Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 307 (2007) 2993 – 3005
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The structure of $K_{3,3}$-subdivision-free toroidal graphs☆

Andrei Gagarin, Gilbert Labelle, Pierre Leroux

Laboratoire de Combinatoire et d'Informatique Mathématique (LaCIM), Université du Québec à Montréal, Montréal, Qué., Canada H3C 3P8

Received 22 November 2004; received in revised form 26 December 2006; accepted 11 March 2007
Available online 21 April 2007

Abstract

We consider the class $\mathscr{T}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the torus. We show that any graph in $\mathscr{T}$ admits a unique decomposition as a basic toroidal graph (the *toroidal core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. The structure theorem provides a practical algorithm to recognize toroidal graphs with no $K_{3,3}$-subdivisions in linear time. Labelled toroidal cores are enumerated, using matching polynomials of cycle graphs. As a result, we enumerate labelled graphs in $\mathscr{T}$ having vertex degree at least two or three, according to their number of vertices and edges. We also show that the number m of edges of graphs in $\mathscr{T}$ satisfies the bound $m \leqslant 3n - 6$, for $n \geqslant 6$ vertices, $n \neq 8$.

Keywords: Toroidal graphs; $K_{3,3}$-subdivisions; 2-Pole networks; Planar networks; Edge substitution; Toroidal core; Structure theorem; Labelled enumeration

1. Introduction

We are interested in the structure of non-planar simple graphs that can be embedded on the torus. By Kuratowski's theorem [17], such a graph G must contain a subdivision of K_5 or $K_{3,3}$. Fig. 1 shows a graph embedded on the torus and containing subdivisions of both K_5 and $K_{3,3}$.

As a first step in the study of toroidal graphs, it is natural to restrict ourselves to the smaller class of graphs with no $K_{3,3}$-subdivisions. Since $K_{3,3}$ is 3-regular, the graphs with no $K_{3,3}$-subdivisions coincide with the graphs with no $K_{3,3}$-minors. These graphs will be referred to as $K_{3,3}$-*free graphs*. By Wagner's theorem [20], a graph G is planar if and only if it does not have a minor isomorphic to K_5 or $K_{3,3}$. Recently, Gagarin et al. [11,12] have extended both Kuratowski's and Wagner's theorems to toroidal $K_{3,3}$-free graphs by giving complete lists of (11) forbidden subdivisions and (four) forbidden minors.

In this paper we give a structural characterization of non-planar 2-connected toroidal $K_{3,3}$-free graphs, using the substitution of strongly planar networks for the edges of certain basic graphs called *toroidal cores*, following an analogous work for projective-planar graphs [9]. We denote by $\mathscr{T}$ the class of non-planar 2-connected toroidal $K_{3,3}$-free graphs. The restriction to 2-connected graphs is natural since a graph G is toroidal if and only if it contains at

☆ With the partial support of NSERC (Canada).

E-mail addresses: gagarin@math.uqam.ca (A. Gagarin), labelle.gilbert@uqam.ca (G. Labelle), leroux.pierre@uqam.ca, leroux@lacim.uqam.ca (P. Leroux).

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.03.083

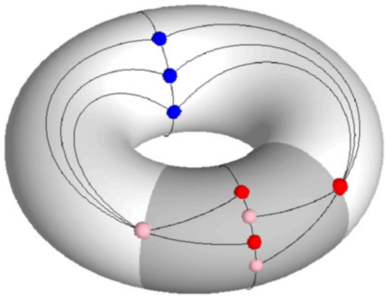

Fig. 1. Graph embedded on the torus.

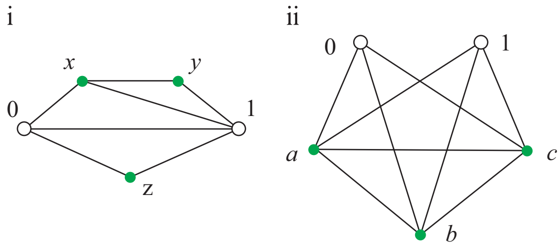

Fig. 2. (i) a series-parallel network, (ii) a $K_5\backslash e$-network.

most one non-planar toroidal 2-connected component, while all the other 2-connected components of G are planar. Our structure theorem is formally stated in Section 2 as the equation

$$\mathscr{T} = \mathscr{T}_{\mathrm{C}} \uparrow \mathscr{N}_{\mathrm{P}}, \tag{1}$$

where $\mathscr{T}_{\mathrm{C}}$ denotes the class of toroidal cores, $\mathscr{N}_{\mathrm{P}}$, that of strongly planar networks and the uparrow $\uparrow$ denotes the operation of substitution of networks for edges (see Definition 1 below). The proof given in Section 3 is based on a refinement of the algorithmic results of Gagarin and Kocay [8]. We also improve known bounds for the number of edges of 2-connected toroidal $K_{3,3}$-free graphs in Section 3. The structure theorem provides a practical algorithm to recognize toroidal graphs with no $K_{3,3}$-subdivisions in linear time.

Labelled toroidal cores are enumerated, using matching polynomials of cycle graphs, in Section 4, and also strongly planar networks. As a result, we enumerate labelled graphs in $\mathscr{T}$ as well as in the subclass of homeomorphically irreducible graphs in $\mathscr{T}$, according to their number of vertices and edges. Computations were carried out using Maple and results are presented in Tables 1–3. Unlabelled graphs in $\mathscr{T}$ have also been enumerated recently in [10] using the structural relationship (1).

2. The structure theorem

We use basic graph-theoretic terminology from Bondy and Murty [5] and Diestel [6]. By convention, the graph K_2 is considered as a 2-connected (non-separable) graph in this paper. A *two-pole network* (or more simply, a *network*) is a connected graph N with two distinguished vertices 0 and 1, such that the graph $N \cup 01$ is 2-connected, where the notation $N \cup ab$ is used for the graph obtained from N by adding the edge ab if it is not already there. The vertices 0 and 1 are called the *poles* of N, and all the other vertices of N are called *internal* vertices. Examples of networks are given by series-parallel networks (see [9] for a formal definition) and by the $K_5\backslash e$-network as illustrated in Fig. 2.

A network N is *strongly planar* if the graph $N \cup 01$ is planar. This means that N can be embedded on the sphere so that the two poles belong to the same face. We denote by $\mathscr{N}_{\mathrm{P}}$ the class of strongly planar networks. Note that the trivial network with no edge should be excluded from $\mathscr{N}_{P}$.

