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The best parameterization for parametric interpolation ☆

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Abstract

We consider the problem of interpolation of curves and surfaces using parametric functions. In this problem the choice of parameters has a strong influence on the form of interpolants. We formulate and prove the necessary and sufficient conditions of the parameterization that provides the best quality of interpolation.

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1. Introduction

The problem of interpolation plays an extremely important role in modern computation mathematics because of many numerical methods are based on the concept of interpolation.

Parametric interpolation has certain advantages over the other interpolation methods. It does not put any restrictions on a set of nodal points and allows to interpolate curves and surfaces of complicated shape. Parametric interpolation is used in computer graphics, in computer-aided design systems, in software for typography, for numerical controlled cutters and in many other applications. The advantages of parametric interpolation are discussed in details in [1–3,6].

The specificity of the parametric interpolation of curves is that usually in practice only ordered arrays of points to be interpolated are given and there is no information on the methods of parameterization needed to construct interpolants. Values of a parameter in nodal points are only need to form a monotone sequence. In [1,2,6] it was established in a study of parametric splines that the choice of a parameter has a critical effect on the interpolation curve, which might even contain loops in some cases. For example, numbers of nodal point can be taken as values of a parameter in nodal points (see [1,2]). The advantage of such parameterization is that it simplify spline manipulations, but the quality of the interpolation will not be good if the nodal points set has a strong variation in the distance between points. For interpolation of ellipses with parametric splines the polar angle was taken as a parameter in [6]. In [1,3,6] it is recommended to use the length of a broken line joining nodal points or an arc length of a curve as a parameter, in [2] it is recommended to use the centripetal parameterization. In [5] it is proved the necessary and sufficient conditions of choosing the best parameter, which is the arc length of interpolation curve.

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For parameterization of the surface it is necessary to set on the surface a grid of parametrical lines defining the directions of measurement of parameters and to set values of parameters in nods of the grid. In [6] in the problem of interpolation of surfaces with parametric splines the length of a broken line joining nodal points used for parameterization of grid lines, in [1,2]—the length of a broken line or numbers of nodal points. In this paper the necessary and sufficient conditions for the best parameterization of the surface are proved. The best parameters are measured along orthogonal curves, laying on the surface, and they are the lengths of arches of these curves.

In Section 2 we consider the best parameterization for the problem of parametric interpolation of curves and in Section 3—for the problem of parametric interpolation of surfaces.

2. Parametric interpolation of curves

The problem of parametric interpolation of curves consists in the construction of two functions

$$x = X(t), \quad y = Y(t), \tag{1}$$

where t is a parameter.

For certain selected values of the parameter t_i, $i = \overline{0, n}$, $t_0 < t_1 < \cdots < t_n$, the functions (1) must satisfy conditions

$$X(t_i) = x_i, \;\; Y(t_i) = y_i, \quad i = \overline{0, n}.$$

In this formulation the choice of the values t_i is not unique. This fact allows to raise the question of choosing the best (in some sense) parameter.

Theorem 1. *In order to formulate the problem of parametric interpolation of the curve with respect to the best parameter, it is necessary and sufficient to choose the arc length of the interpolated curve as this parameter.*

Proof. *Necessity.* If the equation of the curve to be interpolated

$$F(x, y) = 0, \tag{2}$$

which satisfy conditions

$$F(x_i, y_i) = 0, \quad i = \overline{0, n}$$

is known, the parametric equations (1) of the curve can be constructed by the method of continuation of the solution with respect to a parameter.

We introduce a parameter t that is measured along the axis determined by identify vector $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}}$,

$$\alpha_1^2 + \alpha_2^2 = 1. \tag{3}$$

The differential of the parameter can be presented in the form

$$\mathrm{d}t = \alpha_1 \, \mathrm{d}x + \alpha_2 \, \mathrm{d}y. \tag{4}$$

Dividing Eq. (4) by dt and differentiating Eq. (2) with respect to the parameter we obtain the system of continuation equations

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ F_x & F_y \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5}$$

where $F_x = \partial F/\partial x$, $F_y = \partial F/\partial y$, $x_t = \mathrm{d}x/\mathrm{d}t$, $y_t = \mathrm{d}y/\mathrm{d}t$.

