\mathbb{N})}$ such that $p_{\sigma,\tau}(b(u,v))\leqslant p_\sigma(u)q_\tau(v)$ for all $u,v\in\mathbb{R}^{(\mathbb{N})}$. By Remark 2.1, after increasing p_σ and q_τ if necessary, we may assume that they are of the form

$$p_\sigma(u)=\max\bigl\{r_{i,\sigma}|u_i|:\ i\in\mathbb{N}\bigr\}$$

and $q_\tau(v)=\max\{s_{j,\tau}|v_j|:\ j\in\mathbb{N}\}$ with suitable $r_{i,\sigma},s_{j,\tau}>0$. Then

$$P_\sigma(x):=p_\sigma\bigl(\bigl(P_{i,\sigma}(x_i)\bigr)_{i\in\mathbb{N}}\bigr)=\max\bigl\{r_{i,\sigma}P_{i,\sigma}(x_i):\ i\in\mathbb{N}\bigr\}$$

and $Q_\tau(y):=q_\tau((Q_{j,\tau}(y_j))_{j\in\mathbb{N}})$ (for $x\in E:=\bigoplus_{i\in\mathbb{N}}E_i$, $y\in F:=\bigoplus_{j\in\mathbb{N}}F_j$) define continuous seminorms P_σ and Q_τ on E and F, respectively (see Remark 2.1). For all $\sigma,\tau\in\mathbb{N}$ and x, y as before, we obtain

$$\begin{aligned}P_{\sigma,\tau}\bigl(\beta(x,y)\bigr)&\leqslant\sum_{i,j\in\mathbb{N}}P_{\sigma,\tau}\bigl(\beta_{i,j}(x_i,y_j)\bigr)\leqslant\sum_{i,j\in\mathbb{N}}C_{i,j,\sigma,\tau}P_{i,\sigma}(x_i)Q_{j,\tau}(y_j)\\&=p_{\sigma,\tau}\bigl(b\bigl(\bigl(P_{i,\sigma}(x_i)\bigr)_{i\in\mathbb{N}},\bigl(Q_{j,\tau}(y_j)\bigr)_{j\in\mathbb{N}}\bigr)\bigr)\\&\leqslant p_\sigma\bigl(\bigl(P_{i,\sigma}(x_i)\bigr)_{i\in\mathbb{N}}\bigr)q_\tau\bigl(\bigl(Q_{j,\tau}(y_j)\bigr)_{j\in\mathbb{N}}\bigr)=P_\sigma(x)Q_\tau(y).\end{aligned}$$

Hence β admits product estimates. $\square$

Lemma 8.5. *Let A be a countable set and $t_{\alpha,\beta}\in\mathbb{N}_0$ for $\alpha,\beta\in A$. Then there exist $r_\alpha,s_\beta\in\mathbb{N}_0$ for $\alpha,\beta\in A$ such that*

$$(\forall\alpha,\beta\in A)\quad r_\alpha+s_\beta\geqslant t_{\alpha,\beta}.$$

Proof. If A is a finite set, the assertion is trivial. If A is infinite, we may assume that $A=\mathbb{N}$. For $i\in\mathbb{N}$, let $r_i:=\max\{t_{i,j}:\ j\leqslant i\}$. For $j\in\mathbb{N}$, let $s_j:=\max\{t_{i,j}:\ i\leqslant j\}$. If $i,j\in\mathbb{N}$ and $i<j$, we deduce $t_{i,j}\leqslant s_j\leqslant r_i+s_j$. Likewise, $t_{i,j}\leqslant r_i\leqslant r_i+s_j$ if $i\geqslant j$. $\square$

Lemma 8.6. *Let G be a Lie group, E be a locally convex space, $K\subseteq G$ be compact, p be a continuous seminorm on E and $k,\ell\in\mathbb{N}_0$. Then there exists $C>0$ such that $\|\gamma\|^L_{k+\ell,p}\leqslant C\|\gamma\|^{R,L}_{k,\ell,p}=\|\gamma\|^{R,L}_{k,\ell,C\cdot p}$ for all $\gamma\in C^{k+\ell}_K(G,E)$.*