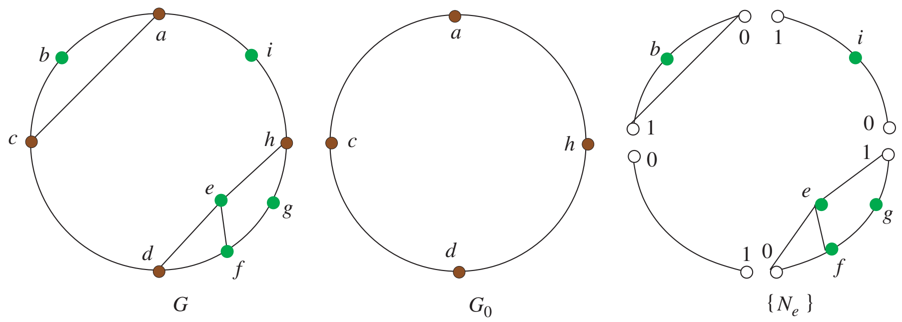

Fig. 3. Example of a $(C_4 \uparrow \mathcal{N})$-structure (G, G_0).

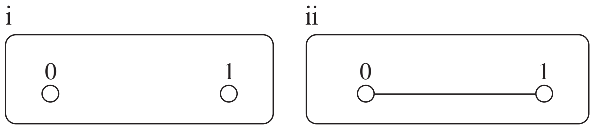

Fig. 4. The trivial networks: (i) $\mathbb{1}$, (ii) $y\mathbb{1}$.

We define an operator τ acting on 2-pole networks, $N \mapsto \tau \cdot N$, which interchanges the poles 0 and 1. A class $\mathcal{N}$ of networks is called *symmetric* if $N \in \mathcal{N} \Longrightarrow \tau \cdot N \in \mathcal{N}$. Examples of symmetric classes of networks are given by the classes $\mathcal{N}_{\mathrm{P}}$, of strongly planar networks, and $\mathcal{R}$, of series-parallel networks (see [9,21]).

Definition 1. Let $\mathcal{G}$ be a class of graphs and $\mathcal{N}$ be a symmetric class of networks. We denote by $\mathcal{G} \uparrow \mathcal{N}$ the class of pairs of graphs (G, G_0), such that

1. the graph G_0 is in $\mathcal{G}$ (called the *core*),
2. the vertex set $V(G_0)$ is a subset of $V(G)$,
3. there exists a family $\{N_e : e \in E(G_0)\}$ of networks in $\mathcal{N}$ (called the *components*) such that the graph G can be obtained from G_0 by substituting N_e for each edge $e \in E(G_0)$, identifying the poles of N_e with the extremities of e according to some orientation.

Such pairs (G, G_0) are called $(\mathcal{G} \uparrow \mathcal{N})$-*structures*.

Notice that the components $\{N_e : e \in E(G_0)\}$ are uniquely determined up to orientation (i.e. pole interchange) by the pair (G, G_0). An example of a $(\mathcal{G} \uparrow \mathcal{N})$-structure (G, G_0), with $\mathcal{G} = C_4$, the class of 4-cycle graphs, and $\mathcal{N} = \{$all networks$\}$, is given in Fig. 3.

As another example, take $\mathcal{G} = K_2$ and $\mathcal{N} = \mathcal{N}_{\mathrm{P}}$, the class of strongly planar networks, then the $(K_2 \uparrow \mathcal{N}_{\mathrm{P}})$-structures consist of graphs G together with two selected (adjacent or not) vertices a and b, such that the graph $G \cup ab$ is 2-connected and planar.

Definition 2. We say that the composition $\mathcal{G} \uparrow \mathcal{N}$ is *canonical* if for any structure $(G, G_0) \in \mathcal{G} \uparrow \mathcal{N}$, the core $G_0 \in \mathcal{G}$ is uniquely determined (and hence also the components) by the graph G.

In the case of a canonical composition, we can identify $\mathcal{G} \uparrow \mathcal{N}$ with the class of resulting graphs G. For example, we can take $\mathcal{G} = K$, the class of complete graphs, $\mathcal{N} = \mathbb{1} + y\mathbb{1}$, the class of trivial networks (see Fig. 4), where the operation "+" denotes the disjoint union. Let $\mathcal{G}_{\mathrm{a}}$ denote the class of all graphs. Then we have

$$K \uparrow (\mathbb{1} + y\mathbb{1}) = \mathcal{G}_{\mathrm{a}},$$

the composition being canonical.

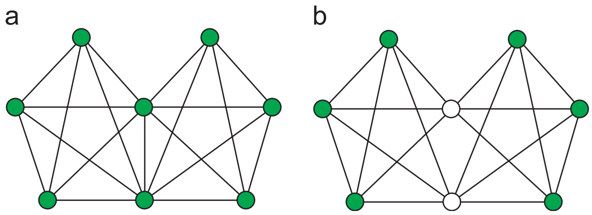

Fig. 5. (a) M-graph, (b) M^*-graph.

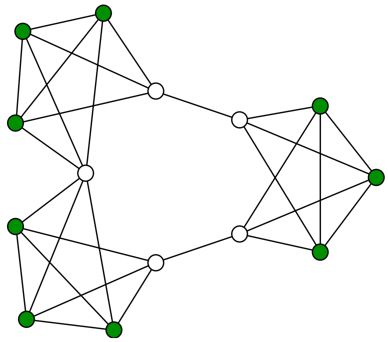

Fig. 6. A toroidal crown obtained from C_5.

In a previous work, we have proved the following structure theorem for 2-connected non-planar projective-planar $K_{3,3}$-free graphs.

Theorem 1 (*Gagarin et al.* [9]). *The class $\mathscr{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs can be expressed as a canonical composition*

$$\mathscr{F} = K_5 \uparrow \mathscr{N}_{\mathrm{P}}, \tag{2}$$

where K_5 denotes the class of complete graphs with five vertices.

Our goal is to give a similar structural result for the class $\mathscr{T}$. In order to do this, more fundamental core graphs have to be considered.

Definition 3. Given two K_5-graphs, the graph obtained by identifying an edge of one of the K_5's with an edge of the other is called an *M-graph* (see Fig. 5a), and, when the edge of identification is deleted, an M^*-*graph* (see Fig. 5b).

Definition 4. A *toroidal crown* is a graph H obtained from a cycle C_i, $i \geqslant 3$, by substituting $K_5\backslash e$-networks for some edges of C_i in such a way that no pair of unsubstituted edges of C_i are adjacent in H (see Fig. 6). We denote by $\mathscr{H}$ the class of toroidal crowns.

Definition 5. A *toroidal core* is a graph H which is isomorphic to either K_5, an M-graph, an M^*-graph, or a toroidal crown. We denote by $\mathscr{T}_{\mathrm{C}}$ the class of toroidal cores. In other words, we have

$$\mathscr{T}_{\mathrm{C}} = K_5 + M + M^* + \mathscr{H}. \tag{3}$$

Note that the graphs K_5 and M^* can be viewed as special cases of toroidal crowns, based on a degenerate cycle C_2. The main result of this paper is the following structure theorem.

