Numerical Investigation of Stable Crack Growth in Ductile Materials Using XFEM

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Abstract

In the present work, extended finite element method (XFEM) has been extended to simulate nonlinear stable crack growth problems. In XFEM, the cracks are modeled by adding enrichment functions into standard finite element approximation. The modeling of large deformations is done using updated Lagrangian approach. Von-Mises yield criterion has been used along with isotropic hardening to check the plasticity. Elastic-predictor and plastic-corrector algorithm has been used for the computation of stress fields. The nonlinear equations are solved by Newton-Raphson iterative scheme. Two problems (crack growth in compact tension and triple point bend specimens) are solved using J-R curve to show the capability of XFEM in modeling large deformation crack growth problems.

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Keywords: XFEM; updated Lagrangian approach; Von-Mises yield criterion; isotropic strain hardening; elastic-predictor and plastic-corrector algorithm; stable crack growth.

1. Introduction

In ductile materials, sudden unstable crack growth does not occur and a substantial amount of stable crack growth takes place before the final failure of the components. The theory of linear elastic fracture mechanics (LEFM) has been widely used by many researchers to investigate the structural component reliability and life
expectancy. However, the validity of LEFM is limited to the brittle materials. Therefore, the nonlinear theory of fracture mechanics needs to be utilized to characterize the plastic behavior of the material. Several parameters have been developed to characterize the ductile materials fracture, for example, crack tip opening displacement (CTOD) [1-2], J-integral [3], tearing modulus [4], and strain energy [5].

Over the years, a number of numerical techniques such as finite element method [6], boundary element method [7], meshfree methods [8] and extended finite element method [9] have been developed to simulate the fracture mechanics problems. Among these, XFEM has become the most widely used numerical method to solve crack growth problems. In XFEM, the conformal meshing is not required hence the modeling of moving discontinuities or crack growth is performed with an ease. The growth of cracks and moving discontinuities are modeled by adding discontinuous functions [9] into standard finite element approximation. Till date, most of the developments in XFEM are focused mainly on the linear elastic behavior of the material involving small deformation. Some investigators studied the elasto-plastic problems with large deformation using finite element method but these studies were quite limited.

In the present work, XFEM is extended to simulate the elasto-plastic behavior of the material with large deformation in an updated Lagrangian framework. Newton-Raphson technique is adopted for the solution of nonlinear equations. Elastic-predictor and plastic-corrector algorithm [10] is employed for stress computations. Von-Mises yield criterion [11] is considered with isotropic hardening to determine the updated value of the yield stress. Two numerical problems of stable crack growth in ductile material are solved in this work to show the capability of XFEM.

2. Numerical Formulation

2.1. Governing Equations

Mainly two types of nonlinearity occur in elasto-plastic large deformation analysis: (1) geometric nonlinearity due to large deformations and (2) material nonlinearity due to plasticity. In the present work, geometric nonlinearity is modeled using updated Lagrangian approach. In this approach, current configuration is used as a reference configuration. The governing equations of elasto-plasticity with large deformation are briefly reviewed, and corresponding equilibrium equation is given as,

$$\nabla \cdot \sigma + b = 0 \quad \text{in} \quad \Omega \quad \text{(1)}$$

where, $\sigma$ is the Cauchy stress tensor and $b$ is the body force per unit volume. In updated Lagrangian approach "Eq. (1)" can be expressed in the variational form [12],

$$\delta \Psi(\mathbf{u}) = \int_\Omega \delta \sigma_{ij} \delta \varepsilon_{ij} \, dV - \int_\Omega \mathbf{b} \cdot \delta \mathbf{u} \, dV - \int_{\Gamma^e} \mathbf{t} \cdot \delta \mathbf{u} \, dS = 0 \quad \text{(2)}$$

where, $\mathbf{b}$ are the body force components, and $\sigma_{ij}$ are updated second Piola–Kirchhoff stress tensor, $\varepsilon_{ij}$ are the updated Green-Lagrange strain tensor. Now using $\sigma_{ij} = \sigma_{ij} + \varepsilon_{ij}$ and $\varepsilon_{ij} = \varepsilon_{ij}^{(E)} + \varepsilon_{ij}^{(D)}$, "Eq. (2)" can be further expressed as,

$$\delta \Psi(\mathbf{u}) = \int_\Omega \delta \sigma_{ij} \delta \varepsilon_{ij}^{(E)} \, dV + \int_\Omega \delta \varepsilon_{ij}^{(D)} \delta \varepsilon_{ij}^{(D)} \, dV - \int_\Omega \mathbf{b} \cdot \delta \mathbf{u} \, dV - \int_{\Gamma^e} \mathbf{t} \cdot \delta \mathbf{u} \, dS \quad \text{(3)}$$