Proof. Let $E_p=E/p^{-1}(0)$ be the corresponding normed space, $\pi:E\to E_p$ be the canonical map and $P:=\|.\|_p$ be the norm on E_p. Because both $\|.\|^L_{k+\ell,P}$ and $\|.\|^{R,L}_{k,\ell,P}$ define the topology of $C^{k+\ell}_K(G,E_p)$ (see Lemma 6.6), there exists $C>0$ such that $\|.\|^L_{k+\ell,P}\leqslant C\|.\|^{R,L}_{k,\ell,P}$. Thus $\|\gamma\|^L_{k+\ell,p}=\|\pi\circ\gamma\|^L_{k+\ell,P}\leqslant C\|\pi\circ\gamma\|^{R,L}_{k,\ell,P}=C\|\gamma\|^{R,L}_{k,\ell,p}$ for all $\gamma\in C^{k+\ell}_K(G,E)$. $\square$

Proof of Lemma 8.3. Let $(h_i)_{i\in\mathbb{N}}$ be a partition of unity for G (smooth if G is a Lie group, continuous if G is merely a locally compact group). Let $\Phi:C^r_c(G,E_1)\to\bigoplus_{i\in\mathbb{N}}C^r_{K_i}(G,E_1)$ and $\Psi:C^s_c(G,E_2)\to\bigoplus_{i\in\mathbb{N}}C^s_{K_i}(G,E_2)$ be the embeddings taking γ to $(h_i\gamma)_{i\in\mathbb{N}}$ (see Lemma 6.2). Let

$$f:\bigoplus_{i\in\mathbb{N}}C^r_{K_i}(G,E_1)\times\bigoplus_{j\in\mathbb{N}}C^s_{K_j}(G,E_2)\to C^t_c(G,F)$$

be the map taking $((\gamma_i)_{i\in\mathbb{N}},(\eta_j)_{j\in\mathbb{N}})$ to $\sum_{i,j\in\mathbb{N}}\gamma_i*_b\eta_j$. Since $\beta_b=f\circ(\Phi\times\Psi)$, we need only to show that f admits product estimates (Lemma 4.6). To verify the latter property, let $P_{\sigma,\tau}$ be continuous seminorms on $C^t_c(G,F)$ for $\sigma,\tau\in\mathbb{N}$.

Before we turn to the general case, let us consider the instructive special case $r=s=t=0$ (whose proof is much simpler). For all $i,j,\sigma,\tau\in\mathbb{N}$, there exists a continuous seminorm $P_{i,j,\sigma,\tau}$ on F such that

$$P_{\sigma,\tau}(\gamma)\leqslant\|\gamma\|_{P_{i,j,\sigma,\tau},\infty}$$

for all $\gamma\in C_{K_iK_j}(G,F)$ (cf. Lemma 6.6 and the lines thereafter). Since $b:E_1\times E_2\to F$ admits product estimates and the set $\mathbb{N}\times\mathbb{N}$ (which contains the (i,σ) and (j,τ)) admits a bijective map $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, there exist continuous seminorms $p_{i,\sigma}$ on E_1 and $q_{j,\tau}$ on E_2 such that

$$P_{i,j,\sigma,\tau}\bigl(b(x,y)\bigr)\leqslant p_{i,\sigma}(x)q_{j,\tau}(y)$$

for all $i, j, \sigma, \tau \in \mathbb{N}$ and $x \in E_1$, $y \in E_2$. Define $S_{i,\sigma} : C_{K_i}(G, E_1) \to [0, \infty[$ and $Q_{j,\tau} : C_{K_j}(G, E_2) \to [0, \infty[$ via $S_{i,\sigma} := \lambda_G(K_i)\|.\|_{p_{i,\sigma},\infty}$ and $Q_{j,\tau} := \|.\|_{q_{j,\tau},\infty}$, respectively. Then

$$P_{\sigma,\tau}(\gamma *_b \eta) \leqslant \|\gamma *_b \eta\|_{P_{i,j,\sigma,\tau},\infty} \leqslant \|\gamma\|_{p_{i,\sigma},\infty}\|\eta\|_{q_{j,\tau},\infty}\lambda_G(K_i) = S_{i,\sigma}(\gamma)Q_{j,\tau}(\eta)$$

for all $(\gamma, \eta) \in C_{K_i}(G, E_1) \times C_{K_j}(G, E_2)$ (using Lemma 6.7 for the second inequality). The hypotheses of Lemma 8.4 are therefore satisfied, whence f (and hence also β_b) admits product estimates.