For realization of continuation process it is necessary to resolve system (5) with respect to derivatives. The solution of the system can be written by Cramer's rule as

$$x_t = \frac{F_y}{\alpha_1 F_y - \alpha_2 F_x}, \quad y_t = \frac{-F_x}{\alpha_1 F_y - \alpha_2 F_x}. \tag{6}$$

As soon as the quality of the parameter t is related with decidability of system (5) with respect to derivatives then it is naturally to connect the quality of the parameter with the conditionality of the matrix of this system.

The parameter providing the best conditionality to the linearized system of continuation equations (5) will be named as the best parameter.

As the measure of conditionality of the matrix we take the module of its determinant divided by product squared measures of its rows (see [4])

$$|D| = \frac{|\alpha_1 F_y - \alpha_2 F_x|}{\sqrt{F_x^2 + F_y^2}}. \tag{7}$$

The better conditionality corresponds to the greater value of $|D|$.

Having find the extremum of the value D as a function of α_1 and α_2 under condition (3) we get that $|D|$ attains its maximum when

$$\alpha_1 = x_t, \quad \alpha_2 = y_t. \tag{8}$$

Substituting values (8) to the expression for differential of parameter (4) and multiplying the equation by dt we get the equality for differential of the arc length

$$(\mathrm{d}t)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2. \tag{9}$$

Sufficiency. Let us take the arc length of the interpolated curve as the parameter t. $\tau = (x_t, y_t)^{\mathrm{T}}$ is the tangential to the curve vector. As it was noted previously the meaning of the vector α is that it determines the local direction of measurement of the parameter. Therefore, on account of the chosen parameter it must be pointed along the tangent to the curve, i.e., the vectors α and τ must be collinear. But they are not only collinear. They are equal because the vector τ is unity vector too. Indeed, being the curve length element, the differential of the chosen parameter must satisfy equality (9). Divided by $(\mathrm{d}t)^2$ this equality gives $x_t^2 + y_t^2 = \tau^2 = 1$.

Since vectors are equal then their components are equal too (8). Equalities (8) ensure the extremes of the measure of conditionality of continuation equation system. The theorem is proved. □

The arc length we will denote by λ .

If Eq. (2) is given, the interpolation functions of the best parameter $x = X(\lambda)$ and $y = Y(\lambda)$ can be obtained as the solution of the system of continuation equations

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2 = 1,$$

$$F_x \frac{\mathrm{d}x}{\mathrm{d}\lambda} + F_y \frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0.$$

This system can be solved with respect to derivatives $\mathrm{d}x/\mathrm{d}\lambda$, $\mathrm{d}y/\mathrm{d}\lambda$ and the problem of constructing interpolating functions can thus be reduced to the solving of the following Cauchy problem:

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda} = \frac{F_y}{\sqrt{F_x^2 + F_y^2}}, \quad x(0) = x_0,$$

$$\frac{\mathrm{d}y}{\mathrm{d}\lambda} = \frac{-F_x}{\sqrt{F_x^2 + F_y^2}}, \quad y(0) = y_0. \tag{10}$$

The parameter λ here is taken from the first interpolation node (x_0, y_0). Some advantages of such a formulation of the Cauchy problem have been noted in [5].

If Eq. (2) of interpolation curve is unknown, the nodal values λ_i cannot be computed before the functions $X(\lambda)$ and $Y(\lambda)$ have been constructed. But, in turn, in order to construct the functions $X(\lambda)$ and $Y(\lambda)$ the nodal values λ_i must be assigned.