Since, in updated Lagrangian approach the incremental displacements $u_i$ are very small hence, the following approximation holds good $\sigma_{ij} \approx \mathbf{D}^{(E)} (\varepsilon_{ij})$ and $\varepsilon_{ij} \approx \varepsilon_{ij}^{(E)}$. Now "Eq. (3)" takes the form,
\[
\int_{\Omega} D^{\nu}(E_{ij}) L \delta(E_{ij}) L d\Omega + \int_{\Gamma} \sigma_{ij} \delta(E_{ij}) M d\Gamma = R_{ext} - R_{int}
\] (4)

where, \( R_{int} = \int_{\Omega} \sigma_{ij} \delta(E_{ij}) L d\Omega \) and \( R_{ext} = \int_{\Omega} b_{ij} \delta u_{j} d\Omega + \int_{\Gamma} t_{ij} \delta u_{j} d\Gamma \)

Now "Eq. (4)" can be further simplified as,
\[
\delta \bar{u}^{T} K^{mat} \bar{u} + \delta \bar{u}^{T} K^{geo} \bar{u} = R_{ext} - R_{int}
\] (5)

where, \( K^{mat} = \int_{\Omega} B^{T} D^{\nu} B d\Omega \) and \( K^{geo} = \int_{\Omega} G^{T} M_{s} G d\Omega \)

Using \( R_{ext} = \delta \bar{u}^{T} f_{ext} \) and \( R_{int} = \delta \bar{u}^{T} f_{int} \), "Eq. (5)" can be modified as,
\[
(K^{mat} + K^{geo}) \bar{u} = f_{ext} - f_{int}
\] (6)

where, external force (\( f_{ext} \)) and internal force (\( f_{int} \)) can be defined as,
\[
f_{ext} = \int_{\Omega} b_{ij} d\Omega + \int_{\Gamma} t_{ij} d\Gamma, \quad f_{int} = \int_{\Omega} B^{T} \sigma d\Omega
\] (7)

where, \( K^{mat} \) and \( K^{geo} \) are the material tangent stiffness matrix and geometric stiffness matrix respectively; \( M_{s} \) is the matrix of Cauchy stress components; and \( B, G \) are Cartesian shape function derivatives matrix.

Von-Mises yield criterion with isotropic strain hardening is used to determine the stress level at which plasticity occurs. During any increment of stress, the change of incremental strain are assumed to be divisible into elastic and plastic components [11],
\[
d \varepsilon = (d \varepsilon)^{e} + (d \varepsilon)^{p}
\] (8)

\[
d \varepsilon_{ij} = \frac{d \sigma_{ij}}{2G} + \frac{(1-2\nu)}{E} \delta_{ij} d \varepsilon_{kk} + d \lambda \frac{\partial f}{\partial \varepsilon_{ij}}
\] (9)

where, \( f \) is the yield function and \( d \lambda \) is a proportionality constant termed as the plastic multiplier. Now the elasto-plastic incremental stress-strain relation is obtained as,
\[
d \sigma = D \varepsilon
\] (10)

where, \( D_{np} = D - \frac{d_{p} d_{p}^{T}}{A + d_{p} d_{p}^{T} A}, \quad d_{p} = a^{T} D, \quad a^{T} = \frac{\partial F}{\partial \sigma}, \quad \text{and} \quad A = -\frac{1}{d \lambda} \frac{\partial F}{\partial k} \)

