On the Maxwell–Dirac Equations with Zero Magnetic Field and their Solution in Two Space Dimensions*

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Submitted by Peter D. Lax

Under the assumption of a vanishing magnetic field (curl $A = 0$), a transformation of variables is exhibited which uncouples the Maxwell–Dirac equations. It is then shown that the Cauchy problem in two space dimensions has a unique solution for $C^\infty$ data with compact support.

1. INTRODUCTION

Consider the problem of finding global solutions to the Cauchy problem for the (classical) coupled Maxwell–Dirac equations

$$(-i\gamma^\mu \partial_\mu + M) \Psi = gA_\nu \gamma^\nu \Psi \quad (M, g > 0) \quad (1.1)$$

$$-\Box A_0 = g\Psi^{\nu} \Psi \quad \left( \Box - \Delta - \frac{\partial^2}{\partial t^2} \right)$$

$$-\Box A_k = g\Psi^{\nu} \gamma^0 \gamma^k \Psi \quad (1 \leq k \leq n) \quad (1.2)$$

$$\frac{\partial A_0}{\partial t} + \sum_{k=1}^{n} \frac{\partial A_k}{\partial x_k} = 0.$$

Here $n$ denotes the spatial dimension, $\Psi$ is the Dirac spinor field (i.e., $\Psi$ is a function on space-time into spin space), and the $A_\nu$'s are the (real) electromagnetic potentials. The $\gamma^\mu$'s are operators on spin space satisfying the usual anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu}$$

where $g^{00} = 1$, $g^{kk} = -1$ for $1 \leq k \leq n$, $g^{\mu \nu} = 0$ for $\mu \neq \nu$. In addition, we have the symmetry relations $\gamma^{0*} = \gamma^0$, $\gamma^{\nu*} = -\gamma^\nu$ ($1 \leq k \leq n$), and $\gamma^\mu_j = g^{\mu \nu} \gamma^\nu$.

* Supported by NSF Grant GP 37630.
The solution of this problem in one dimension has been obtained by one of the authors in [2], and its asymptotic behavior was examined in [3]. General conditions for the existence of global solutions to these equations in higher dimensions are unknown (the local problem has been treated by Gross [7]).

The purpose of this article is twofold. We show in Section 2 that the Maxwell-Dirac equations can be uncoupled under the assumption of a vanishing magnetic field. Specifically, we assume that

$$\mathbf{H} = \text{curl} \mathbf{A} = 0$$

where \( \mathbf{A} = (A_1, A_2, A_3) \) (when \( n = 3 \)). We perform a gauge-transformation; that is, a change of variables of the form

$$\Psi = e^{i\varphi(x, t)} \psi,$$

where \( \varphi = \varphi(x, t) \) is an appropriately chosen real scalar function. The resulting nonlinearity in the spinor equation is of the form

$$i\varphi^2 \varphi(\psi) \psi,$$

where \( \varphi(\psi) \) is the Newtonian potential of the charge density \( \psi^\dagger \psi \), and the equations for the \( A_\mu \)'s are invariant. This procedure is independent of the spatial dimension, but sufficient conditions on the Cauchy data to guarantee the vanishing of \( \mathbf{H} \) are unknown at present.

In Section 3, we shall formulate the Maxwell-Dirac equations when \( n = 2 \). In this case, the spinor has two components and the \( \gamma^k \)'s are essentially the Pauli matrices. We shall prove that, given \( C^\infty \) Cauchy data of compact support, these two-dimensional equations have unique global solutions in \( H^2(\mathbb{R}^2) \) (i.e., they have second derivatives lying in \( L^2(\mathbb{R}^2) \)) provided \( \text{curl} \mathbf{A} = 0 \).