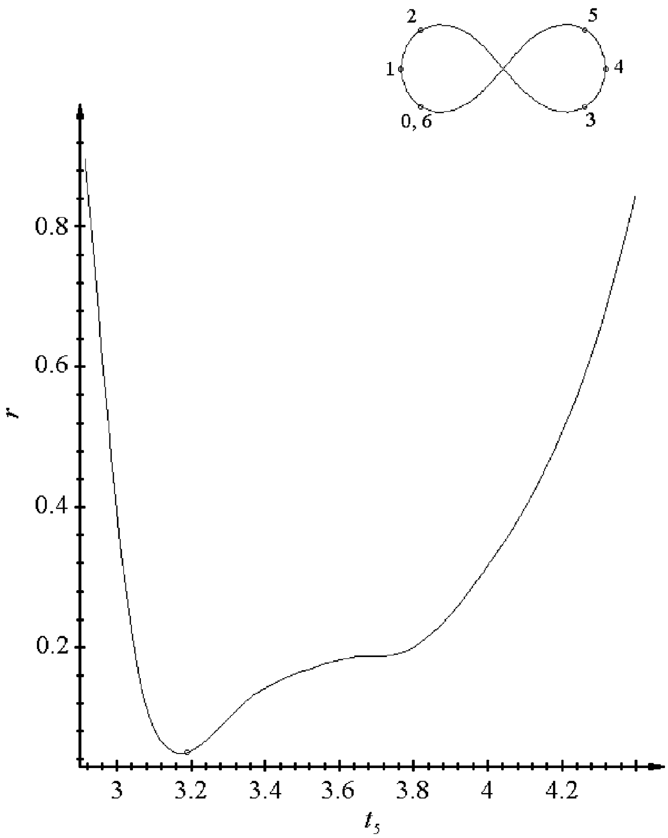

Fig. 1.

In this situation to a first approximation the nodal values of the parameter $\lambda_i^{(1)},\ i=\overline{0,n}$, can be taken as the length of the broken line joining the points $(x_i, y_i),\ i=\overline{0,n}$, in the x, y plane, starting from point (x_0, y_0), that is, by the recurrence formula

$$\lambda_0^{(1)}=0,\ \lambda_i^{(1)}=\lambda_{i-1}^{(1)}+\Delta\lambda_i,\ \Delta\lambda_i=\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2},\quad i=\overline{1,n}.$$

Once interpolation functions of the first approximation $X^{(1)}(\lambda),\ Y^{(1)}(\lambda)$ have been constructed, the accuracy of the nodal values λ_i can be improved:

$$\lambda_0^{(2)}=0,\ \lambda_i^{(2)}=\lambda_{i-1}^{(2)}+\int_{\lambda_{i-1}^{(1)}}^{\lambda_i^{(1)}}\sqrt{(\mathrm{d}X^{(1)}/\mathrm{d}\lambda)^2+(\mathrm{d}Y^{(1)}/\mathrm{d}\lambda)^2}\,\mathrm{d}\lambda,\quad i=\overline{1,n},$$

and interpolation functions of the second approximation $X^{(2)}(\lambda)$, $Y^{(2)}(\lambda)$ can be constructed, and so on.

In most cases for actual use the first approximation $X^{(1)}(\lambda)$, $Y^{(1)}(\lambda)$ is quite sufficient.

Example 2. As a test example we present the interpolation of Bernoulli lemniscate with parametrical cubic splines using different parameters.

The implicit Bernoulli lemniscate's equation in coordinates (x, y) is

$$F(x,y)=(x^2+y^2)^2-(x^2-y^2)=0. \tag{11}$$

The interpolation realized by seven nodal points designated in Fig. 1. The last point coincides with the first.

The estimations of closeness of the sets of points of the parametrical curves to the set of points of the initial curve that is given by Eq. (11) was calculated in the Hausdorff metric. The Hausdorff distance between two bounded sets of points E and F is calculated by the formula

$$r(E, F) = \max\left[\max_{P\in E}\ \min_{Q\in F}\ \rho(P, Q), \max_{P\in F}\ \min_{Q\in E}\ \rho(P, Q)\right]. \tag{12}$$

Here $\rho(P, Q)$ is the Euclidean distance between points P and Q.

The error of interpolation with using the length of broken line as a parameter ($t_0 = 0,\ t_i = t_{i-1} + \rho(P_{i-1}, P_i),\ i = \overline{1, n}$) is $r = 0.05$, with using the numbers of nodal points as values of parameter in nodal points ($t_i = i,\ i = \overline{0, n}$) is $r = 0.15$ and with using the centripetal parameterization ($t_0 = 0,\ t_i = t_{i-1} + \sqrt{\rho(P_{i-1}, P_i)},\ i = \overline{1, n}$) is $r = 0.11$. I.e. at the use of the close to best parameterization the interpolation error more than twice less than the interpolation error at the use of centripetal parameterization and in three times less then the interpolation error at the use of integer parameterization.