2.2. Displacement Approximation for Cracks

In XFEM, standard finite element approximation is locally enriched to model the discontinuities. A shifted enrichment is used to obtain the Kronecker delta property in the enriched elements. In 2-D, at a particular node of interest \( x \), the displacement approximation can be written as [13],
\begin{equation}
\mathbf{u}^i(\mathbf{x}) = \sum_{j=1}^{n} N_j(x) \left[ \mathbf{u}_j + \left( \mathbf{H}(\mathbf{x}) - \mathbf{H}(\mathbf{x}_j) \right) \mathbf{a}_j + \sum_{\eta=1}^{4} \left[ \mathbf{\beta}_\eta(\mathbf{x}) - \mathbf{\beta}_\eta(\mathbf{x}_j) \right] \mathbf{b}_\eta \right] \quad (11)
\end{equation}

where, \( \mathbf{\bar{u}}_j \) is the nodal displacement vector associated with the continuous part of the FE solution, \( n \) is the set of all nodes in the mesh; \( n_e \) is the set of nodes belonging to those elements which are completely cut by the crack and \( n_d \) is the set of nodes belonging to those elements which are partially cut by the crack. \( \mathbf{a}_j \) is the nodal enriched degrees of freedom associated with discontinuous Heaviside function \( \mathbf{H}(\mathbf{x}) \), and \( \mathbf{b}_\eta^\alpha \) is the nodal enriched degree of freedom vector associated with crack tip enrichment \( \mathbf{\beta}_\eta(\mathbf{x}) \).

Heaviside function, \( \mathbf{H}(\mathbf{x}) \) is used to model the discontinuity in the displacement due to the presence of a crack, and can be expressed as,

\begin{equation}
\mathbf{H}(\mathbf{x}) = \begin{cases} 1 & \text{if } \chi(\mathbf{x}) \geq 0 \\ -1 & \text{otherwise} \end{cases} 
\end{equation}

where, \( \chi(\mathbf{x}) \) is the level set function.

Asymptotic crack tip enrichment functions, \( \mathbf{\beta}_\eta(\mathbf{x}) \) are used to model the singular stress field near the crack tip. Considering the local polar coordinates system \( r \) and \( \Theta \) at the crack tip, these functions can be written as [14],

\begin{equation}
\mathbf{\beta}_\eta(\mathbf{x}) = \left\{ \beta_1, \beta_2, \beta_3, \beta_4 \right\} = \left[ r^l \cos \frac{\Theta}{2}, r^l \sin \frac{\Theta}{2}, r^l \cos \frac{\Theta}{2} \sin \Theta, r^l \sin \frac{\Theta}{2} \sin \Theta \right] \quad (13)
\end{equation}

For LEFM, the value of exponent \( l = 0.5 \), and for EPFM, the value of exponent \( l = (1/1+\bar{n}) \), where \( \bar{n} \) is the hardening exponent, which depends on the material.

The elemental matrices, \( \mathbf{\bar{K}}_r \) and \( \mathbf{f} \) are obtained by substituting the approximation function, defined in "Eq. (11)" into "Eq. (6)" and "Eq. (7)",

\begin{equation}
\mathbf{\bar{K}}_{ij} = \begin{bmatrix} K_{ij}^{aa} & K_{ij}^{ab} \\ K_{ij}^{ba} & K_{ij}^{bb} \end{bmatrix}, \quad \text{and} \quad \mathbf{f}^b = \left\{ \mathbf{f}_i^m, \mathbf{f}_i^a, \mathbf{f}_i^b, \mathbf{f}_i^{b1}, \mathbf{f}_i^{b2}, \mathbf{f}_i^{b3}, \mathbf{f}_i^{b4} \right\}^T \quad (14)
\end{equation}

The sub-matrices and vectors that appear in the foregoing equations are given as,

\begin{equation}
\mathbf{\bar{K}}_r^a = \int \int (\mathbf{B}_s^r)^T \mathbf{D}_{rs}^{\alpha} \mathbf{B}_s^r \, h d\Omega + \int \int (\mathbf{G}_s^r)^T \mathbf{M}_{rs} \mathbf{G}_s^r \, h d\Omega \quad \text{where} \quad r, s = u, a, b 
\end{equation}

\begin{equation}
\mathbf{f}_s^a = \int \int N_s \mathbf{b} \, d\Omega + \int \mathbf{N}_s \mathbf{\bar{d}} \, d\Gamma 
\end{equation}