Theorem 2. *The class $\mathscr{T}$ of 2-connected non-planar $K_{3,3}$-free toroidal graphs is characterized by the relation*

$$\mathscr{T} = \mathscr{T}_{\mathrm{C}} \uparrow \mathscr{N}_{\mathrm{P}}, \tag{4}$$

the composition being canonical.

The proof of Theorem 2 is given in Section 3. It is clear that if G is a 2-connected non-planar $K_{3,3}$-free graph then so is any graph obtained from G by replacing its edges by strongly planar networks. The theorem states first that any toroidal core is a 2-connected non-planar $K_{3,3}$-free graph, so that any graph G arising from a $(\mathscr{T}_{\mathrm{C}} \uparrow \mathscr{N}_{\mathrm{P}})$-structure (G, G_0) is in $\mathscr{T}$, and that moreover, this is the only possibility and the decomposition of a graph in $\mathscr{T}$ as a $(\mathscr{T}_{\mathrm{C}} \uparrow \mathscr{N}_{\mathrm{P}})$-structure is unique.

3. Proof of the structure theorem

We first give an overview of the structural results for toroidal graphs described in [8]. Following Diestel [6], a K_5-subdivision is denoted by TK_5. The vertices of degree 4 in TK_5 are the *corners* and the vertices of degree 2 are the *inner vertices* of TK_5. For a pair of corners a and b, the path P_{ab} between a and b with all other vertices being inner vertices is called a *side* of the K_5-subdivision.

Let G be a non-planar graph containing a fixed K_5-subdivision TK_5. A path p in G with one endpoint an inner vertex of TK_5, the other endpoint on a different side of TK_5, and all other vertices and edges in $G\backslash TK_5$ is called a *short cut* of the K_5-subdivision. A vertex $u \in G\backslash TK_5$ is called a 3-*corner vertex* with respect to TK_5 if $G\backslash TK_5$ contains internally disjoint paths connecting u with at least three corners of the K_5-subdivision.

Proposition 1 (*Asano [1], Fellows and Kaschube [7], Gagarin and Kocay [8]*). *Let G be a non-planar graph with a K_5-subdivision TK_5 for which there is either a short cut or a* 3*-corner vertex. Then G contains a $K_{3,3}$-subdivision.*

Proposition 2 (*Fellows and Kaschube [7], Gagarin and Kocay [8]*). *Let G be a* 2*-connected graph with a TK_5 having no short cut or* 3*-corner vertex. Let K denote the set of corners of TK_5. Then any connected component C of $G\backslash K$ contains inner vertices of at most one side of TK_5, and C is connected in G to exactly two corners of TK_5.*

Given a graph G satisfying the hypothesis of Proposition 2, a *side component* of TK_5 is defined as the subgraph of G induced by a pair of corners a and b in K and the connected components of $G\backslash K$ which are connected to both a and b in G.

Corollary 1 (*Fellows and Kaschube [7], Gagarin and Kocay [8]*). *For a* 2*-connected graph G with a TK_5 having no short cut or* 3*-corner vertex, two side components of TK_5 in G have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of TK_5.*

Thus we see that a graph G satisfying the hypothesis of Proposition 2 can be decomposed into side components corresponding to the sides of TK_5. Each side component S contains exactly two corners a and b corresponding to a side of TK_5. The graph $S \cup ab$ obtained by adding the edge ab to S if it is not already there is called an *augmented side component* of TK_5 in G. Side components of a subdivision of an M-graph are defined in a similar manner.

A side component S of TK_5 in G with two corners a and b is called *cylindrical* if the edge $ab \notin S$, S is planar but the augmented side component $S \cup ab$ is non-planar. It is easy to see that a cylindrical side component S is embeddable in a cylindrical section of the torus, with the vertices a and b on opposite sides. Indeed, given an embedding of S on the sphere, two distinct faces, one containing the vertex a and the other, the vertex b, can be chosen and declared to be the two external sides of a cylindrical embedding of S. Fig. 1 shows an embedding of K_5 on the torus, where one edge has been replaced by a cylindrical side component $S = K_{3,3}\backslash e$ (the shaded area).

Now, if a graph G is $K_{3,3}$-free, then Proposition 2 and its corollary can be applied, in virtue of Proposition 1. In this case, we have the following result:

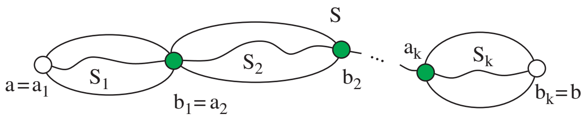

Fig. 7. Block-cutpoint decomposition for the cylindrical side component *S*.

Proposition 3 (*Gagarin and Kocay* [8]). *A* 2-*connected* $K_{3,3}$-*free graph G containing a* K_5-*subdivision* TK_5 *is toroidal if and only if*:

(i) *all the augmented side components of* TK_5 *in G are planar graphs*, *or*

(ii) *nine augmented side components of* TK_5 *in G are planar*, *and the remaining side component S is cylindrical*, *or*

(iii) *G contains a subdivision TM of an M-graph*, *and all the augmented side components of TM in G are planar.*

Further analysis of the cylindrical side component *S* of Proposition 3(ii) will provide a proof of Theorem 2. A side component *S* having two corners *a* and *b* can be considered as a network, with poles *a* and *b* instead of 0 and 1. We use the notation $\mathrm{Int}(S)$ to denote the *interior* of *S*, that is the subgraph generated by the internal vertices of the network.

By analogy with side components, a network *N* is called *cylindrical* if $01 \notin N$, *N* is a planar graph, but $N \cup 01$ is non-planar. Recall that a network *N* is called strongly planar if $N \cup 01$ is planar. Thus a planar network is either strongly planar or cylindrical.

A *block* is a maximal 2-connected subgraph of a graph. The *block-cutpoint* tree $\mathrm{bc}(G)$ of a connected graph *G* is the graph whose set of vertices is the union of the set of blocks and the set of cutpoints of *G*, with two vertices adjacent if one corresponds to a block of *G* and the other to a cutpoint of *G* in that block. See for example [3,6].

Proposition 4. *Let G be a* 2-*connected non-planar toroidal* $K_{3,3}$-*free graph satisfying Proposition* 3(ii), *with the cylindrical side component S having corners a and b. Then the block-cutpoint tree* $\mathrm{bc}(S)$ *forms a path*

$$(a = a_1)S_1(b_1 = a_2)S_2(b_2 = a_3)\ldots S_k(b_k = b),$$

as in a series composition of networks, *with* $k \geqslant 1$ (*see* Fig. 7). *Moreover*, *the networks* S_i, *with poles* a_i, b_i, *for* $i = 1, \ldots, k$, *are either cylindrical or strongly planar*, *with at least one cylindrical*, *and the cylindrical networks are formed of a* $K_5\backslash e$-*network where the edges have been replaced by strongly planar networks*.