We now complete the proof of the lemma in full generality. In the case $t < \infty$, we choose $k, \ell \in \mathbb{N}_0$ with $k \leqslant r$, $\ell \leqslant s$ and $k + \ell = t$. For all $i, j \in \mathbb{N}$, there exists a continuous seminorm $P_{i,j,\sigma,\tau}$ on F such that

$$P_{\sigma,\tau}(\gamma) \leqslant \|\gamma\|^{R,L}_{k,\ell,P_{i,j,\sigma,\tau}}$$

for $\gamma \in C^t_{K_iK_j}(G, F)$ (Lemma 6.6). Set $r_{i,\sigma} := k$ and $s_{j,\tau} := \ell$ for $i, j, \sigma, \tau \in \mathbb{N}$.

In the case $r = s = t = \infty$, there exist $t_{i,j,\sigma,\tau} \in \mathbb{N}_0$ and continuous seminorms $Q_{i,j,\sigma,\tau}$ on F such that $P_{\sigma,\tau}(\gamma) \leqslant \|\gamma\|^L_{t_{i,j,\sigma,\tau},Q_{i,j,\sigma,\tau}}$ for all $\gamma \in C^\infty_{K_iK_j}(G, F)$. Using Lemma 8.5, we find $r_{i,\sigma}, s_{j,\tau} \in \mathbb{N}_0$ such that

$$r_{i,\sigma} + s_{j,\tau} \geqslant t_{i,j,\sigma,\tau}$$

for all $i, j, \sigma, \tau \in \mathbb{N}$. Then $\|.\|^L_{t_{i,j,\sigma,\tau},Q_{i,j,\sigma,\tau}} \leqslant \|.\|^L_{r_{i,\sigma}+s_{j,\tau},Q_{i,j,\sigma,\tau}} \leqslant \|.\|^{R,L}_{r_{i,\sigma},s_{j,\tau},P_{i,j,\sigma,\tau}}$ on $C^\infty_{K_iK_j}(G, F)$, with some positive multiple $P_{i,j,\sigma,\tau}$ of $Q_{i,j,\sigma,\tau}$ (Lemma 8.6).

In either case, since $b : E_1 \times E_2 \to F$ admits product estimates, there exist continuous seminorms $p_{i,\sigma}$ on E_1 and $q_{j,\tau}$ on E_2 such that

$$P_{i,j,\sigma,\tau}\big(b(x, y)\big) \leqslant p_{i,\sigma}(x)q_{j,\tau}(y)$$

for all $i, j, \sigma, \tau \in \mathbb{N}$ and $x \in E_1$, $y \in E_2$. Define $S_{i,\sigma} : C^r_{K_i}(G, E_1) \to [0, \infty[$ and $Q_{j,\tau} : C^s_{K_j}(G, E_2) \to [0, \infty[$ via $S_{i,\sigma} := \lambda_G(K_i)\|.\|^R_{r_{i,\sigma},p_{i,\sigma}}$ and $Q_{j,\tau} := \|.\|^L_{s_{j,\tau},q_{j,\tau}}$, respectively. Then

$$\begin{aligned} P_{\sigma,\tau}(\gamma *_b \eta) &\leqslant \|\gamma *_b \eta\|^{R,L}_{r_{i,\sigma},s_{j,\tau},P_{i,j,\sigma,\tau}} \leqslant \|\gamma\|^R_{r_{i,\sigma},p_{i,\sigma}}\|\eta\|^L_{s_{j,\tau},q_{j,\tau}}\lambda_G(K_i) \\ &= S_{i,\sigma}(\gamma)Q_{j,\tau}(\eta) \end{aligned}$$

for all $(\gamma, \eta) \in C^r_{K_i}(G, E_1) \times C^s_{K_j}(G, E_2)$ (using Lemma 6.7). As the hypotheses of Lemma 8.4 are satisfied, f (and thus β_b) admits product estimates. □

Proof of Theorem C. *Case* 1: *G is a finite group*. Then G is compact and hence β_b is always continuous [3, Corollary 2.3]. If β_b admits product estimates, then also b admits these (Lemma 8.1). If b admits product estimates, then β_b admits product estimates, by Lemma 8.3 (note that any (r, s, t) can be replaced with $(0, 0, 0)$ without changing the function spaces).