\begin{equation}
\mathbf{f}_s^d = \int \int N_i (\mathbf{H}(\mathbf{x}) - \mathbf{H}(\mathbf{x}_i)) \mathbf{b} \, d\Omega + \int N_i (\mathbf{H}(\mathbf{x}) - \mathbf{H}(\mathbf{x}_i)) \mathbf{\bar{d}} \, d\Gamma
\end{equation}

\begin{equation}
\mathbf{f}_s^{\alpha} = \int \int N_i \mathbf{\beta}_\alpha((\mathbf{x}) - (\mathbf{x}_i)) \mathbf{b} \, d\Omega + \int N_i (\mathbf{\beta}_\alpha(\mathbf{x}) - (\mathbf{x}_i)) \mathbf{\bar{d}} \, d\Gamma
\end{equation}
where, $\alpha = 1, 2, 3, 4$ and $N_i$ are finite element shape function, $B_i^x$, $B_i^y$, $B_i^\alpha$ and $B_i^\beta$ are the matrices of shape function derivatives and $G_i^u$, $G_i^a$, $G_i^b$ and $G_i^{ba}$ are the matrices of Cartesian shape function derivatives.

2.3. Computation of J-integral

The domain form of $J$-integral for a homogeneous cracked body [15] is given as,

$$J = \int_A \left[ \sigma_i^0 \frac{du_i}{dx_1} - W \delta_{ij} \right] \frac{\partial q_i}{\partial x_j} \, dA$$

(19)

where, $W = \int_0^\sigma \sigma \, d\varepsilon$ is the strain energy density, $\varepsilon$ is the stress.

3. Numerical Simulation

In this section, two examples of crack growth are solved by XFEM. The material properties and the specimen geometry are given in the problem. Ramberg-Osgood material model [14] is used for modeling nonlinear behavior of the material.

$$\varepsilon = \frac{\sigma}{\sigma_0} + \bar{\alpha} \left( \frac{\sigma}{\sigma_0} \right)^\pi$$

(20)

3.1. Stable Crack Growth in Compact Tension Specimen

The dimensions ($W = 72$ mm, $L = 1.25W$, $B = 36$ mm, $a = 38$ mm) and material data of a compact tension (CT) specimen (as shown in Fig. 1) are taken from the literature (pressure vessel steel 20 Mn Mo Ni 55) [16]. Young modulus, $E = 187.7$ GPa, yield strength, $\sigma_0 = 428$ MPa and Poisson’s ratio $\mu = 0.30$. The Ramberg-Osgood dimensionless parameters ($\bar{n} = 6.838$ and $\bar{\alpha} = 1.757$) are derived from the true stress and true strain diagram. CT specimen is discretized with uniform mesh of 56 equally distributed nodes in $x$-direction and 50 equally distributed nodes in $y$-direction. $J-R$ criterion is used for the crack growth and the initial critical value of $J$-integral ($J_{cr}$) is found as 304 N/mm. Whenever this criterion is satisfied, the crack length is increased by the specified amount (0.50 mm per step).

![Fig. 1. Compact tension specimen with dimensions](image-url)
The final deformed shape of the CT specimen is provided in Fig. 2. The actual variation of load with the crack mouth opening displacement (CMOD) is depicted in Fig. 3. As the condition for the first critical $J$-integral satisfies, the crack length is increased by the specified amount. Therefore, a drop in the load level occurs. The maximum load before the first crack growth indicates the crack initiation load. The drop in the load is shown in Fig. 3. The difference between the upper and lower values of the load is very small; therefore only upper values are used for the comparison with the literature results. The load versus CMOD plot for the literature and XFEM results is plotted in Fig. 4, and a very good agreement is achieved between the two. The percentage error in the maximum load ($P_{\text{max}}$) and CMOD corresponding to the maximum load is provided in the Table 1. Fig. 5 shows the variation of load with $J$-integral.

![Numerical deformed configuration](image1)

![Load vs. CMOD](image2)

**Table 1. Comparison of predicted maximum load and CMOD with literature results**

<table>
<thead>
<tr>
<th></th>
<th>Literature</th>
<th>XFEM</th>
<th>% Error</th>
</tr>
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<tbody>
<tr>
<td>$P_{\text{max}}$ (kN)</td>
<td>132.98</td>
<td>117.78</td>
<td>11.43</td>
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<tr>
<td>CMOD (mm) at $P_{\text{max}}$</td>
<td>2.97</td>
<td>2.65</td>
<td>10.77</td>
</tr>
</tbody>
</table>

![Comparison of XFEM and literature results](image3)

![Variation of load with $J$-integral](image4)
The variation of load with the crack growth is shown in Fig. 6, which indicates that a continuous increase in load is required for further stable crack extension. For the CT specimen, the stable crack growth is found as 0.5 mm. The stress contour plot of $\sigma_{yy}$ is shown in Fig. 7 for the final stage of loading. The maximum tensile stress occurs at the crack tip, and a compressive stress region is developed due to the nature of loading.