Proof. Since *G* is 2-connected, each cut-vertex of *S* belongs to exactly two blocks and lies on the corresponding side P_{ab} of TK_5. Therefore the blocks of *S* form a path as in Fig. 7. Suppose that each network S_i is strongly planar. Then, clearly, $S \cup ab$ is planar as well. Hence the fact that *S* is cylindrical implies that at least one of the networks S_i is itself a cylindrical network.

Suppose that the network S_j is cylindrical. Then, by Kuratowski's theorem, adding the (new) edge a_jb_j to S_j, the graph $S_j \cup a_jb_j$ contains a K_5-subdivision TK_5'. Clearly, the edge a_jb_j is a side of this TK_5'. Now replace the edge a_jb_j by a more complex side component: $G\backslash\mathrm{Int}(S_j)$, thus transforming TK_5' into a new K_5-subdivision in *G*, denoted by TK_5'' to avoid confusion.

Since *G* is toroidal and the side component $G\backslash\mathrm{Int}(S_j)$ of TK_5'' is cylindrical, all the other side components of TK_5'' in *G* must be strongly planar networks by Proposition 3(ii). But these are precisely the side components of TK_5', except one provided by a_jb_j. This concludes the proof. □

Proposition 5. *Any graph G in the class* $\mathscr{T}_{\mathrm{C}}$ *of toroidal cores is non-planar*, 2-*connected*, $K_{3,3}$-*free*, *and toroidal*.

Proof. Recall that $\mathscr{T}_{\mathrm{C}} = K_5 + M + M^* + \mathscr{H}$, where $\mathscr{H}$ denotes the class of toroidal crowns. Certainly, all those graphs are non-planar, 2-connected, and $K_{3,3}$-free. Fig. 1 shows that K_5 is toroidal. It also shows how to replace an edge of K_5 in this embedding by a cylindrical network, in that case a $K_{3,3}\backslash e$-network. It is clear that any other cylindrical network, for example a $K_5\backslash e$-network can be embedded in this way on the torus, and in fact that several cylindrical networks can also be embedded in series. Thus the graph M^* and the toroidal crowns are toroidal. There remains to see that the graph *M* is toroidal. This fact is illustrated in Fig. 8, where the torus is represented as a rectangle with opposite sides identified. □

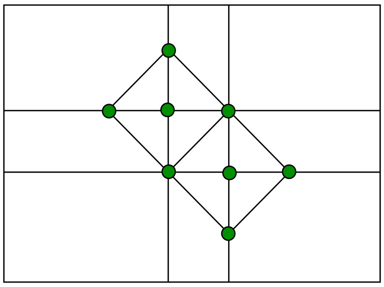

Fig. 8. Embedding of the graph M on the torus.

Proof of Theorem 2. Since all graphs in $\mathscr{T}_{\mathrm{C}}$ are non-planar, 2-connected, $K_{3,3}$-free, and toroidal, by Proposition 5, it is clear that any graph G arising from a $\mathscr{T}_C \uparrow \mathscr{N}_{\mathrm{P}}$-structure (G, G_0) is in $\mathscr{T}$.

Conversely, we have to show that any graph G in $\mathscr{T}$ admits a representation as a core graph G_0 in $\mathscr{T}_{\mathrm{C}}$, where the edges are replaced by strongly planar networks and that moreover the core G_0 is uniquely determined by G. So let G be a 2-connected non-planar $K_{3,3}$-free toroidal graph. By Kuratowski's theorem, G contains a K_5-subdivision TK_5 and Proposition 3 can be applied. Clearly, the sets of graphs corresponding to the cases (i), (ii) and (iii) of this proposition are mutually disjoint.

In case (i), the core is K_5 itself and the side components are strongly planar networks. The unicity of the core follows from the fact that the corners of the TK_5 are uniquely determined, by Corollary 1. Case (iii) is similar: the core is an M-graph and it is easily seen that the set of corners of the M-graph-subdivision is uniquely defined as in Corollary 1.

In case (ii), there is a unique cylindrical side component S of TK_5 in G. Notice that $G\backslash\mathrm{Int}(S)$ itself is a cylindrical network of the form $K_5\backslash e \uparrow \mathscr{N}_{\mathrm{P}}$. Applying Proposition 4, the block-cutpoint decomposition of S forms a path (series composition) of networks $S_1, S_2, \ldots, S_k$, $k \geqslant 1$, as in Fig. 7, and at least one of the networks S_i is cylindrical. In this path we can regroup maximal series of consecutive strongly planar networks into single strongly planar networks so that at most one strongly planar network N' is separating two cylindrical networks in the resulting path, and the poles of the strongly planar network N' are uniquely defined by maximality. By Proposition 4, the cylindrical networks in the path are of the form $K_5\backslash e \uparrow \mathscr{N}_{\mathrm{P}}$ and the corners are uniquely defined with respect to the corresponding K_5-subdivision TK_5'' in G. Therefore, globally taken, the unique set of corners of G completely defines a toroidal core G_0, where G_0 is either a M^*-graph or a toroidal crown so that $G \in M^* \uparrow \mathscr{N}_{\mathrm{P}}$ or $G \in \mathscr{H} \uparrow \mathscr{N}_{\mathrm{P}}$. $\square$

Euler's formula for connected graphs on an orientable surface of genus g (see, for example [19]), $n+f-m=2-2g$, where f is the number of faces, implies that a connected planar graph with $n \geqslant 3$ vertices can have at most $3n-6$ edges. Let us state this for 2-connected planar graphs with n vertices and m edges as follows:

$$m \leqslant \begin{cases} 3n-5 & \text{if } n=2, \\ 3n-6 & \text{if } n \geqslant 3. \end{cases} \tag{5}$$

In fact, if $n=2$, $m=3n-5=1$.

Euler's formula also implies that a connected toroidal graph G with n vertices can have at most $3n$ edges. However, an arbitrary $K_{3,3}$-free graph G is known to have at most $3n-5$ edges (see [1]). The following proposition shows that toroidal graphs with no $K_{3,3}$-subdivisions satisfy a stronger relation, which is analogous to planar graphs. An analogous result for projective-planar graphs can be found in [9]. Also note that Corollary 8.3.5 of [6] implies that graphs with no K_5-minors can have at most $3n-6$ edges.

