New results on chromatic index critical graphs

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In this paper, we prove several new results on chromatic index critical graphs. We also prove that if $G$ is a $\Delta(\geq 4)$-critical graph, then
\[ n_{\Delta} \geq 2 \sum_{j=2}^{\Delta-1} \frac{n_j}{j-1} + \frac{1}{2} n_2, \]
where $n_j$ is the number of vertices having degree $j$ in $G$.

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1. Introduction

All graphs considered here are simple. Terminology and notation not introduced here are given in the books by Bondy and Murty [2] and by Yap [11].

Suppose that $G$ is a graph and $v \in V(G)$. If $v$ is of degree $\Delta(G)$, then $v$ is called a major vertex; otherwise $v$ is called a minor vertex. The set of all vertices adjacent to $v$ is denoted by $N_G(v)$ or simply by $N(v)$. We say $A \subset B$ if $A$ is a subset of the set $B$. For $S \subset V(G)$, let $N(S) = \bigcup_{v \in S} N(v)$ and $\overline{N}(S) = N(S) \setminus S$. The symbol $G[S]$ denotes the subgraph induced by $S$.

The order (size) of a graph $G$, denoted by $|G|$ ($e(G)$), is the number of vertices (edges) of $G$. For disjoint subsets $A$ and $B$ of $V(G)$, $[A, B]_G$ denotes the set of edges with one end in $A$ and the other in $B$, and $m_G[A, B]$ denotes the number of edges in $[A, B]_G$. We usually write $[A, B]_G$ and $m_G[A, B]$ if $B = \{b\}$. If no confusion will be caused, we will not distinguish between a subset $A$ of $V(G)$ and the subgraph of $G$ with the vertex set $A$.

A $k$-(edge) coloring of a graph $G$ is a map $\chi : E(G) \to \{1, 2, \ldots, k\}$ such that no two adjacent edges of $G$ have the same image. The chromatic index of $G$ is $\chi'(G) = \min\{k \mid G$ has a $k$-coloring$\}$. Vizing [10] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

If $\chi'(G) = \Delta(G)$, $G$ is said to be of class 1, otherwise $G$ is said to be of class 2.

A graph $G$ is said to be (chromatic index) critical if $G$ is connected, of class 2, and $\chi'(G-e) < \chi'(G)$ for any edge $e$ of $G$. If $G$ is critical and $\Delta(G) = \Delta$, then $G$ is said to be $\Delta$-critical.

In the next section, we prove several new results on critical graphs and give a new upper bound on the size of chromatic index critical graphs of even order. In Section 3, we prove an inequality involving the number of vertices $n_i$ of degree $i$ in a critical graph. Finally, in Section 4, we make a conjecture.

The following known results will be used in this paper:

Lemma 1 (Vizing’s Adjacency Lemma). Let $G$ be a $\Delta$-critical graph and let $uv \in E(G)$ where $d(v) = k$. The following hold:

(1) If $k < \Delta$, then $u$ is adjacent to at least $\Delta - k + 1$ major vertices.

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Lemma 1. Let G be a Δ-critical graph and let u ∈ V(G). By Lemma 1, u is adjacent to at least two major vertices.

Remark 1. Every vertex of N(x, y) \ {x, y} is a major vertex.

Lemma 2 (Zhang [12]). Let G be a Δ-critical graph, xy ∈ E(G) and d(x) + d(y) = Δ + 2. The following hold:

1. Every vertex of N(x, y) \ {x, y} is a major vertex.
2. Every vertex of N(N(x, y)) \ {x, y} is at least of degree Δ − 1.
3. If d(x), d(y) < Δ, then every vertex of N(N(x, y)) \ {x, y} is a major vertex.

2. Properties of Δ-critical graphs

In this section, we always assume that G is a Δ(≥ 3)-critical graph, xy ∈ E(G) and π is a Δ-coloring of G − xy having image set Γ = {1, 2, . . . , Δ}. Given a k-coloring π of G having image {1, 2, . . . , k}, we note Cπ(v) = {π(e) | e ∈ E(G), e is incident with v} and C′π(v) = {1, 2, . . . , k} \ Cπ(v). Thus, Cπ(x) ≠ ∅ and C′π(y) ≠ ∅, and so Cπ(x) ∪ C′π(y) ≠ ∅. It is clear that k ∈ Cπ(x) ∩ C′π(y) if and only if k ∈ Cπ(x) ∩ C′π(y) for any k ∈ Γ, and

|Cπ(x) ∩ C′π(y)| = d(x) + d(y) − Δ − 2.

