A Smooth and Unified Proof of Cases 6, 5 and 3 of the Ringel–Youngs Theorem

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A proof of the Heawood conjecture for Cases 6, 5, and 3 is given; “unified” means that the same “geometry” and “zigzag” are used in each of the three cases; “smooth” means that the zigzag is of simplest possible type and that the related “chord problems” are trivial or nearly so.

1. INTRODUCTION

This paper is not self-contained. The proof of the Heawood conjecture has already occupied 125 pages of this journal [11, 13, 14, 17, 18] and many more elsewhere. For brevity we lean heavily on earlier papers, especially [17] with which we assume the reader to be quite familiar. On page 205 of this paper, Youngs wrote “It should be mentioned that the geometry for Case 6 is suitable for Cases 3 and 5. However, because of the simplicity of the solutions presented for those cases, no attempt has been made to solve them using the geometry designed for Case 6. Such a unification of geometry, however, may be desirable.” On pages 203–204 he also wrote “A solution [of Case 6] for $s \geq 3$ was found in the spring of 1966, and no changes have been made since then in the predetermined part of the current graph. However, the related chord and zigzag problems have been solved in a much more expeditious fashion. For example, the chord problem was originally solved in four cases depending on the residue class of $s$ mod 4. A unified solution is presented here. Similar simplifications have occurred in the solution of the zigzag problem. We

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† This work was done when the first author visited Santa Cruz in May, 1970; the final selection of the solution given here was made in a letter from Ted Youngs, dated 19 June. He was to have written this paper, but died a month later before doing so.

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make these comments to emphasize a conviction that the solution is not yet in final form, and to suggest a further exercise for the reader." This paper presents a unification of the geometry for Cases 6, 5, and 3 and a solution much nearer to final form. The zigzag is the same for each case and is of the simplest possible type. The solutions of the three related chord problems are trivial or almost so.

2. **Outline of Method**

We establish the Heawood conjecture when \( n = 12s + 6, 12s + 5 \) or \( 12s + 3 \) by proving that the genus of the complete graph, \( K_n \), on \( n \) points is

\[
\gamma(K_n) = \frac{(n - 3)(n - 4)}{12}
\]

in the cases mentioned. We do this by exhibiting triangular imbeddings of \( K_{12s+6} - K_3 \), \( K_{12s+5} - K_2 \), and \( K_{12s+3} \) in orientable surfaces of genus \( s(12s + 5) \), \( s(12s + 3) \), and \( s(12s - 1) \), respectively. The additional adjacencies needed in Cases 6 and 5 can be made on one extra handle, using the method of Ringel \([7; 17, pp. 177-178]\) in the former case.

The method is due to Gustin \([2]\) and uses "current graphs" \([17, pp. 179-186]\). Gustin's geometry is not suitable for the "index 3" solutions \([17, p. 179]\) needed in the cases considered here for which Youngs developed the theory of "vortices" \([17, p. 199]\). Our geometry consists of a Möbius ladder \([3]\) with \( 2s - 5 \) ordinary rungs, \( 4 \) \( Y \)-like rungs, or \( yungs \), and \( 2s - 1 \) "globular rungs" or \( ringles \).

The points of \( K_n \) are the members of the additive group \( \mathbb{Z}_{12s+3} \) in Case 3, together with \((X\) and\) \( Y \) and \( Z \) in Cases (6 and) 5. As the solution is of index 3, we work with the "quotient" group \( \mathbb{Z}_{12s+3}/\mathbb{Z}_3 \), which is isomorphic to \( \mathbb{Z}_{4s+1} \). The three disjoint circuits which between them traverse each edge of the current graph once in each direction, represent the points \( \{3k : 0 \leq k \leq 4s\} \) of this quotient group, denoted by \([0]\), and its two cosets \( \{3k + 1 : 0 \leq k \leq 4s\} \) and \( \{3k + 2 : 0 \leq k \leq 4s\} \) denoted by \([1]\) and \([2]\).