However, the error of interpolation attains the least value $r = 10^{-5}$ when functions (1) satisfy the initial value problem (10), i.e., when we use the best parameter λ.

Now we shall review the case, when in one of the nodal points the value of the parameter is differ from the best and the values of the parameter in other points are the best. The graphic on the Fig. 1 represents the dependence of the calculated by formula (12) interpolation error from the value of the parameter at the point 5. The length of the broken line, connecting the nods 0–5 is $\lambda_5 = 3.19$. The graphic shows that the interpolation error achieves its minimum $r = 0.05$ when $t_5 = \lambda_5$.

3. Parametric interpolation of surfaces

The problem of parametric interpolation of surfaces consists in the construction of functions of two parameters

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v), \tag{13}$$

satisfying conditions

$$x_{ij} = X(u_i, v_j),\ y_{ij} = Y(u_i, v_j),\ z_{ij} = Z(u_i, v_j), \quad i = \overline{0, n},\ j = \overline{0, n}.$$

We prove the following theorem.

Theorem 3. *In order to formulate the problem of parametric interpolation of the surface with respect to the best parameters, it is necessary and sufficient to choose as these parameters the lengths of arches calculated along two orthogonal curves, laying on this surface.*

Proof. *Necessity*. If the equation

$$F(x, y, z) = 0 \tag{14}$$

of the surface is given, parameterization (13) of this surface can be obtained by the method of parametric continuation.

We introduce parameters u, v, such that variables x, y, z are differentiable functions of these parameters. Then differentials of parameters can be presented in the form

$$\mathrm{d}u = \alpha_1\,\mathrm{d}x + \alpha_2\,\mathrm{d}y + \alpha_3\,\mathrm{d}z, \quad \mathrm{d}v = \beta_1\,\mathrm{d}x + \beta_2\,\mathrm{d}y + \beta_3\,\mathrm{d}z. \tag{15}$$

Here vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ and $\beta = (\beta_1, \beta_2, \beta_3)^{\mathrm{T}}$ define the directions, along which the parameters u and v are measured. To ensure that all the directions are equivalent, we will define them by unit vectors, so we have

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \tag{16}$$

If we shall write down the expression for differential of Eq. (14) together with Eqs. (15), we shall obtain the system of the continuation equations

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ F_x & F_y & F_z \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \\ 0 \end{pmatrix}, \tag{17}$$

where $F_x = \partial F/\partial x,\ F_y = \partial F/\partial y,\ F_z = \partial F/\partial z$.

To realize the process of continuation, it is necessary to solve this system with respect to differentials. The better conditioned the matrix of system (17), the more effective the process of solving this system for differentials.

The measure of conditionality of system (17) is calculated by the formula

$$|D| = \frac{|\Delta|}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \tag{18}$$

where Δ is determinant of the matrix of system (17).

Having research the measure of conditionality (18) for an extremum as the function of components of the vectors α and β under conditions (16) we obtain that the maximum of the measure of conditionality is achieved at the values

$$\alpha_i = \frac{A_{1i}}{\Delta},\ \ \beta_i = \frac{A_{2i}}{\Delta}, \quad i = 1, 2, 3, \tag{19}$$

where A_{1i}, A_{2i} are cofactors of elements α_i, β_i.

We shall determine the arrangement with respect to the interpolating surface of vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ and $\beta = (\beta_1, \beta_2, \beta_3)^{\mathrm{T}}$, which components satisfy equalities (19). Calculating the scalar products (N, α) and (N, β), where $N = (F_x, F_y, F_z)^{\mathrm{T}}$ is a normal vector of the surface, we get

$$(N, \alpha) = F_x\alpha_1 + F_y\alpha_2 + F_z\alpha_3 = \frac{F_x A_{11} + F_y A_{12} + F_z A_{13}}{\Delta},$$

$$(N, \beta) = F_x\beta_1 + F_y\beta_2 + F_z\beta_3 = \frac{F_x A_{21} + F_y A_{22} + F_z A_{23}}{\Delta}. \tag{20}$$

The sums in numerators of the fractions in the right-hand sides of equalities (20) represent the sums of products of elements of the third line by cofactors of elements of the first and the second line. By virtue of the property of determinants all such sums are equal to zero. Hence, the vectors α and β are orthogonal to the normal vector N and they are situated in the tangent to the surface plane.