Let u0u1 ∈ E(G) and let π be a Δ-coloring of H = G − u0u1. A path P = u0u1u2 . . . un of G is said to be π-acceptable if for each i > 0, dπ(u(i)) < Δ and π(ei) ∈ (−1)i Cπ(u(i)), where ei = u(i)+1 ∈ E(G).

Lemma 3 (Zhang [12]). Let G be a Δ-critical graph and let u0u1 ∈ E(G). Then for every Δ-coloring of G − u0u1 and every π-acceptable path u0u1u2 . . . un, Cπ(u(i)) ∩ C′π(u(i)) = ∅ if i ≠ j.

Lemma 4 (Zhang [12]). Let G be a Δ-critical graph and let u0u1 ∈ E(G). Let π be a Δ-coloring of G − u0u1 and let P = u0u1u2 . . . un be a path. If P is π-acceptable, then ∑n i=0 dπ(u(i)) ≥ nΔ + 2.

Lemma 5 (Zhang [12]). Let yuw (w ≠ x) be a path of G and let {π(yu), π(uw)} ⊆ (Cπ(x) ∪ C′π(y)). Then

1. Cπ(x) ∩ Cπ(y) = C′π(y) and; or
2. σ(e) = σ(e) if (e) ∈ Cπ(x) ∩ Cπ(y), or σ(e) ∈ Cπ(x) ∩ C′π(y) if (e) ∈ Cπ(x) ∩ C′π(y), for any edge e of G − xy.

Clearly, this similarity relation in Δ-colorings of G − xy is an equivalence relation.

Lemma 6 (Zhang [12]). Let w ∈ V(G) \ {x, y} and let e be an edge incident with w. Let (S ∪ {π(e)}) ⊆ (Cπ(x) ∪ C′π(y)) and let k ∈ S. If 2 ≤ |S| ≤ |Cπ(x) ∪ C′π(y)|, then G − xy has Δ-coloring σ similar to π such that σ(e) = k and C′π(w) ⊆ (S \ {k}).

Lemma 7 (Zhang [12]). Let w ∈ V(G) \ {x, y} and let e be an edge incident with w. Let (S ∪ {π(e)}) ⊆ (Cπ(x) ∪ C′π(y)) and let k ∈ S. If d(x), d(y) < Δ, and 2 ≤ |S| ≤ |Cπ(x) ∪ C′π(y)|, then G − xy has a Δ-coloring σ similar to π such that σ(e) = k and C′π(w) ⊆ (S \ {k}). We now apply above lemmas to prove the following theorems.

Theorem 8. Let y ∈ V(G); x, w ∈ N(y) and d(x) ≤ 4. If |N(w) ∩ N(y)| ≥ 2, then d(x) + d(w) ≥ Δ + 2.

Proof. If d(x) = 2 or xw ∈ E(G), then this result follows by Lemma 1. We now assume that xw ∉ E(G) and 3 ≤ d(x) ≤ 4. Let π be a Δ-coloring of G − xy. Since |Cπ(x) ∩ Cπ(y)| = d(x) + d(y) − Δ − 2 and d(x) ≤ 4, |Cπ(x) ∩ Cπ(y)| ≤ 2.

If d(x) + d(w) < Δ + 2, then d(w) ≤ Δ − 2 and d(x) + d(y) + d(w) ≤ 2Δ + 1. By Lemma 4, xy is not π-acceptable. By the definition of the π-acceptable path, π(yw) ∈ Cπ(x) ∩ Cπ(y) and so |Cπ(x) ∩ Cπ(y)| ≤ 1. Since |Cπ(w) ∩ Cπ(x) ∩ Cπ(y)| = 1 means that d(x) = 4 and d(w) ≤ Δ − 3, |Cπ(w) ∩ Cπ(y)| ≥ 2 in any case. Let u, v ∈ N(N(w) ∩ N(y)). Then yuw and yuw both are paths of G. By Lemma 5(1), neither π(yw), π(uw) nor π(yw), π(uw) is included in C′π(x) ∪ C′π(y). Since |Cπ(x) ∩ Cπ(y)| ≤ 2 and π(yw) ∈ Cπ(x) ∩ Cπ(y), either π(yw) = π(uw) or else π(yw) = π(uw) is in Cπ(x) ∩ Cπ(y).

Suppose that π(yw) = π(uw) ∈ Cπ(x) ∩ Cπ(y). (The proof for the other case is similar.) Then

{π(yw), π(uw)} ⊆ C′π(x) ∩ C′π(y).