The uncoupled equations presented in Section 2 seem to provide an excellent model for the unrestricted Maxwell-Dirac equations, (1.1) and (1.2). This is evidence by our inability at the present time to establish global existence for the corresponding curl-zero version when \( n = 3 \). Although the equations uncouple, the problem of finding a bound for \( \| \nabla \Psi(t) \|_{L^2} \) remains essentially the same as for the full equations, (1.1) and (1.2). This difficulty has prompted the authors to investigate similar nonrelativistic versions of equations with "potential" nonlinearities such as the time-dependent Hartree equations

$$i \frac{\partial \psi}{\partial t} = \frac{1}{2} \Delta \psi + \frac{2}{|x|} \psi - \varphi(\psi) \psi$$

and

$$i \frac{\partial \psi_j}{\partial t} = \frac{1}{2} \Delta \psi_j + \frac{z}{|x|} \psi_j - V_{\alpha\rho} \psi_j,$$
Global existence theorems for these equations have been obtained by the authors and will appear elsewhere (cf. [5]).

We shall employ the usual notation for \( L^p(\mathbb{R}^n) \)-norms:

\[
\| u \|_p = \left[ \int_{\mathbb{R}^n} |u(x)|^p \, dx \right]^{1/p};
\]

\[
\| u \|_\infty = \text{ess sup} \, |u(x)|.
\]

The space \( H^m(\mathbb{R}^n) \) is the Sobolev space of functions with derivatives of order \( \leq m \) lying in \( L^2(\mathbb{R}^n) \) with the norm

\[
\| u \|_{H^m} = \left\{ \sum_{|k| \leq m} \int_{\mathbb{R}^n} |D^k u(x)|^2 \, dx \right\}^{1/2}.
\]

When (1.1) and (1.2) are written as a system, the desired solutions are pairs

\[
\left( \Psi(t), \left( \frac{\partial A_k}{\partial t} / \partial t \right) \right)
\]

lying in certain of the spaces \( H^m \oplus (H^m \oplus H^{m-1}) \) for each \( t \). Ordinary spatial convolution will be denoted by \( * \):

\[
f \ast g(x) = \int_{\mathbb{R}^n} f(y) g(x - y) \, dy.
\]

2. The Case of Zero Magnetic Field

We prefer to write Eqs. (1.1) and (1.2) in a slightly different form. Left-multiplying (1.1) by \( \gamma^\rho \) and using \( (\gamma^\rho)^2 = I \), we obtain

\[
\Psi_t = \sum_k \alpha_k \Psi_{x_k} + M \beta \Psi + igA_0 \Psi + ig \sum_k A_k \alpha_k \Psi
\]

\[
-\Box A_0 = g^{\rho} \Psi \Psi \tag{2.1}
\]

\[
-\Box A_k = -g^{\rho} \Psi \alpha_k \Psi \quad (k = 1, 2 \text{ or } k = 1, 2, 3) \tag{2.2}
\]

\[
\frac{\partial A_0}{\partial t} + \sum_k \frac{\partial A_k}{\partial x_k} = 0
\]

where \( \alpha_k = -\gamma^\rho \gamma^k \), \( \beta = -i\gamma^0 \).
Consider the effect of a change of variables of the form

$$\psi = e^{i\varphi} \Psi,$$

where $\varphi = \varphi(x, t)$ is a real scalar-valued function to be chosen later. A direct calculation gives

$$\psi_t = \sum_k \alpha_k \psi_{x_k} + M \beta \psi + i g \left( \varphi_t - \sum_k \varphi_{x_k} \alpha_k \right) \psi + e^{i \varphi} \left( \Psi_t - \sum_k \alpha_k \Psi_{x_k} - M \beta \Psi \right).$$

Substituting from (2.1) in the last term we obtain

$$\psi_t = \sum_k \alpha_k \psi_{x_k} + M \beta \psi + i g (A_0 + \varphi_t) \psi + i g \sum_k (A_k - \varphi_{x_k}) \psi.$$

Now suppose that no magnetic field is present, i.e., $H = \text{curl } A = 0$. Then, taking $n = 3$ for definiteness, we know that there exists a function $\Phi$ such that $A = \nabla \Phi$. We choose $\Phi = \varphi$ so that the above equation becomes

$$\psi_t = \sum_k \alpha_k \psi_{x_k} + M \beta \psi + i g (A_0 + \varphi_t) \psi.$$