A minor blemish is that our proof holds only for \( s \geq 3 \). This is inevitable in Case 6; indeed \( s = 2 \) \((n = 30)\) was the very last case of the Heawood conjecture to yield; it was solved independently and simultaneously on opposite sides of the world by Mayer \([5]\) and Youngs, with nonisomorphic solutions. Mayer also gave a solution for \( s = 1 \) \((n = 18)\). The difficulty of this case is exemplified even by \( s = 0 \) \((n = 6)\). A solution was given by Heffter \([4]\). It is not possible to imbed \( K_6 - K_3 \) in the plane, since it
contains $K_{3,3}$ as a subgraph. It is possible to imbed $K_6 - 3K_2$ (octahedron) or $K_6 - P_4$, where $P_4$ is a path of four vertices and three edges in the plane. It is, of course, possible to imbed $K_6$ (indeed $K_2$) in the torus.

Cases 5 and 3 were first solved by Ringel [6, p. 200; 7]; there is no particular difficulty for small values of $s$, but the geometry we use is too complicated to cover them. For $s = 0 (n = 5$ and 3), both $K_6 - K_2$ and $K_3$ can be imbedded in the plane; $s = 1 (n = 17$ and 15) is illustrated in detail in [17, pp. 198–201, 179–183]; $s = 2$ has to be covered by a simpler geometry, such as that given there.

3. ANALYSIS OF THE GEOMETRY

We analyze the geometry used in [17] for Case 6; Fig. 1 to 4 show the parts of the current graph containing the four yungs, and the corresponding parts of the three circuits are indicated in the figures. The circuit for the members of the quotient group [0] is indicated by a dotted line, that for the coset [1] by a continuous line, and that for the coset [2] by a pecked line. The vertices of the current graph are indicated as small open or black circles. These determine the routes of the circuits; the former are right handed or U.S.-type traffic circles, the latter are sinister or U.K.-type. Those marked with Roman capitals are visited by all three circuits and are potential vortices. Lower case Roman letters and Greek letters are elements of $Z_{4s+1} \cong Z_{12s+3}/Z_3$. Greek letters and letters in the latter half of the Roman alphabet are underlined and represent elements of the subgroup $[0] = \{3k: 0 \leq k \leq 4s\}$ of $Z_{12s+3}$; the actual currents are obtained by multiplication by 3. Letters in the first half of the

![Figure 1](image-url)
alphabet are in parentheses and represent elements of the coset 
\([1] = \{3k + 1 : 0 \leq k \leq 4s\}\); the actual currents are found by multiplying by 3 and adding 1.

The elements of \([0]\) occur on the rungs (except the rung in Fig. 1), and on the vertical parts of the ringles and yungs. The elements of \([1]\) occur
on the stales of the ladder, on the rung in Fig. 1, on the curved parts of
the ringles and on the oblique parts of the yungs. The elements of \([2]\) are
obtained by traversing those of \([1]\) in the opposite direction, since
\([2] = [-1]\).

The six Roman capitals in Fig. 1 and 2 denote vortices [17, pp. 199–201]
and the Greek letters on the arrows issuing from them are their "strengths." In
Case 6 just three of them are nonzero; in the solution we give here these
are \(\alpha, \gamma, \delta\), associated with the three points \(X, Y, Z\) of \(K_{12s+6}\). In Case 5,
just two are nonzero and in Case 3 all are zero. The nonzero strengths
must each generate the whole group \(Z_{4s+1}\), i.e., \((\alpha, 4s + 1) = 1\), etc. for
all values of \(s\), so they must be chosen from \(\{\pm 1, \pm 2, \pm 4, \ldots, \pm s, \pm 2s\}\).
Note that all three circuits visit each vortex; this choice will ensure that
the corresponding point is adjacent to all the points of \(K_{12s+6}\) or \(K_{12s+5}\)
which are members of \(Z_{12s+3}\).