Proposition 6. *The number m of edges of a non-planar $K_{3,3}$-free toroidal n-vertex graph G satisfies $m \leqslant 3n-5$ if $n=5$ or 8, and*

$$m \leqslant 3n-6 \quad \text{if } n \geqslant 6 \text{ and } n \neq 8. \tag{6}$$

Proof. It is sufficient to prove the result when G is 2-connected, so that G is in $\mathscr{T}$ and Proposition 3 can be applied. If G belongs to the cases (i) or (ii), each side component S_i of TK_5 in G, $i=1,2,\ldots,10$, satisfies condition (5) with

$n = n_i$, the number of vertices, and $m = m_i$, the number of edges of S_i, $i = 1, 2, \ldots, 10$. Since each corner of TK_5 is in precisely 4 side components, we have $\sum_{i=1}^{10} n_i = n + 15$ and we obtain, by summing these 10 inequalities,

$$m = \sum_{i=1}^{10} m_i \leqslant \begin{cases} 3\sum_{i=1}^{10} n_i - 50 = 3(n+15) - 50 = 3n - 5 & \text{if } n = 5, \\ 3\sum_{i=1}^{10} n_i - 51 = 3(n+15) - 51 = 3n - 6 & \text{if } n \geqslant 6, \end{cases}$$

since $n = 5$ iff $n_i = 2$, $i = 1, 2, \ldots, 10$, and $n \geqslant 6$ if and only if at least one $n_j \geqslant 3$, $j = 1, 2, \ldots, 10$.

Similarly, in case (iii), each side component S_i of TM in G, $i = 1, 2, \ldots, 19$, satisfies the condition (5) with $n = n_i$, the number of vertices, and $m = m_i$, the number of edges of S_i, $i = 1, 2, \ldots, 19$. Since 2 vertices of TM are in precisely 7 side components, 6 vertices of TM are in precisely 4 side components, and all the other vertices of G are in a unique side component, we have $\sum_{i=1}^{19} n_i = n + 30$ and we obtain, by summing these 19 inequalities,

$$m = \sum_{i=1}^{19} m_i \leqslant \begin{cases} 3\sum_{i=1}^{19} n_i - 95 = 3(n+30) - 95 = 3n - 5 & \text{if } n = 8, \\ 3\sum_{i=1}^{19} n_i - 96 = 3(n+30) - 96 = 3n - 6 & \text{if } n \geqslant 9, \end{cases}$$

since $n = 8$ iff $n_i = 2$, $i = 1, 2, \ldots, 19$, and $n \geqslant 9$ if and only if at least one $n_j \geqslant 3$, $j = 1, 2, \ldots, 19$. □

4. Counting labelled $K_{3,3}$-subdivision-free toroidal graphs

Now let us consider the question of the labelled enumeration of toroidal graphs with no $K_{3,3}$-subdivisions according to the numbers of vertices and edges. First, we review some basic notions and terminology of labelled enumeration together with the counting methods and technique used in [9,21]. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). For example, see [3,14,18,22].

By a *labelled* graph, we mean a simple graph $G = (V, E)$ where the set of vertices $V = V(G)$ is itself the set of labels and the labelling function is the identity function. V is called the *underlying set* of G. An edge e of G then consists of an unordered pair $e = uv$ of elements of V and $E = E(G)$ denotes the set of edges of G. If W is another set and $\sigma : V \tilde{\to} W$ is a bijection, then any graph $G = (V, E)$ on V, can be transformed into a graph $G' = \sigma(G) = (W, \sigma(E))$, where $\sigma(E) = \{\sigma(e) = \sigma(u)\sigma(v) \mid e \in E\}$. We say that G' is obtained from G by *vertex relabelling* and that σ is a graph *isomorphism* $G \tilde{\to} G'$. An *unlabelled graph* is then seen as an isomorphism class γ of labelled graphs. We write $\gamma = \gamma(G)$ if γ is the isomorphism class of G. By the *number of ways to label* an unlabelled graph $\gamma(G)$, where $G = (V, E)$, we mean the number of distinct graphs G' on the underlying set V which are isomorphic to G. Recall that this number is given by $n!/|\mathrm{Aut}(G)|$, where $n = |V|$ and $\mathrm{Aut}(G)$ denotes the automorphism group of G.

A *species* of graphs is a class of labelled graphs which is closed under vertex relabellings. Thus any class $\mathscr{G}$ of unlabelled graphs gives rise to a species, also denoted by $\mathscr{G}$, by taking the set union of the isomorphism classes in $\mathscr{G}$. For any species $\mathscr{G}$ of graphs, we introduce its *mixed* (*exponential*) *generating function* $\mathscr{G}(x, y)$ as the formal power series

$$\mathscr{G}(x, y) = \sum_{n \geqslant 0} g_n(y) \frac{x^n}{n!} \quad \text{with } g_n(y) = \sum_{m \geqslant 0} g_{n,m} y^m, \tag{7}$$

where $g_{n,m}$ is the number of graphs in $\mathscr{G}$ with m edges over a given set of vertices V_n of size n. Here y is a formal variable which acts as an edge counter. For example, for the species $\mathscr{G} = K = \{K_n\}_{n \geqslant 0}$ of complete graphs, we have

$$K(x, y) = \sum_{n \geqslant 0} y^{\binom{n}{2}} x^n / n!, \tag{8}$$

while for the species $\mathscr{G} = \mathscr{G}_a$ of all simple graphs, we have

$$\mathscr{G}_a(x, y) = K(x, 1 + y). \tag{9}$$

A species of graphs is *molecular* if it contains only one isomorphism class. For a molecular species $\gamma = \gamma(G)$, where G has n vertices and m edges, we have $\gamma(x, y) = y^m n!/(|\mathrm{Aut}(G)|) x^n / n! = y^m x^n / |\mathrm{Aut}(G)|$. For example,

$$K_5(x, y) = \frac{x^5 y^{10}}{5!}. \tag{10}$$

Also, for the graphs M and M^* described in Section 2, we have

$$M(x, y) = 280\frac{x^8y^{19}}{8!}, \quad M^*(x, y) = 280\frac{x^8y^{18}}{8!}, \tag{11}$$

since $|\mathrm{Aut}(M)| = |\mathrm{Aut}(M^*)| = 144$.

For the enumeration of networks, we consider that the poles 0 and 1 are not labelled, or, in other words, that only the internal vertices form the underlying set. Hence the mixed generating function of a class (or species) $\mathscr{N}$ of networks is defined by

$$\mathscr{N}(x, y) = \sum_{n\geqslant 0} v_n(y)\frac{x^n}{n!} \quad \text{with } v_n(y) = \sum_{m\geqslant 0} v_{n,m}y^m, \tag{12}$$

where $v_{n,m}$ is the number of networks in $\mathscr{N}$ with m edges over a given set of size n, V_n, of internal vertices. For example, we have

$$(K_5\backslash e)(x, y) = \frac{x^3y^9}{3!}. \tag{13}$$

Lemma 1 (*Gagarin et al. [9], Walsh [21]*). *Let $\mathscr{G}$ be a species of graphs and $\mathscr{N}$ be a symmetric species of networks. Then the following generating function identity holds*:

$$(\mathscr{G} \uparrow \mathscr{N})(x, y) = \mathscr{G}(x, \mathscr{N}(x, y)). \tag{14}$$

As a simple example, Eq. (9) reflects the fact that $\mathscr{G}_a = K \uparrow (\mathbb{1} + y\mathbb{1})$. We can now concentrate on the labelled enumeration of $K_{3,3}$-free toroidal graphs. By Theorem 2, we have the following corollary.