Case 2: *G is an infinite discrete group*. If β_b is continuous, then G is countable and b admits product estimates, by [3, Proposition 6.1]. If G is countable and b admits product estimates, then β_b admits product estimates, by Lemma 8.3 (note that any (r, s, t) can be replaced with $(0, 0, 0)$ without changing the function spaces). If β_b admits product estimates, then β_b is continuous, as observed in the introduction.

Case 3: *G is an infinite compact group* (and hence not discrete). Then β_b is always continuous, by [3, Corollary 2.3]. If β_b admits product estimates, then also b admits these (by Lemma 8.1), and if $t = \infty$, then also $r = s = \infty$ (see Lemma 8.2). Thus (b) and (c) from Theorem A are satisfied. If, conversely, (b) and (c) are satisfied, then β_b admits product estimates, by Lemma 8.3.

Case 4: *G is neither compact nor discrete.* If β_b admits product estimates, then β_b is continuous and hence (a)–(c) hold by Theorem A. If, conversely, (a)–(c) are satisfied, then β_b admits product estimates, by Lemma 8.3. □

9. Product estimates on spaces without norm

If we start with a continuous bilinear map $b : E_1 \times E_2 \to F$ on a product of normed spaces, then it satisfies product estimates (by Proposition 4.2), and can be fed into Theorem C, to obtain bilinear maps β on function spaces that admit product estimates. Since E_1 and E_2 are normed, also the function spaces admit a continuous norm. However, the existence of a continuous norm on the domain $E_1 \times E_2$ is not necessary for the existence of product estimates, as the trivial example $\beta : E_1 \times E_2 \to \mathbb{R}$, $\beta(x, y) := 0$ shows. The situation does not change if one assumes that β is non-degenerate in the sense that $\beta(x, .) \neq 0$ and $\beta(., y) \neq 0$ for all $0 \neq x \in E_1$ and $0 \neq y \in E_2$, as illustrated by the following example.

Example 9.1. Let M be an uncountable set and $E := \mathbb{R}^{(M)}$ be the set of all functions $\gamma : M \to \mathbb{R}$ with finite support, equipped with the (unusual!) locally convex topology $\mathcal{O}$ which is initial with respect to the restriction maps

$$\rho_C : E \to \mathbb{R}^{(C)}, \quad \gamma \mapsto \gamma|_C$$

for all countable subsets $C \subseteq M$, where $\mathbb{R}^{(C)}$ is equipped with the finest locally convex topology (turning $\mathbb{R}^{(C)}$ into the locally convex direct sum $\bigoplus_{j\in C}\mathbb{R}$). Hence the seminorms

$$p_v : E \to [0,\infty[\,, \quad p_v(\gamma) := \max\{v(m)|\gamma(m)|:\ m \in M\}$$

define the locally convex topology on E, for v ranging through the set $\mathcal{V}$ of all functions $v : M \to [0,\infty[$ such that $\{m \in M:\ v(m) > 0\}$ is countable. Since none of these p_v is a norm, we conclude that E does not admit a continuous norm. Consider the map $\beta : E \times E \to E$, $(\gamma,\eta) \mapsto \gamma\eta$ taking γ and η to their pointwise product $\gamma\eta$, given by $(\gamma\eta)(m) := \gamma(m)\eta(m)$. Then β is bilinear and non-degenerate, as $\gamma\gamma \neq 0$ for each $\gamma \in E \setminus \{0\}$. The pointwise multiplication map $\beta_C : \mathbb{R}^{(C)} \times \mathbb{R}^{(C)} \to \mathbb{R}^{(C)}$ is bilinear and hence continuous (see Corollary 4.3), for each countable set $C \subseteq M$. Since $\rho_C \circ \beta = \beta_C \circ (\rho_C \times \rho_C)$ is continuous, also β is continuous. Because E has the cnp by [5, §1.4], we deduce with Proposition 4.1 that β admits product estimates.

We mention that E is complete, being the projective limit of the spaces $\mathbb{R}^{(C)}$.

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