The required triangular imbedding of the graph in the appropriate
orientable surface is ensured by Kirchhoff’s current law [17, pp. 182–183]
suitably modified at the vortices. Application of this to the vertices of
Fig. 1 to 4 (with \(a' = a, d' = d\) in Cases 6 and 5) gives the following,
mod \(4s + 1\):

\[
\begin{align*}
\alpha + \beta + \gamma + \delta &= 0 \quad (1.1) \\
k_0 + a + e + \delta + 1 &= 0 \\
k_0 + a + e - \alpha + 1 &= 0 \quad (1.2) \\
a + b + f + \beta + 1 &= 0 \quad (1.3) \\
k_1 + c + f - \gamma + 1 &= 0 \quad (1.4) \\
k_2 - d + t &= 0 \quad (1.5) \\
b - c + t &= 0 \quad (1.6) \\
e + \zeta &= 0 \quad (1.7) \\
k_2 + c + g - e + 1 &= 0 \quad (2.1) \\
k_3 + b + g + \zeta + 1 &= 0 \quad (2.2) \\
k_2 - k_3 + t &= 0 \quad (2.3) \\
u + v + w &= 0 \quad (2.4) \\
k_4 - d + u &= 0 \quad (3.1) \\
k_5 - d - v &= 0 \quad (3.2) \\
k_5 - k_4 + w &= 0 \quad (3.3) \\
x + y + z &= 0 \quad (3.4) \\
k_6 - a - x &= 0 \quad (4.1) \\
k_7 - a + y &= 0 \quad (4.2) \\
k_8 - k_7 - z &= 0 \quad (4.3) \\
k_6 - k_7 - z &= 0 \quad (4.4)
\end{align*}
\]
In Case 3 there are no nonzero vortices, so we write $a = d = A$ in Eqs. (1.2)-(1.4) and (1.6) and $a' = d' = D$ in Eqs. (3.2), (3.3), (4.2), and (4.3). Not all of these equations are independent, and the number of unknowns far exceeds the number of equations. However, the number of suitable solutions is not large; there are many coincidences to be avoided between the values of the unknowns, and if we are to obtain simple solutions to the zigzag [17, pp. 190–193] and chord [17, pp. 195–197] problems, we must observe the following desiderata.

The zigzag determines the order of the currents from [1] which occur on the stales of the ladder and their relation with the currents from [0] which occur on the rungs and ringles between them. If it is to be both simple and operative for all $s \geq 3$, then the values of $a, b, c, d$ should be nearly equal, and the omitted steps $u, v, x, y$ and the special steps $t, w, z$ should be small. Since the sum of the first six integers is odd, the smallest possible solutions for $(u, v, w), (x, y, z)$ are $(1, 3, -4), (2, 5, -7)$ or their negatives, in some order. The ends of the zigzag, $k_0$ and $k_1$, should differ by about $2s$.

We examined many possibilities and selected the following values as being consistent with the same simple zigzag for all three cases. It is

![Diagram of zigzag sequence and values](image-url)

Figure 5.
RINGEL–YOUNGS THEOREM

pictured (for s = 5) in Fig. 5 (Cases 6 and 5) and Fig. 6 (Case 3); it is valid for s ≥ 3.

\[ a, A b c d, D \quad k_0 \quad k_1 k_4 k_5 u v w k_8 k_7 x y z \]

Cases 6 & 5

\[ 0 \quad -2 \quad 1 \quad 2 \quad 2s + 1 \quad -1 \quad -3 \quad 4 \quad 5 \quad 2 \quad -7 \quad -1 \quad 3 \quad -1 \quad -3 \quad 4 \]

Case 3

\[ -2 \quad 1 \quad 0 \quad 2 \quad 2s + 1 \quad -1 \quad -3 \quad 4 \quad 5 \quad 2 \quad -7 \quad -1 \quad 3 \quad -3 \quad -1 \quad 4 \]

This zigzag gives the assemblage of Fig. 1, 3, and 4, together with a rung (ringle) between the last two, shown in Fig. 7. This part of the geometry is the same for all three cases; the currents are now shown as members of \( Z_{4s+1} \).