Proposition 7. *The mixed generating function $\mathscr{T}(x, y)$ of labelled 2-connected non-planar $K_{3,3}$-free toroidal graphs is given by*

$$\mathscr{T}(x, y) = (\mathscr{T}_{\mathrm{C}} \uparrow \mathscr{N}_{\mathrm{P}})(x, y) = \mathscr{T}_{\mathrm{C}}(x, \mathscr{N}_{\mathrm{P}}(x, y)), \tag{15}$$

where $\mathscr{T}_{\mathrm{C}}$ denotes the species of toroidal cores (*see Definition* 5).

Let P denote the species of 2-connected planar graphs. Then the mixed generating function of $\mathscr{N}_{\mathrm{P}}$, the associated class of strongly planar networks, is given by

$$\mathscr{N}_{\mathrm{P}}(x, y) = (1 + y)\frac{2}{x^2}\frac{\partial}{\partial y}P(x, y) - 1 \tag{16}$$

(see [9,21]). Methods for computing the generating function $P(x, y)$ of labelled 2-connected planar graphs are described in [2,4]. Formula (16) can then be used to compute $\mathscr{N}_{\mathrm{P}}(x, y)$.

Recall that $\mathscr{T}_{\mathrm{C}} = K_5 + M + M^* + \mathscr{H}$, where $\mathscr{H}$ denotes the class of toroidal crowns. There remains only to compute the mixed generating function $\mathscr{H}(x, y)$ for toroidal crowns. This will be done using matching polynomials. Recall that a *matching* μ of a finite graph G is a set of disjoint edges of G. We define the *matching polynomial* of G as

$$M_G(y) = \sum_{\mu\in\mathscr{M}(G)} y^{|\mu|}, \tag{17}$$

where $\mathscr{M}(G)$ denotes the set of matchings of G. In particular, the matching polynomials $U_n(y)$ and $T_n(y)$ for paths and cycles of size n are well known (see [13]). They are closely related to the Chebyshev polynomials. To be precise, let P_n denote the path graph (V, E) with $V = [n] = \{1, 2, \ldots, n\}$ and $E = \{\{i, i + 1\}|\ i = 1, 2, \ldots, n - 1\}$ and C_n denote the cycle graph with $V = [n]$ and $E = \{\{i, i + 1 \pmod n\}|\ i = 1, 2, \ldots, n\}$. Then we have

$$U_n(y) = \sum_{\mu\in\mathscr{M}(P_n)} y^{|\mu|}, \quad T_n(y) = \sum_{\mu\in\mathscr{M}(C_n)} y^{|\mu|}. \tag{18}$$

Table 1
The number t_n (resp. i_n) of labelled non-planar 2-connected $K_{3,3}$-free toroidal (resp. and irreducible) graphs having n vertices

n	t_n	i_n
5	1	1
6	120	0
7	10 920	420
8	989 520	37 520
9	99 897 840	3 656 520
10	11 940 037 200	454 406 400
11	1 737 017 325 120	67 651 907 400
12	307 410 206 405 280	11 713 973 686 800
13	64 915 089 945 797 520	2 309 360 318 565 300
14	15 941 442 348 672 800 960	509 886 615 053 415 600
15	4 446 392 119 411 980 978 240	124 470 953 623 133 617 500
16	1 382 470 831 306 742 435 905 920	33 253 861 507 512 510 664 800
17	472 436 578 501 629 382 684 767 360	9 642 802 738 009 988 846 098 800
18	175 569 440 215 502 279 529 214 410 240	3 014 293 919 820 242 935 601 325 600
19	70 373 115 034 109 453 975 811 430 602 240	1 009 949 253 303 428 292 707 750 898 000
20	30 226 304 060 184 007 557 277 939 796 259 840	360 931 928 359 726 264 215 290 579 964 000

The dichotomy caused by the membership of the edge $\{n-1, n\}$ in the matchings of the path P_n leads to the recurrence relation

$$U_n(y) = yU_{n-2}(y) + U_{n-1}(y), \tag{19}$$

for $n \geqslant 2$, with $U_0(y) = U_1(y) = 1$. It follows that the ordinary generating function of the matching polynomials $U_n(y)$ is rational. In fact, it is easily seen that

$$\sum_{n \geqslant 0} U_n(y)x^n = \frac{1}{1 - x - yx^2}. \tag{20}$$

Now, the dichotomy caused by the membership of the edge $\{1, n\}$ in the matchings of the cycle C_n leads to the relation

$$T_n(y) = yU_{n-2}(y) + U_n(y), \tag{21}$$

for $n \geqslant 3$. It is then a simple matter, using (20) and (21) to compute their ordinary generating function, denoted by $G(x, y)$. We find

$$G(x, y) = \sum_{n \geqslant 3} T_n(y)x^n = \frac{x^3(1 + 3y + yx + 2y^2x)}{1 - x - yx^2}. \tag{22}$$

In fact, we also need to consider the *homogeneous matching polynomials*

$$T_n(y, z) = z^n T_n\left(\frac{y}{z}\right) = \sum_{\mu \in \mathscr{M}(C_n)} y^{|\mu|} z^{n-|\mu|}, \tag{23}$$

where the variable z marks the edges which are not selected in the matchings, whose generating function $G(x, y, z) = \sum_{n \geqslant 3} T_n(y, z)x^n$ is given by

$$G(x, y, z) = G\left(xz, \frac{y}{z}\right) = \frac{x^3z^2(z + 3y + xyz + 2xy^2)}{1 - xz - x^2yz}. \tag{24}$$

We now introduce the species C^{m} of pairs (c, μ), where c is a cycle graph of length $n \geqslant 3$ and μ is a matching of c, with weight $y^{|\mu|}z^{n-|\mu|}$. Since there are $(n-1)!/2$ non-oriented cycles on a set of size $n \geqslant 3$, and all these cycles admit