Now from the gauge-invariance we get

$$0 = \frac{\partial A_0}{\partial t} + \sum_{k=1}^3 \frac{\partial A_k}{\partial x_k} = \frac{\partial A_0}{\partial t} + \Delta \varphi.$$

Hence,

$$\varphi = \frac{1}{4\pi r} \frac{\partial A_0}{\partial t},$$

where $r = |x|$. Then using the first of Eqs. (2.2), we find

$$\varphi_t - \frac{1}{4\pi r} \frac{\partial^2 A_0}{\partial t^2} = \frac{1}{4\pi r} \Delta A_0 + \frac{g}{4\pi r} * |\psi|^2$$

$$= \Delta \left( \frac{1}{4\pi r} \right) * A_0 + \frac{g}{4\pi r} * |\psi|^2$$

$$= -\delta * A_0 + \frac{g}{4\pi r} * |\psi|^2.$$

Thus, $\varphi_t + A_0 = (g/(4\pi r)) * |\psi|^2$ and the equation for $\psi$ becomes,

$$\psi_t = \sum_k \alpha_k \psi_{x_k} + M \beta \psi + i g^2 c \psi,$$

\( (2.3) \)
where
\[ v = \frac{1}{4\pi r} \ast | \psi |^2 = \frac{1}{4\pi} \int \frac{|\psi(x, t)|^2}{||x - y||} dy. \]

Notice also that since \( \varphi \) is a real scalar function, we have
\[ \Psi^\dagger \Psi = \psi^\dagger \psi \quad \text{and} \quad \Psi^\dagger \alpha_k \Psi = \psi^\dagger \alpha_k \psi \]
so that the equations for the \( A_k \)'s are invariant. Of course, when \( n = 2 \), we will have
\[ v = \frac{1}{2\pi} \log \frac{1}{r} \ast | \psi |^2. \]

Since Eq. (2.3) is independent of the \( A_k \)'s, it will suffice to solve it, and then treat Eqs. (2.2) as nonhomogeneous wave equations. In summary, the "curl zero" forms of (1.1) and (1.2) are
\[ \psi_k = \sum_k \alpha_k \psi_{x_k} + M \beta \psi + ig \psi \quad \text{(2.3)} \]
\[ -\Delta A_0 = g \psi^\dagger \psi, \quad -\Delta A_k = -g \psi^\dagger \alpha_k \psi \quad \text{(2.4)} \]
where
\[ v = \frac{1}{4\pi r} \ast | \psi |^2 \quad \text{if} \quad n = 3 \]
\[ = \frac{1}{2\pi} \log \frac{1}{r} \ast | \psi |^2 \quad \text{if} \quad n = 2. \]

We remark that it is impossible to obtain the linear Dirac equation for \( \psi \) for non-trivial solutions. It is easily seen that simultaneous satisfaction of the equations
\[ \varphi_t + A_0 = 0, \quad \varphi_{x_k} - A_k = 0 \quad (k = 1, 2, 3) \]
implies \( g \psi^\dagger \psi = 0. \)

3. Formulation of the Two-Dimensional Problem and Its Solution

When \( n = 2 \), the Maxwell–Dirac equations retain the same form as (1.1) and (1.2). In this case, the spinor \( \Psi \) has two components
\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]
and the $\gamma$'s are $2 \times 2$ matrices satisfying the anticommutation and symmetry relations. Specifically, consider the Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
We may take $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$. It is easily verified that these choices satisfy the requisite conditions imposed above. These particular choices lead to
\[
\beta = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]
for Eqs. (2.1) and (2.2). However, the results obtained below are independent of the particular representation chosen.