For a simple solution to the chord problem, visualize the members of \( Z_{4s+1} \) at the vertices of a regular \((4s + 1)\)-gon. The unused vertices e, f, g should be at angular distances which are nearly multiples of a right angle, and the omitted chord length, \( t \), should be small or near \( 2s \). The solution to the chord problem determines which pairs of members of \([1]\) (ends of chords) are associated with the curved parts of the ringles, whose vertical parts carry the corresponding members of \([0]\) ("lengths" of the chords). The following values of the remaining unknowns lead to three chord problems, solutions of which are illustrated (for \( s = 8 \); the solutions are
valid for \( s \geq 3 \) in Figs. 8, 9 and 10. The values of \( h \) and \( i \) are the ends of the chord of length 6, corresponding to the second step of the zigzag, between \( k_7 = 3 \) and \( k_4 = -3 \), and to the ringle in Fig. 7.

\[
\begin{array}{cccccccccccccccc}
\text{t} & \text{\( k_2 \)} & \text{\( k_3 \)} & \alpha & \beta & \gamma & \delta & \epsilon & \zeta & e & f & g & h & i \\
\hline
\text{Case 6} & 3 & 2s & 1 & 2s & 1 & 2 & 0 & 0 & 2s & 1 & 1 & 2 & 2 & 4 \\
\text{Case 5} & 3 & 2s-1 & 2s+2 & 0 & 0 & 2 & -2 & 0 & 0 & 2s-1 & 1 & 2s & 2s-3 & 2s+3 \\
\text{Case 3} & -1 & 2s+1 & 2s & 0 & 0 & 0 & 0 & 0 & 0 & 2s+1 & 0 & 2s-1 & 2s-3 & 2s+3 \\
\end{array}
\]

The parts of the resulting geometries to be attached (letters \( E, F \)) to the right of Fig. 7 and connected (letters \( B, C \)) as a Möbius strip, are illustrated (for \( s = 4 \)) in Fig. 11 (Case 6), 12 (Case 5), and 13 (Case 3). The left hand version of Fig. 2 occurs as the last rung but two of Fig. 11 and 12, and the right hand version forms the final inverted yung in Fig. 13.

The cyclic order of the vertices adjacent to a given vertex in the imbedding is now obtained by reading round the appropriate circuit on the current graph. For example, \( n = 54 \) (Case 6, \( s = 4 \)): the vertices adjacent to vertex 0 are in the cyclic order given by circuit [0], i.e., starting at \( E \) in Fig. 7:

0: 13 6 36 43 10 12 9 50 48 39 49 Z 47 5 1 X 23 (we have reached \( B \) and continue on Fig. 11) 26 20 46 4 29 17 32 18 37 16 40 27 38 30 45 7 15 21 8 41 42 3 2 44 Y 28 25 31 22 34 19 33 14 35 11 24 (and we have returned to \( E \)).

To find the cyclic order of the vertices adjacent to any other member of \([0]\), say \( 3k \), we add \( 3k \mod 12s + 3 = 51 \) to each member of the cycle for 0, keeping \( X, Y, Z \) fixed. To find the order of the adjacencies to the member \( 3k + 1 \) of \([1]\) we add \( 3k + 1 \mod 12k + 3 \) to the cycle given by circuit [1] viz. (starting at \( C \)): 23 Y 28 X 50 4 Z 2 41 33 46 13 18 8 44 38 11 27 43 16 24 35 21 40 19 30 14 32 15 37 22 36 17 39 49 10 12 29 9 5 25 47 42 20 45 1 7 6 26 3 34 31 48, e.g. 1: 24 Y 29 X 0 5 Z 3 42.... Similarly the
Figure 8 (top), 9 (center), and 10 (bottom).

circuit [2] provides the cycles of adjacencies to members of [2] after addition of $3k + 2$. Finally, the adjacencies to $X$, $Y$, $Z$ can be deduced from the information now available; for example (cf. [17, pp. 199-201]) the cycle for $X$ is $X: 23 0 1 29 6 7 35 12 13...$; the complete cycle is generated from the first triple (one member from each of [2], [0], and [1]) by adding $3\alpha = 6$. Similarly $Y: 28 0 44 16 39 32 4 27 20...$ is generated from any three consecutive members by adding $3\delta = -12$, and $Z: 47 0 49 2 6 4 8 12 10...$ by adding $3\gamma = 6$. 
Case 5 similar, except that there are only two points, $Y$ and $Z$, in addition to those of $Z_{12s+3}$; their cycles are generated by addition of $3\delta = -6$ and $3\gamma = 6$. Case 3 is an "exact" case and uses just the members of $Z_{12s+3}$.

![Figure 11](image1)

![Figure 12](image2)

![Figure 13](image3)

REFERENCES