Table 2
The number $t_{n,m}$ of labelled non-planar 2-connected $K_{3,3}$-free toroidal graphs having n vertices and m edges

n	m	$t_{n,m}$	n	m	$t_{n,m}$	n	m	$t_{n,m}$
5	10	1	11	16	1 664 863 200	13	18	1 261 490 630 400
6	11	60	11	17	17 556 739 200	13	19	21 330 659 750 400
6	12	60	11	18	78 956 539 200	13	20	159 781 461 840 000
7	12	2310	11	19	202 084 621 200	13	21	713 882 464 495 200
7	13	5250	11	20	334 016 949 420	13	22	2 168 012 582 255 520
7	14	3150	11	21	387 916 512 060	13	23	4 843 734 946 530 480
7	15	210	11	22	336 903 576 240	13	24	8 380 128 998 022 210
8	13	73 920	11	23	223 779 124 800	13	25	11 537 956 984 129 290
8	14	283 920	11	24	109 666 533 900	13	26	12 710 849 422 805 820
8	15	380 240	11	25	36 500 148 300	13	27	11 091 197 779 962 300
8	16	205 520	11	26	7 300 200 600	13	28	7 523 040 609 294 210
8	17	40 320	11	27	671 517 000	13	29	3 868 223 230 962 090
8	18	5320	12	17	45 664 819 200	13	30	1 454 441 069 881 800
8	19	280	12	18	617 512 896 000	13	31	376 789 239 426 600
9	14	2 162 160	12	19	3 642 195 110 400	13	32	60 029 345 376 000
9	15	12 383 280	12	20	12 576 897 194 400	13	33	4 429 660 435 200
9	16	27 592 740	12	21	28 943 910 959 040	14	19	35 321 737 651 200
9	17	30 616 740	12	22	48 151 723 490 640	14	20	732 123 289 497 600
9	18	18 419 940	12	23	61 179 019 743 600	14	21	6 797 952 466 905 600
9	19	6 706 980	12	24	60 949 737 367 200	14	22	38 137 563 765 100 800
9	20	1 771 560	12	25	47 362 199 346 000	14	23	147 357 768 378 300 480
9	21	244 440	12	26	27 882 539 962 200	14	24	423 704 585 721 296 880
10	15	60 540 480	12	27	11 911 924 840 200	14	25	952 194 383 913 853 080
10	16	481 572 000	12	28	3 475 786 545 000	14	26	1 719 165 782 299 705 740
10	17	1 578 301 200	12	29	620 188 569 000	14	27	2 519 330 273 617 857 700
10	18	2 810 039 400	12	30	50 905 562 400	14	28	2 992 115 301 780 284 680
10	19	3 055 603 320				14	29	2 857 231 696 936 256 640
10	20	2 220 031 800				14	30	2 168 091 732 460 633 980
10	21	1 170 779 400				14	31	1 286 490 621 084 248 580
10	22	447 867 000				14	32	583 196 381 484 116 400
10	23	104 781 600				14	33	194 805 099 201 913 200
10	24	10 521 000				14	34	45 144 355 587 130 800
						14	35	6 478 057 100 314 800
						14	36	433 347 847 732 800

the same homogeneous matching polynomial $T_n(y,z)$, the generating function of labelled C^{m}-structures is

$$\begin{aligned} C^{\mathrm{m}}(x,y,z) &= \sum_{n\geqslant 3} \frac{(n-1)!}{2} T_n(y,z)\frac{x^n}{n!} \\ &= \frac{1}{2}\sum_{n\geqslant 3} T_n(y,z)\frac{x^n}{n} \\ &= \frac{1}{2}\int_0^x \frac{1}{t} G(t,y,z)\,\mathrm{d}t \\ &= -\frac{2xz+2x^2zy+x^2z^2+2\ln(1-xz-x^2yz)}{4}. \end{aligned} \tag{25}$$

Proposition 8. *The mixed generating series $\mathscr{H}(x,y)$ of toroidal crowns is given by*

$$\mathscr{H}(x,y) = -\frac{12x^4y^9+12x^5y^{10}+x^8y^{18}+72\ \ln(1-x^4y^9/6-x^5y^{10}/6)}{144}. \tag{26}$$

Table 3
The number $i_{n,m}$ of labelled non-planar toroidal 2-connected $K_{3,3}$-free graphs with no vertex of degree 2, having n vertices and m edges

n	m	$i_{n,m}$	n	m	$i_{n,m}$	n	m	$i_{n,m}$
5	10	1	12	21	2 025 777 600	15	25	7 205 830 632 000
7	14	210	12	22	44 347 564 800	15	26	923 081 887 728 000
7	15	210	12	23	321 609 657 600	15	27	21 992 072 494 392 000
8	15	3360	12	24	1 163 345 198 400	15	28	226 161 061 676 550 000
8	16	13 440	12	25	2 451 538 504 800	15	29	1 307 818 406 288 394 000
8	17	15 120	12	26	3 217 869 547 200	15	30	4 820 304 955 001 936 400
8	18	5320	12	27	2 679 196 027 200	15	31	12 127 842 733 266 760 500
8	19	280	12	28	1 380 518 785 800	15	32	21 656 100 829 701 838 500
9	16	15 120	12	29	402 617 061 000	15	33	28 004 986 553 485 441 500
9	17	257 040	12	30	50 905 562 400	15	34	26 380 390 080 138 944 850
9	18	948 780	13	22	5 772 967 200	15	35	17 958 176 171 082 897 750
9	19	1 377 180	13	23	462 940 077 600	15	36	8 617 081 622 936 787 000
9	20	861 840	13	24	7 019 020 008 000	15	37	2 767 688 443 795 275 000
9	21	196 560	13	25	45 947 694 592 800	15	38	534 515 180 727 528 000
10	18	2 116 800	13	26	167 149 913 931 000	15	39	46 965 224 818 512 000
10	19	23 511 600	13	27	378 523 016 071 200	16	27	5 038 469 339 904 000
10	20	85 453 200	13	28	563 775 314 152 050	16	28	277 876 008 393 984 000
10	21	145 681 200	13	29	564 008 061 667 050	16	29	5 018 911 980 001 920 000
10	22	129 124 800	13	30	376 553 969 391 600	16	30	45 918 223 239 784 896 000
10	23	57 997 800	13	31	161 308 779 231 600	16	31	254 992 509 548 208 432 000
10	24	10 521 000	13	32	40 176 176 040 000	16	32	945 457 303 873 642 560 000
11	19	6 652 800	13	33	4 429 660 435 200	16	33	2 474 573 372 205 558 624 000
11	20	301 039 200	14	24	2 746 116 172 800	16	34	4 727 240 139 887 673 408 000
11	21	2 559 249 000	14	25	100 222 343 020 800	16	35	6 716 649 016 178 905 003 200
11	22	9 235 749 600	14	26	1 207 927 570 449 600	16	36	7 153 242 188 461 303 334 400
11	23	17 763 669 000	14	27	7 362 531 794 217 600	16	37	5 696 806 114 991 150 359 200
11	24	19 766 508 300	14	28	26 961 446 454 742 800	16	38	3 346 766 076 216 793 230 000
11	25	12 824 865 900	14	29	64 693 543 016 302 200	16	39	1 408 983 208 995 652 290 000
11	26	4 522 656 600	14	30	106 495 506 315 198 000	16	40	402 452 373 740 088 672 000
11	27	671 517 000	14	31	122 840 238 008 287 800	16	41	69 902 581 386 429 792 000
			14	32	99 468 461 823 330 600	16	42	5 576 572 329 584 256 000
			14	33	55 515 218 486 527 800			
			14	34	20 373 871 298 180 400			
			14	35	4 431 553 979 252 400			
			14	36	433 347 847 732 800			