Let us now consider the existence of global solutions to (2.1) and (2.2) for $n = 2$ under the condition $\text{curl } A = 0$. Throughout this section we shall assume that the Cauchy data
\[
A_k(0), \quad \psi(x, 0) = \psi_0(x)
\]
and $\Psi(x, 0) \equiv \Psi_0(x)$ lie in $C^n_{\text{loc}}(\mathbb{R}^2)$. The integrated forms of (2.1) and (2.2) are
\[
\Psi(t) = \Psi^{(0)}(t) + ig \int_0^t D(t - \tau) \ast A_0(\tau) \Psi(\tau) \, d\tau + \sum_{\kappa=1}^2 A_\kappa(\tau) \alpha_\kappa \Psi(\tau) \, d\tau \quad (3.1)
\]
\[
A_k(t) = A_k^{(0)}(t) - g \int_0^t M(t - \tau) \ast \Psi^* \alpha_k \Psi(\tau) \, d\tau. \quad (3.2)
\]
($k = 0, 1, 2; \alpha_0 = -I$). Here $D(\cdot)$ and $M(\cdot)$ are the Dirac and Maxwell propagators, and $\Psi^{(0)}(t)$, $A_k^{(0)}(t)$ are the free solutions with the same Cauchy data as $\Psi$ and the $A_k$'s.

**Lemma 1.** Equations (3.1) and (3.2) have unique local solutions in $H^m \oplus (H^m \oplus H^{m-1})$ for all $m \geq 2$, and thus are $C^\infty$ on their interval of existence.

**Proof.** The existence result follows immediately from Segal's theorem [8] because the Lipschitz character of the nonlinearities above on $H^m \oplus (H^m \oplus H^{m-1})$ is a consequence of the Sobolev inequalities when $n = 2$ and [3, Lemma 2.2]. We omit the details of the argument since they are contained essentially in [3]; see also [2].
Now consider the global problem. In view of Segal's theorem \cite{8} we need to show that the norms

\[ \| \Psi'(t) \|_{L^2} \quad \text{and} \quad \left\| \frac{A_k(t)}{\partial t} \right\|_{L^2 \oplus L^1} \]

remain finite at each \( t \). In order to do this, we make the change of variables as in Section 2 and examine Eqs. (2.3) and (2.4). From the well-known charge conservation

\[ \| \Psi'(t) \|_2 = \| \Psi(0) \|_2 \]

(see \cite{2}) we also have

\[ \| \psi(t) \|_2 = \| \psi(0) \|_2 . \]

**Lemma 2.** Let \( \psi \) be a solution of (2.3). Then

\[ \frac{d}{dt} \| \nabla \psi(t) \|_2^2 = -2g^2 \sum_{j=1}^{2} \text{Im} \int v_{x_j} \psi^*_x \psi dx. \]

**Proof.** Differentiating (2.3) we get

\[ \psi_{tx_j} = \sum_k \alpha_k \psi_{x_k x_j} + M \beta \psi_{x_j} + ig^2 v_{x_j} \psi + ig^2 v_{x_{x_j}} \]

so that

\[ \psi_{x_j} \psi_{tx_j} = \sum_k \psi^*_{x_j} \alpha_k \psi_{x_k x_j} + M \psi_{x_j} \beta \psi_{x_j} + ig^2 v_{x_j} \psi^*_{x_j} \psi + ig^2 v_{x_j} | \psi_{x_j} |^2 . \]

We take the conjugate transpose of this and add the result to the above equation. Since \( \alpha_k^* = -\alpha_k, \beta^* = -\beta \) we obtain

\[ \frac{\partial}{\partial t} | \psi_{x_j} |^2 = \sum_k \frac{\partial}{\partial x_k} (\psi_{x_j x_k} \psi_{x_j}) - 2g^2 \text{Im} v_{x_j} \psi^*_{x_j} \psi . \]

An integration over \( \mathbb{R}^2 \) then gives the result.