Then we shall prove that vector α is orthogonal to vector β. The scalar product of these vectors

$$(\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = \frac{A_{11}\beta_1 + A_{12}\beta_2 + A_{13}\beta_3}{\Delta} \tag{21}$$

is equal to zero, because of the numerator of the fraction in the right-hand side of equality (21) is equal to zero by virtue of the property of determinants. Hence, determining in everyone point of the surface the directions of the lines of the best parameters vectors α and β are orthogonal.

Now we shall prove that the best parameters u and v are the lengths of arches of parametric lines $v = \text{const}$ and $u = \text{const}$ accordingly.

At the lines $v = \text{const}$ the differential of the parameter v is equal to zero, and the continuation equations system takes the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ F_x & F_y & F_z \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \mathrm{d}u \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

If we will divide all equations of system (22) by $\mathrm{d}u$, we will obtain the system of the linear equations with respect to derivatives $x_u = \mathrm{d}x/\mathrm{d}u$, $y_u = \mathrm{d}y/\mathrm{d}u$, $z_u = \mathrm{d}z/\mathrm{d}u$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ F_x & F_y & F_z \end{pmatrix} \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{23}$$

Written by Cramer's rule solution of system (23) coincides with the expressions for ensuring the best conditionality to the system of continuation equations components of the vector α (19). That is, when $v = \text{const}$, the equalities

$$\alpha_1 = x_u, \quad \alpha_2 = y_u, \quad \alpha_3 = z_u \tag{24}$$

are taking place.

Substituting values (24) to the first of expressions (15) and multiplying the obtained equality by du, we get the equality

$$(\mathrm{d}u)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{25}$$

Thus the parameter u is the arc length of the parametric line $v = \text{const}$.

Similarly it can be shown that when $u = \text{const}$ the differential of the parameter v satisfies to the equality

$$(\mathrm{d}v)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{26}$$

Sufficiency. Let us take the arc lengths of two orthogonal parametric lines $v = \text{const}$ and $u = \text{const}$ as the parameters u and v.

The vectors $r_u = (x_u, y_u, z_u)^{\mathrm{T}}$ and $r_v = (x_v, y_v, z_v)^{\mathrm{T}}$ are tangential to the parametric lines $v = \text{const}$ and $u = \text{const}$. The meaning of the vectors α and β is that they determine the local directions of measurement of the parameters. On account of the chosen parameters α and β must be pointed along the tangent to the curves $v = \text{const}$ and $u = \text{const}$. Therefore, the vector α is collinear to the vector r_u and the vector β is collinear to the vector r_v.

The differentials of the chosen parameters satisfy equalities (25), (26). Dividing equality (25) by $(\mathrm{d}u)^2$ and equality (26) by $(\mathrm{d}v)^2$ we obtain the equalities $x_u^2 + y_u^2 + z_u^2 = r_u^2 = 1$, $x_v^2 + y_v^2 + z_v^2 = r_v^2 = 1$. Therefore, the vectors r_u and r_v are unity as the vectors α and β. Since $r_u = \alpha$ and $r_v = \beta$ then the components of these vectors are equal too, and equalities (19) ensure the best conditionality of the continuation equation system (17). $\square$

4. Conclusions

Optimal parameterization conditions are obtained using the parametric continuation method. It is shown that the optimal parameter is the length of the curve to be interpolated. If the length of the polyline is used as a parameter, the parameterization is close to the optimal one. For the parametric approximation of a surface, the optimal parameterization at each surface point is given by two orthogonal curves lying on the surface and passing through this point. The optimal parameters are the lengths of the arcs of those curves.

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