Proof. Notice that in a toroidal crown, the unsubstituted edges are not adjacent, by definition, and hence form a matching of the underlying cycle, while the substituted edges are replaced by $K_5\backslash e$-networks. We can thus write

$$\mathscr{H} = C^{\mathrm{m}}\uparrow_z(K_5\backslash e), \tag{27}$$

where the notation $\uparrow_z$ means that only the edges marked by z are replaced by $K_5\backslash e$-networks. Hence we have, by analogy with Lemma 1,

$$\mathscr{H}(x, y) = C^{\mathrm{m}}(x, y, (K_5\backslash e)(x, y)), \tag{28}$$

which implies (26) using (25) and (13). $\square$

A substitution of the generating function $\mathscr{N}_{\mathrm{P}}(x, y)$ (16) counting the strongly planar networks for the variable y in (10), (11) and (26) gives the generating function for labelled 2-connected non-planar toroidal graphs with no $K_{3,3}$-subdivisions, i.e.

$$\mathscr{T}(x, y) = K_5(x, \mathscr{N}_{\mathrm{P}}(x, y)) + M(x, \mathscr{N}_{\mathrm{P}}(x, y)) + M^*(x, \mathscr{N}_{\mathrm{P}}(x, y)) + \mathscr{H}(x, \mathscr{N}_{\mathrm{P}}(x, y)). \tag{29}$$

Numerical results are presented in Tables 1 and 2, where

$$\mathscr{T}(x, y) = \sum_{n \geqslant 5} \sum_{m} t_{n,m} x^n y^m / n!$$

and $t_n = \sum_m t_{n,m}$ count labelled graphs in $\mathscr{T}$. Notice that the term $K_5(x, \mathscr{N}_P(x, y))$ in (29) also enumerates non-planar 2-connected $K_{3,3}$-free *projective-planar graphs* and that corresponding tables are given in [9].

The homeomorphically irreducible graphs in $\mathscr{T}$, i.e. the graphs having no vertex of degree two, can be counted by using several methods described in detail in Section 4 of [9]. We used the approach of Proposition 8 of [9] to obtain the numerical data presented in Tables 1 and 3 for labelled homeomorphically irreducible graphs in $\mathscr{T}$.

5. Concluding remarks

Notice that graphs with six or more vertices satisfying Proposition 3 are not 3-connected. Therefore a 3-connected non-planar toroidal graph different from K_5 must contain a $K_{3,3}$-subdivision, a result also obtained by Asano [1].

Theorems 1 and 2 imply that a projective-planar graph with no $K_{3,3}$-subdivisions is toroidal. However an arbitrary projective-planar graph can be non-toroidal. For an example, see [16, p. 368].

The characterization of Theorem 2 can be used to detect if a graph is toroidal and $K_{3,3}$-free in linear time. The implementation of this algorithm can be derived from [8] by using a breadth-first or depth-first search technique for the decomposition and by doing a linear-time planarity testing. The linear-time complexity follows from the linear-time complexity of the decomposition and from the fact that each vertex of the initial graph can appear in at most seven different components.

References

[1] T. Asano, An approach to the subgraph homeomorphism problem, Theoret. Comput. Sci. 38 (1985) 249–267.
[2] E.A. Bender, Zh. Gao, N.C. Wormald, The number of labeled 2-connected planar graphs, Electron. J. Combin. 9 (2002) Research Paper 43, 13pp (electronic).
[3] F. Bergeron, G. Labelle, P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge University Press, Cambridge, 1998.
[4] M. Bodirsky, C. Gröpl, M. Kang, Generating labeled planar graphs uniformly at random, in: J.C.M. Baeten, J.K. Lenstra, J. Parrow, G.J. Woeginger (Eds.), Automata, Languages and Programming, Lecture Notes in Computer Science, vol. 2719, Springer, Berlin, 2003, pp. 1095–1107.
[5] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, New York, 1976.
[6] R. Diestel, Graph Theory, second ed., Springer, Berlin, 2000.
[7] M. Fellows, P. Kaschube, Searching for $K_{3,3}$ in linear time, Linear and Multilinear Algebra 29 (1991) 279–290.
[8] A. Gagarin, W. Kocay, Embedding graphs containing K_5-subdivisions, Ars Combin. 64 (2002) 33–49.
[9] A. Gagarin, G. Labelle, P. Leroux, The structure and labelled enumeration of $K_{3,3}$-subdivision-free projective-planar graphs, Pure Math. Appl. 16 (3) (2005) 267–286, preprint available at: ⟨http://arxiv.org/abs/math.CO/0406140⟩.
[10] A. Gagarin, G. Labelle, P. Leroux, Counting unlabelled toroidal graphs with no $K_{3,3}$-subdivisions, Adv. in Appl. Math. 39(1) (2007) 51–75, preprint available at: ⟨http://arxiv.org/abs/math.CO/0509004⟩.
[11] A. Gagarin, W. Myrvold, J. Chambers, Forbidden minors and subdivisions for toroidal graphs with no $K_{3,3}$'s, Proceedings of the seventh International Colloquium on Graph Theory (ICGT'05), Electron. Notes Discrete Math. 22 (2005) 151–156.
[12] A. Gagarin, W. Myrvold, J. Chambers, The obstructions for toroidal graphs with no $K_{3,3}$'s, Discrete Math., to appear, preprint available at: ⟨http://arxiv.org/abs/math.CO/0411488⟩.
[13] C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993.
[14] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.
[16] W. Kocay, D.L. Kreher, Graphs, Algorithms and Optimization, Chapman & Hall, CRC, Boca Raton, FL, 2005.
[17] K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. 15 (1930) 271–283.
[18] R.P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth Brooks, Cole, Pacific Grove, CA, 1986. Reedited in: Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.
[19] C. Thomassen, The Jordan–Schönflies theorem and the classification of surfaces, Amer. Math. Monthly 99 (2) (1992) 116–131.
[20] K. Wagner, Über eine Erweiterung eines Satzes von Kuratowski, Deutsche Math. 2 (1937) 280–285 (in German).
[21] T.R.S. Walsh, Counting labelled three-connected and homeomorphically irreducible two-connected graphs, J. Combin. Theory Ser. B 32 (1982) 1–11.
[22] H.S. Wilf, Generatingfunctionology, Academic Press, New York, 1990.