**Lemma 3.** Let \( \psi(x, t) \) be a solution of (2.3) with \( \psi(x, 0) = \psi_0(x) \in C_0^\infty(\mathbb{R}^2) \). Then there exists a locally bounded function \( R_1(\cdot) \) such that

\[ \| \psi(t) \|_{L^2} \leq \| \nabla \psi(0) \|_2 \text{ exp} \left( \int_0^t R_1(\xi) d\xi \right) \quad \text{for all } t \geq 0. \]

**Proof.** We know that \( \| \psi(t) \|_2 = \| \psi_0 \|_2 \). To obtain a bound on \( \| \nabla \psi(t) \|_2 \) we employ Lemma 2:

\[ \frac{d}{dt} \| \nabla \psi(t) \|_2^2 \leq \text{const.} \int | \nabla \psi(t) | | \nabla \psi(t) | | \psi(t) | dx. \]

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Now \( \psi(x, t) \) has compact support in \( x \) for each fixed \( t \). Thus, we may employ the Sobolev inequality (cf. [1, p. 220, part b] with \( q_1 = \infty, \lambda = 1, n = 2, q = 2 \)) to estimate \( \| \nabla \psi(t) \|_\infty \):

\[
\| \nabla \psi(t) \|_\infty = \text{const.} \left\| \frac{1}{r} \ast |\psi|^2 \right\|_\infty \leq \text{const.} R(t) \| \psi \|^{\frac{q}{q-1}}(t) 2
\]

where \( R(t) \) is a locally bounded function depending on the support of \( \psi(x, t) \).

By the ordinary Sobolev inequality for \( n = 2 \),

\[
\| \psi \|^{2}(t) \leq \text{const.} \| \nabla (|\psi|^2) \|_1 \leq \text{const.} \| \psi(t) \|_2 \| \nabla \psi(t) \|_2 \\
\leq \text{const.} \| \nabla \psi(t) \|_2 .
\]

Thus, \( \| \nabla \psi(t) \|_\infty \leq \text{const.} R(t) \| \nabla \psi(t) \|_2 \). Returning to (3.3), we then find

\[
\frac{d}{dt} \| \nabla \psi(t) \|_2^2 \leq \text{const.} \| \nabla \psi(t) \|_\infty \| \nabla \psi(t) \|_2 \| \psi(t) \|_2 \\
\leq \text{const.} R(t) \| \nabla \psi(t) \|_2^2 \\
= R_1(t) \| \nabla \psi(t) \|_2^2 .
\]

By Gronwall's inequality,

\[
\| \nabla \psi(t) \|_2^2 \leq \| \nabla \psi(0) \|_2^2 \exp \left( \int_0^t R_1(\xi) \, d\xi \right)
\]

which proves the result.

We remark that it is precisely the special convolution form of \( \psi \) which allows us to estimate \( \| \nabla \psi(t) \|_\infty \) in term of \( \| \nabla \psi(t) \|_2 \). This is not possible for the unrestricted versions (2.1), (2.2) even with \( n = 2 \). Moreover, when \( n = 3 \) the Sobolev inequality

\[
\int \frac{|\psi|^2}{|x-y|^2} \, dy \leq 4 \| \nabla \psi \|_2^2
\]

(cf. [6], p. 446) gives

\[
\| \nabla \psi \|_\infty = \text{const.} \left\| \nabla \left( \frac{1}{4\pi r} \ast |\psi|^2 \right) \right\|_\infty \\
\leq \text{const.} \| \nabla \psi(t) \|_2^2 ;
\]

this estimate is clearly not strong enough and precludes our applying Gronwall’s inequality as above. Thus, when \( n = 3 \), a bound on \( \| \nabla \psi(t) \|_2 \) for a solution \( \psi \) of (2.3) remains unknown.
LEMMA 4. Let $\psi$ be as above and let the $A_k$ ($k = 0, 1, 2$) be solutions of (3.2) (or (2.4)). Then the norms

$$\| \frac{\partial A_k}{\partial t} (t) \|_2^2 + \| A_k(t) \|_{H^1}$$

are finite for each $t \geq 0$.