![Fig. 6. Variation of load with $\Delta a$](image1)

![Fig. 7. Stress contour plot ($\sigma_{yy}$) for CT Specimen](image2)

### 3.2. Stable Crack Growth in Triple Point Bend Specimen

Triple point bend (TPB) specimen dimensions (Fig. 8) and material properties for cast steel 1343 are taken from the literature [17] i.e. $W=50\text{mm}$, $B=25\text{mm}$, $a/W=0.52$, $E=210.2\text{ GPa}$, $\sigma_y^{\theta}=394\text{MPa}$ and $\mu=0.284$. Ramberg-Osgood dimensionless parameters ($\bar{\eta}=7.088$ and $\bar{\alpha}=4.705$) are derived from the true-stress and true-strain diagram. TPB specimen is discretized using a uniform mesh of 80 equally distributed nodes in $x$-direction and 20 equally distributed nodes in $y$-direction. $J$-$R$ criterion is used for the crack growth. The initial critical value of $J$-integral ($J_{cr}$) is taken as 95 MPa.mm. The plane stress condition is assumed for the simulation. Whenever, the calculated $J$-integral approaches to the current value of $J_{cr}$, the crack length is increased by specified amount (0.50 mm per step).

The load-load line displacement (LLD) values obtained from XFEM are compared with the literature, and a very good agreement is found as shown in Fig. 9. The error in maximum load and LLD corresponding to the maximum load are found as 7.28% and 16.80% respectively, and are given in Table 2. For TPB specimen, the stable crack growth is found as 2.5 mm.

![Fig. 8. Triple point bend specimen with dimensions](image3)
The variation of $J$-integral with load is depicted in Fig. 10, and the value of $J$-integral at maximum load is found as 270.84 MPa-mm. The stress component $\sigma_{xx}$ plays a dominant role for the triple point bend specimen. The maximum value of tensile stress occurs at the crack tip, and a compressive stress zone is obtained at the loading point. The stress contour plot of $\sigma_{xx}$ and $\sigma_{yy}$ are shown in Fig. 11 and Fig. 12 respectively.

![Graph 1](image1.png)  ![Graph 2](image2.png)

**Fig. 9.** Comparison of XFEM and Literature results  **Fig. 10.** Variation of load with $J$-integral

<table>
<thead>
<tr>
<th></th>
<th>Literature</th>
<th>XFEM</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{max}$ (kN)</td>
<td>38.921</td>
<td>36.088</td>
<td>7.28</td>
</tr>
<tr>
<td>CMOD (mm) at $P_{max}$</td>
<td>2.08</td>
<td>2.50</td>
<td>16.8</td>
</tr>
</tbody>
</table>

**Table 2.** Comparison of predicted maximum load and CMOD with literature results

![Stress Contour 1](image3.png)

**Fig. 11.** Stress contour plot ($\sigma_{xx}$) for TPB specimen

![Stress Contour 2](image4.png)

**Fig. 12.** Stress contour plot ($\sigma_{yy}$) for TPB specimen
4. Conclusions

In the present work, XFEM is used for the simulation of stable crack growth in CT and TPB specimens under plane stress condition. Crack growth is modeled using $J-R$ criterion. The load versus CMOD plots obtained by XFEM show a very good agreement with the literature data for both kinds of specimens. These simulations show that the stable crack growth for CT and TPB specimens is found as 0.50 mm and 2.50 mm respectively. On the basis of simulations, it can be concluded that XFEM is an effective tool for the modeling of stable crack growth in ductile materials as it does not require physical modeling of a crack in the domain. The nonlinearity due to large deformation and material plasticity can be easily modeled hence this work can be further extended to simulate the stable crack problems under mixed-mode loading.

References