Let S = {k, π(yw)} where k ∈ C′π(y). By Lemma 6, we may assume that π(uw) = k and π(yw) ∈ C′π(y). We choose σ as follows: σ(yw) = π(uw), σ(wv) = π(yw), σ(yu) = π(uw), σ(wu) = π(yw), σ(xy) = π(yw), and σ(e) = π(e) for every other edge e. Then σ is a Δ-coloring of G, which is a contradiction. □

Theorem 9. Let y ∈ V(G), w ∈ N(y) and d(w) ≤ Δ − 2. If N(y) ∩ N(w) ≠ ∅, then (N(y) \ {w}) ∩ (S2 ∪ S3) ≠ ∅.
Theorem 10. Let \( |G| = 2k \) and let \( e(G) \geq (k - 1)\Delta + 1 \). We have

(1) \( d(x) + d(y) \geq \Delta + l + 1 \) if \( xy \in E(G) \);

(2) Suppose that \( B = \{x, y, z\} \) is a set of minor vertices of \( G \) and let \( G[B] \cong K_s \). If \( l = 3 \) and \( d(x) + d(y) = \Delta + 4 \), then \( (N(x) \cup N(y)) \setminus \{x, y\} = \emptyset \).

Proof. (1) If \( xy \in E(G) \), then let \( \pi \) be a \( \Delta \)-coloring of \( G - xy \) with color classes \( E_1, E_2, \ldots, E_\Delta \), and let \( |E_1| \geq |E_2| \geq \cdots \geq |E_\Delta| \). If \( l \leq 1 \), then by Lemma 1(3), \( d(x) + d(y) \geq \Delta + l + 1 \). Now we assume that \( l \geq 2 \). Since \( |G| = 2k \) and \( e(G) \geq (k - 1)\Delta + l \), \( |E_1| = |E_2| = \cdots = |E_{l-1}| = k \). Hence, \( \{1, 2, \ldots, l - 1\} \subset C_1(x) \cap C_{l-1}(y) \). Since \( |C_1(x) \cap C_{l-1}(y)| = d(x) + d(y) - \Delta - 2 \), \( d(x) + d(y) \geq \Delta + l + 1 \).

(2) If \( (N(x) \cup N(y) \cup N(z)) \setminus \{x, y\} \neq \emptyset \), then let \( w \in (N(x) \cup N(y) \cup N(z)) \setminus \{x, y\} \). Without loss of generality, we assume that \( xwz \) or \( ywz \) is a path of \( G \). Let \( \pi \) be a \( \Delta \)-coloring of \( G - xy \) with color classes \( E_1, E_2, \ldots, E_\Delta \), and let \( |E_1| \geq |E_2| \geq \cdots \geq |E_\Delta| \). Since \( e(G) \geq (k - 1)\Delta + 3 \), \( |E_1| = |E_2| = k \). Since \( d(x) + d(y) \geq \Delta + 4 \), \( |C_1(x) \cap C_{l-1}(y)| = d(x) + d(y) - \Delta - 2 \), and so \( C_1(x) \cap C_{l-1}(y) \) is a \( \Delta \)-coloring of \( G \), which is a contradiction. \( \square \)

Theorem 11. Let \( G \) be a \( \Delta \)-critical graph of order \( 2k \) and let \( xy \in E(G) \). Then

\[ e(G) \leq (k - 2)\Delta + d(x) + d(y) - 1. \]

Proof. Let \( \pi \) be a \( \Delta \)-coloring of \( G - xy \) with color classes \( E_1, E_2, \ldots, E_\Delta \). Note that \( |C_1(x) \cap C_{l-1}(y)| = d(x) + d(y) - \Delta - 2 \). Let \( |C_1(x) \cap C_{l-1}(y)| = l \). If this theorem is false, then \( e(G) \geq (k - 2)\Delta + d(x) + d(y) = (k - 1)\Delta + d(x) + d(y) - \Delta = (k - 1)\Delta + l + 2 \). Hence, there exist \( l + 1 \) color classes having edge number \( k \). It follows that \( |C_1(x) \cap C_{l-1}(y)| \geq l + 1 \), which is a contradiction. The proof is now complete. \( \square \)

Let \( \delta = \Delta(G) \) and let \( v \) be a vertex of \( G \) having degree \( \delta \). Let \( u \in N(v) \); then \( d(u) \leq \Delta \) and so \( d(u) + d(v) \leq \Delta + \delta \). By Theorem 11, \( e(G) \leq (k - 1)\Delta + \delta - 1 \). Thus, we have arrived at the following result:

Corollary 12 (Fiorini and Wilson [6]). Let \( G \) be a \( \Delta \)-critical graph of order \( 2k \). Then \( e(G) \leq (k - 1)\Delta + \delta - 1 \).