Proof. For the moment let us set

$$\| A_k(t) \|_e^2 = \| \frac{\partial A_k}{\partial t} (t) \|_2^2 + \| \nabla A_k(t) \|_2^2.$$

From (3.2), we have

$$A_k(t) = A_k^{(0)}(t) - g \int_0^t M(t - \tau) * \psi^* \alpha_k \psi(t) \, d\tau.$$

It is well known that

$$\| M(t - \tau) * F(t) \|_e = \| F(\tau) \|_e$$

(cf. [9]). Hence,

$$\| A_k(t) \|_e \leq \| A_k^{(0)}(t) \|_e + g \int_0^t \| \psi^* \alpha_k \psi(\tau) \|_2 \, d\tau$$

$$\leq \text{const.} \left( 1 + \int_0^t \| \psi^* \psi(\tau) \|_2 \, d\tau \right).$$

The Sobolev inequality gives

$$\| \psi^* \psi(\tau) \|_2 \leq \text{const.} \| \nabla (\psi^* \psi)(\tau) \|_1$$

$$\leq \text{const.} \| \nabla \psi(\tau) \|_2 \| \psi(\tau) \|_2$$

$$\leq \text{const.} \| \nabla \psi(\tau) \|_2.$$

Therefore, $\| A_k(\tau) \|_e$ is an exponentially bounded function of $t$. To obtain a bound on $\| A_k(t) \|_2^2$ we multiply any of equations (2.4) by $A_k$ and integrate to get

$$\frac{1}{2} \frac{d^2}{dt^2} \| A_k(t) \|_2^2 = \int \left[ \left( \frac{\partial A_k}{\partial t} (t) \right)^2 - \| \nabla A_k(t) \|_2^2 - g A_k \psi^* \alpha_k \psi(t) \right] \, dx.$$

By the above argument, $\| A_k(t) \|_e^2$ is exponentially bounded. Furthermore we have

$$\| A_k \psi^* \alpha_k \psi(t) \|_e \leq \text{const.} \| A_k(t) \|_e \| \psi(t) \|_2 \| \psi(t) \|_2$$

$$\leq \text{const.} \| \nabla A_k(t) \|_2 \| \psi(t) \|_2 \| \psi(t) \|_2 \| \nabla \psi(t) \|_2 \| \psi(t) \|_2.$$


Two integrations then show that $\| A_k(t) \|_2$ is exponentially bounded, and this proves the lemma. We have now proved

**Theorem 1.** Let

$$\left( \psi(t), \left( \frac{\partial A_k}{\partial t}(t) \right) \right)$$

be a solution of the system (2.3), (2.4) with $n = 2$ and $C^\infty$ Cauchy data with compact support. Then the norms

$$\| \psi(t), A_k(t), \frac{\partial A_k}{\partial t}(t) \|_{H^1(H^3 \oplus L^2)}$$

are finite at each $t \geq 0$.

It remains for us to show that the norm $\| \nabla \Psi(t) \|_2$ of the original solution remains finite, and that appropriate $H^2$-bounds can be found.

**Lemma 5.** Under the above conditions, $\| \Psi(t) \|_{H^1}$ is finite at each $t \geq 0$.

**Proof.** We need only consider $\| \nabla \Psi(t) \|_2$. By definition we have

$$\Psi = e^{-i\varphi_\omega t}\psi$$

where $\varphi_\omega = \lambda_k, k = 1, 2$. Thus,

$$\| \nabla \Psi(t) \|_2 \leq \text{const.}(\| \nabla \varphi(t) \|_2 + \| \nabla \psi(t) \|_2)$$

$$\leq \text{const.}(\| A_k(t) \phi(t) \|_2 + \| \nabla \psi(t) \|_2)$$

$$\leq \text{const.}(\| A_k(t) \|_2 \| \phi(t) \|_2 + \| \nabla \psi(t) \|_2)$$

$$\leq \text{const.}(\| \nabla A_k(t) \|_2 \| \psi(t) \|_2^{1/2} + \| \nabla \psi(t) \|_2^{1/2} + \| \nabla \psi(t) \|_2).$$

All of these quantities are appropriately bounded by the results of Lemmas 3 and 4.