3. An inequality involving the number of major vertices of critical graphs

Theorem 13. Let \( \Delta \geq 4 \) be an integer. If \( G \) be a \( \Delta \)-critical graph, then

\[ n_\Delta \geq 2 \sum_{j=2}^{\Delta-1} \frac{n_j}{j-1} + \frac{1}{2} n_3. \]

Proof. To each major vertex \( v \) in \( G \), assign a \((\Delta - 2)\)-tuple \((i_2, i_3, \ldots, i_{\Delta-1})\) where \( i_t \) is the number of vertices of degree \( t \) adjacent to \( v \). We denote by \( \Phi \) the set of all such \((\Delta - 2)\)-tuples associated with each major vertex of \( G \). We first prove that if \((i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi \), then

\[ \sum_{j=2}^{\Delta-1} \frac{i_j}{j-1} \leq 1. \]  \[ \text{(1)} \]

Let \( q \) be the smallest index of all non-zero elements of the \((\Delta - 2)\)-tuple \((i_2, i_3, \ldots, i_{\Delta-1})\), and let \( v \) be a major vertex associated with this \((\Delta - 2)\)-tuple. By Lemma 1(1), \( v \) is adjacent to at least \( \Delta - q + 1 \) major vertices, and so it must be adjacent to at most \( \Delta - (\Delta - q + 1) = q - 1 \) minor vertices. Thus, \( \sum_{j=q}^{\Delta-1} \frac{i_j}{q-1} \leq 1 \), and hence

\[ \sum_{j=2}^{\Delta-1} \frac{i_j}{j-1} \leq \sum_{j=2}^{\Delta-1} \frac{i_j}{q-1} \leq 1. \]
Let \(n(i_2, i_3, \ldots, i_{\Delta-1})\) be the number of major vertices of \(G\) associated with the \((\Delta - 2)\)-tuple \((i_2, i_3, \ldots, i_{\Delta-1})\), and let \(M\) be the set of all major vertices of \(G\). Then, for each \(j \in \{2, 3, \ldots, \Delta - 1\},\)

\[
m_G[S_j, M] \leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} i_j n(i_2, i_3, \ldots, i_{\Delta-1}).
\]

Hence

\[
\sum_{j=2}^{\Delta-1} \frac{m_G[S_j, M]}{j - 1} \leq \sum_{j=2}^{\Delta-1} \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} \frac{i_j}{j - 1} n(i_2, i_3, \ldots, i_{\Delta-1})
\]

\[
= \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j - 1}.
\]

Since each vertex of \(G\) is adjacent to at least two major vertices,

\[2n_j \leq m_G[S_j, M].\]

By Lemma 1(3), \(N(S_2) \subset S_\Delta \cup S_{\Delta-1}\). Let \(R = \{x \in S_3 \mid N(x) \cap S_{\Delta-1} \neq \emptyset\}\). Then \(N(S_3 \setminus R) \subset S_\Delta\). Let \(r = |R|\). We have

\[2r + 3(n_3 - r) = m_G[S_3, M].\]

Note that \(d(x) + d(u) = \Delta + 2\) if \(x \in S_{\Delta-1}\) and \(u \in S_3\). Then, for every vertex \(u \in R\), \(|N(u) \cap S_{\Delta-1}| = 1\) and \(N(N(u) \cap S_{\Delta-1}) \subset (S_{\Delta-1} \cup \{u\})\), by Lemma 2(1). Hence, \(|N(R) \cap S_{\Delta-1}| = r\), and so

\[r(\Delta - 2) + 2(n_{\Delta-1} - r) \leq m_G[S_{\Delta-1}, M].\]

By these inequalities (2)-(5), we have

\[2n_2 + \frac{2r + 3(n_3 - r)}{2} + \sum_{j=4}^{\Delta-2} \frac{2n_j}{j - 1} + \frac{r(\Delta - 2) + 2(n_{\Delta-1} - r)}{\Delta - 2} \leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j - 1},\]

which can be simplified as

\[
\sum_{j=2}^{\Delta-1} \frac{2n_j}{j - 1} + \frac{1}{2} n_3 + \frac{r(\Delta - 4)}{\Delta - 2} - \frac{r}{2} \leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j - 1}.
\]

If \(r = 0\), then applying (1) to (6), we have

\[2 \sum_{j=2}^{\Delta-1} \frac{n_j}{j - 1} + \frac{1}{2} n_3 \leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j - 1}\]

\[\leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) = n_\Delta.
\]

If \(r \neq 0\), then let \(x \in R\), and we find that there is a vertex \(y\) in \(S_{\Delta-1}\) such that \(xy \in E(G)\). By Lemma 2(1), every vertex of \(N(x, y) \setminus \{x, y\}\) is a major vertex. By Lemma 2(3), every vertex of \(N(N(x, y)) \setminus \{x, y\}\) is a major vertex. Then every vertex \(N(x) \setminus \{y\}\) is associated with the \((\Delta - 2)\)-tuple \((0, 1, 0, \ldots, 0)\). Hence, the \((\Delta - 2)\)-tuple \((0, 1, 0, \ldots, 0)\) is a major vertex. Then \(n(0, 1, 0, \ldots, 0) \geq r\). Let \(\Phi' = \Phi \setminus \{(0, 1, 0, \ldots, 0)\}\). Then, by (1) and (6), we have

\[
\sum_{j=2}^{\Delta-1} \frac{2n_j}{j - 1} + \frac{1}{2} n_3 + \frac{r(\Delta - 4)}{\Delta - 2} - \frac{r}{2} \leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j - 1}\]

\[\leq \sum_{(i_2, i_3, \ldots, i_{\Delta-1}) \in \Phi} n(i_2, i_3, \ldots, i_{\Delta-1}) + n(0, 1, 0, \ldots, 0) - \frac{1}{2} r,
\]

from which we can derive the required inequality. The proof of Theorem 13 is now complete. □

4. A conjecture

In 1974, Jakobsen [9] conjectured that there are no critical graphs of even order. Over five years later, Gol'dberg [7] constructed an infinite family of 3-critical graphs of even order. The smallest known critical graph of even order which
is a 4-critical graph of order 18. This critical graph was found by Fiol (see Yap [11, p. 49]) in 1980. Since there are no critical graphs of even order less than 12, Yap [11] posed the following problem.

**Problem.** Does there exist a critical graph of order 12, 14 or 16?

With the aid of a computer, G. Brinkmann and E. Steffen prove that there are no critical graphs of order 12 (see [4]). D. Bokal, G. Brinkmann and S. Grunewald also prove that there are no critical graphs of order 14, with the help of a computer (see [3]).

Using the newly proven properties of $\Delta$-critical graphs, it can also be proven theoretically that there are no critical graphs of order 12.

The structural characteristics of critical graphs of order not greater than 11 have been understood.

We always denote by $S_i$, the set of vertices having degree $i$ in $G$ and let $n_i = |S_i|$. The degree-list of a graph $G$ is $1^{n_1}2^{n_2}\ldots k^{n_k}$. If $n_k = 0$, the factor $j^0$ is usually omitted from the degree-list.

**Lemma 14.** Let $G$ be a critical graph of order $n$. We have

1. (see [11]) if $n = 5$, then the degree-list of $G$ is $2^5, 3^34^1$;
2. (see [1]) if $n = 7$, then the degree-list of $G$ is $2^7, 23^1, 24^1, 3^24^1, 25^1, 345^1, 4^25^1, 45^26^1$ or $4^46^1$;
3. (see [5]) if $n = 9$ and $\Delta \geq 4$, then $e(G) = 4\Delta + 1$;
4. (see [4,8]) if $n = 11$ and $\Delta > 4$, then $e(G) = 5\Delta + 1$;
5. (see [3]) if $n = 13$ and $\Delta \geq 4$, then $e(G) = 6\Delta + 1$.

Using this Lemma 14 we can immediately deduce the following lemma.

**Lemma 15.** Let $\Delta \geq 4$ and $k \leq 6$ be integers. If $G$ is a $\Delta$-critical graph of order $2k + 1$, then $e(G) = k\Delta + 1$.

Prompted by these results in Lemma 14, we make the following conjecture.

**Conjecture.** Let $G$ be a 2-connected graph of order $2k + 1$ with maximum degree of at least $2\lfloor \frac{\Delta}{2} \rfloor + 1$. Then $G$ is $\Delta$-critical if and only if $e(G) = k\Delta + 1$.

Suppose that $G$ is a 2-connected graph of order $2k + 1$. It is clear that $G$ is $\Delta$-critical if and only if $e(G) = k\Delta + 1$ when $1 \leq k \leq 2$. Beineke and Fiorini [1] prove if $k = 3$, then $G$ is $\Delta$-critical if and only if $e(G) = 3\Delta + 1$. Chetwynd and Yap [5] prove if $k = 4$ and $\Delta \geq 5$, then $G$ is $\Delta$-critical if and only if $e(G) = 4\Delta + 1$. Huang and Zhang [8] prove if $k = 5$ and $\Delta \geq 5$, then $G$ is $\Delta$-critical if and only if $e(G) = 5\Delta + 1$. So that this conjecture is true when $k \leq 5$.

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