Finally, we show that the second derivatives can be estimated in $L^2(\mathbb{R}^2)$:

**Lemma 6.** Under the above conditions, the norms $\| \phi(t) \|_{H^2}, \| A_k(t) \|_{H^2}$, and $\| \Psi(t) \|_{H^2}$ are finite at each $t \geq 0$.

**Proof.** We formally apply the $\Delta$ operator to (2.3) to get

$$\Delta \psi_t = \sum_k \alpha_k \Delta \varphi_{\psi_k} + M_\beta \Delta \psi + i g^2 \Delta (v \psi)$$

$$= \sum_k \alpha_k \Delta \varphi_{\psi_k} + M_\beta \Delta \psi + i g^2 (\Delta v \psi + 2 \nabla v \cdot \nabla \psi + v \Delta \psi).$$
This calculation is justified by a limiting procedure since $\Delta v$ exists. Left multiplication by $\Delta \psi^*$ then yields

$$\Delta \psi^* \Delta \psi = \sum_k \Delta \psi^* \gamma_k \Delta \psi_{x_k} + M \Delta \psi^* \beta \Delta \psi + ig^2 \Delta \psi^* \psi$$

$$+ 2ig^2 \Delta \psi^* \nabla v \cdot \nabla \psi + ig^2 v | \Delta \psi|^2.$$

We take the conjugate transpose of this and add the result to the above equation; this gives

$$\frac{\partial}{\partial t} | \Delta \psi |^2 = \sum_k \frac{\partial}{\partial x_k} (\Delta \psi^* \gamma_k \Delta \psi) + ig^2 \Delta v (\Delta \psi^* \psi - \psi^* \Delta \psi)$$

$$+ 2ig^2 (\Delta \psi^* \nabla v \cdot \nabla \psi - \nabla v \cdot \nabla \psi^* \Delta \psi).$$

An integration produces

$$\frac{d}{dt} \| \Delta \psi(t) \|_6^2 = 2g^2 \text{Im} \int \psi^* \Delta \psi \Delta v \, dx + 4g^2 \text{Im} \int \nabla v \cdot \nabla \psi^* \Delta \psi \, dx.$$

Now $\Delta v = \text{const.} \| \psi \|^2$ so that

$$\| \psi^* (t) \Delta \psi(t) \Delta v(t) \|_1 \leq \text{const.} \| \psi(t) \|^3 \| \Delta \psi(t) \|_1$$

$$\leq \text{const.} \| \psi(t) \|^2 \| \Delta \psi(t) \|_2$$

$$\leq \text{const.} (\| \psi(t) \|_6^2 + \| \Delta \psi(t) \|_6^2).$$

Now $\| \psi(t) \|_6$ is exponentially bounded by the Sobolev inequality and Lemma 3. Moreover, we have

$$\| \nabla v(t) \cdot \nabla \psi^* (t) \Delta \psi(t) \|_1 \leq \| \nabla v(t) \|_\infty \| \nabla \psi(t) \|_2 \| \Delta \psi(t) \|_2$$

$$\leq \text{const.} (\| \psi(t) \|_6^2 + \| \Delta \psi(t) \|_6^2).$$

by the proof of Lemma 3. Splitting this as above we obtain

$$\| \nabla v(t) \cdot \nabla \psi^* (t) \Delta \psi(t) \|_1 \leq \text{const.} (R \| \psi(t) \|_6^2 + \| \Delta \psi(t) \|_6^2).$$

Now Gronwall's inequality can be applied and yields the finiteness of $\| \Delta \psi(t) \|_2$, and hence, also $\| \psi(t) \|_{H^2}$, for all $t \geq 0$. Again using (3.2) we find

$$\left\| \frac{\partial}{\partial x_j} A_k(t) \right\|_e \leq \left\| \frac{\partial}{\partial x_j} A^{(0)}_k(t) \right\|_e + g \int_0^t \left\| \frac{\partial}{\partial x_j} \psi^* \psi \right\|_2 \, d\tau$$

$$\leq \text{const.} \left(1 + \int_0^t \| \psi \nabla \psi \|_2 \, d\tau \right)$$

$$\leq \text{const.} \left(1 + \int_0^t \| \psi(\tau) \|_e \| \nabla \psi(\tau) \|_2 \, d\tau \right).$$
Now \(\| \phi(t) \|_{\infty} \leq \text{const.} \| \phi(t) \|_{L^2} \) by the Sobolev inequality. It follows that 
\( \| (\partial / \partial x_j) A_k(t) \|_{\infty} \) is appropriately bounded for \( k = 0, 1, 2 \) and \( j = 1, 2 \). Thus, 
\( \| A_k(t) \|_{L^2} \) is finite at each \( t \geq 0 \).

Finally, to obtain \( L^2 \)-bounds on the second derivatives of \( \phi \), we revert to its definition and find

\[
\Delta \phi = \Delta (e^{-i\phi t}) \phi + 2 \nabla (e^{-i\phi t}) \cdot \nabla \phi + e^{-i\phi t} \Delta \phi.
\]

Now \( \| e^{-i\phi t} \Delta \phi(t) \|_{L^2} = \| \Delta \phi(t) \|_{L^2} \) is finite by the above argument. Also \( \nabla e^{-i\phi t} = -i \nabla e^{-i\phi t} \) so that

\[
\| \nabla (e^{-i\phi t}) \cdot \nabla \phi(t) \|_{L^2} \leq \text{const.} \| \nabla \phi(t) \|_{L^2} \leq \text{const.} \| A_k(t) \nabla \phi(t) \|_{L^2} \leq \text{const.} \| A_k(t) \|_{L^2} \| \nabla \phi(t) \|_{L^2} \leq \text{const.} \| A_k(t) \|_{L^2} \| \phi(t) \|_{H^1}.
\]

For the first term,

\[
\Delta (e^{-i\phi t}) = -i g \nabla \cdot (\nabla e^{-i\phi t})
\]

\[
= -i g (\Delta e^{-i\phi t} - i g e^{-i\phi t} \nabla \phi^2)
\]

\[
= i g e^{-i\phi t} \left( \frac{\partial A_0}{\partial t} + i g \sum_{k=1}^2 |A_k|^2 \right).
\]

Hence

\[
\| \Delta (e^{-i\phi t}) \phi(t) \|_{L^2} \leq \text{const.} \left( \left\| \frac{\partial A_0}{\partial t} (t) \phi(t) \right\|_{L^2} + \| A_k^2(t) \phi(t) \|_{L^2} \right)
\]

\[
\leq \text{const.} \left( \| \phi(t) \|_{L^2} (\| A_0(t) \|_{L^2} + \| A_k^2(t) \|_{L^2}) \right).
\]

All of the above quantities are finite at each \( t \geq 0 \). This concludes the proof of the lemma.

Putting all of the above results together, we can state

**Theorem 2.** Let \( n = 2 \). If \( \text{curl} \, A = 0 \) and the Cauchy data

\[
\left( \psi_0, \left( \frac{A_k(0)}{\partial t}, \frac{A_k(0)}{\partial x_j} \right) \right)
\]
is $C^\infty$ with compact support, then the (integrated form of the) coupled Maxwell–Dirac equations (2.1), (2.2) have a unique global solution in $H^2 \oplus (H^2 \oplus H^1)$.

This higher-dimensional existence result complements that obtained by the authors in [4] for the coupled Klein–Gordon–Dirac equations. We conjecture that a similar uncoupling is possible for those equations where the new nonlinearity $v = v(\psi)$ should take the form of the Yukawa potential

\[ v = \frac{e^{-r}}{r} \ast |\psi|^2. \]

REFERENCES